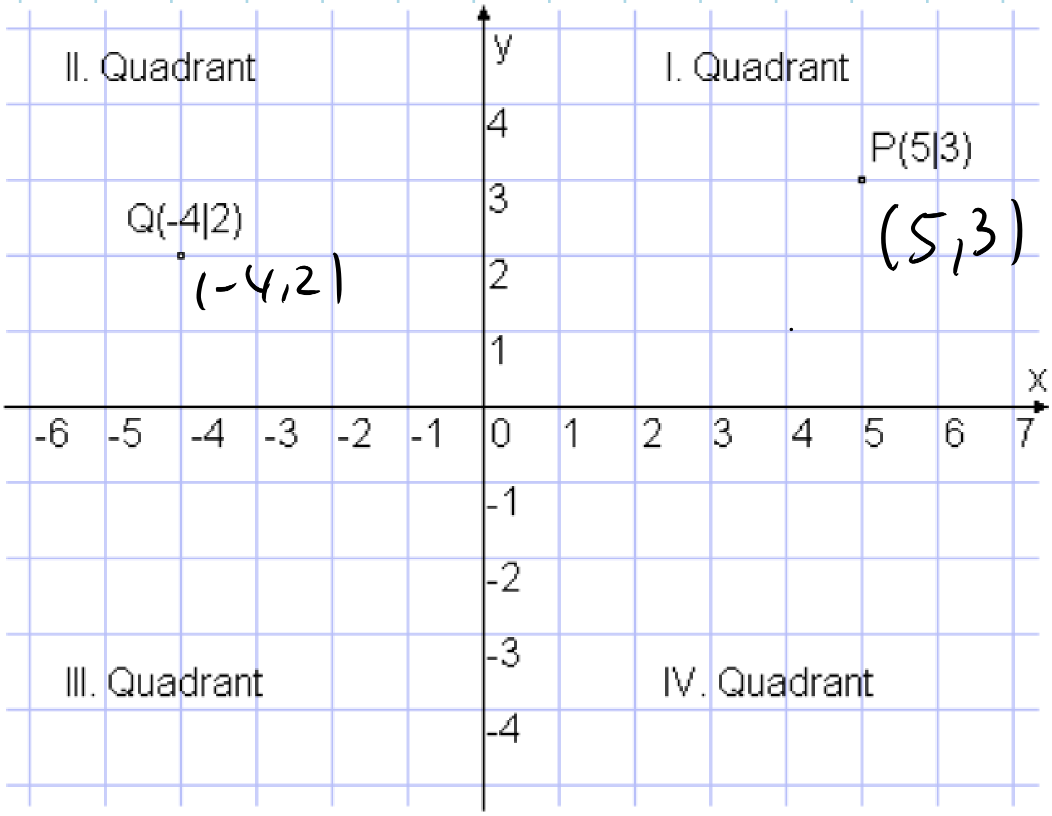
3. Vorlesung 25.10.10

1.3 Die Ebene

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x, y \in \mathbb{R}\}$

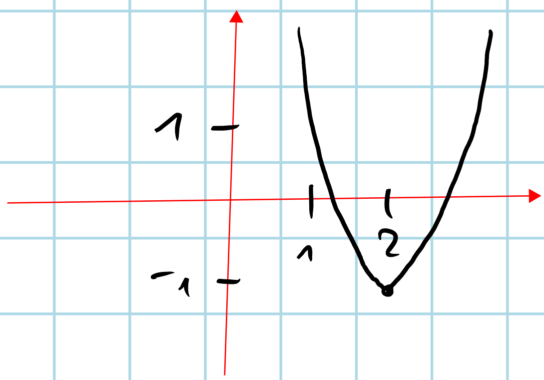


Häufig auftretende Teilmengen der Ebene

- Graph einer Funktion $f: \underbrace{I}_{\text{Intervall}} \longrightarrow \mathbb{R}$

$G_f = \{(x,y) \mid y = f(x), x \in I\}$

z.B. $f(x) = 3(x-2)^2 - 1$



- Lösungsmengen von Gleichungen

$$\mathbb{L} = \{(x,y) \mid F(x,y) = 0\}$$

$F: \mathbb{R}^2 \to \mathbb{R}$ eine Funktion

z.B. 1) $F(x,y) = (x-1)^2 + (y+2)^2 - 4$

ist ein Kreis vom Radius 2 um $(1,-2)$

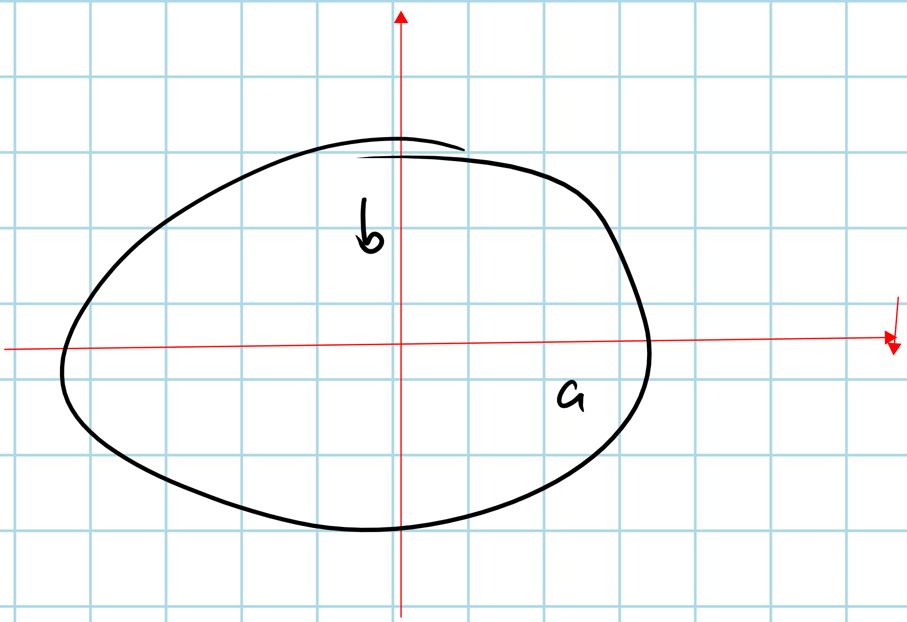
$\tilde{x} = x-1$ $\quad F(x,y) = 0 \iff \tilde{x}^2 + \tilde{y}^2 = 2^2$

$\tilde{y} = y+2$

Kreis um $\tilde{x} = \tilde{y} = 0$

2) $F(x,y) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1$

$\mathbb{L}$ ist eine Ellipse

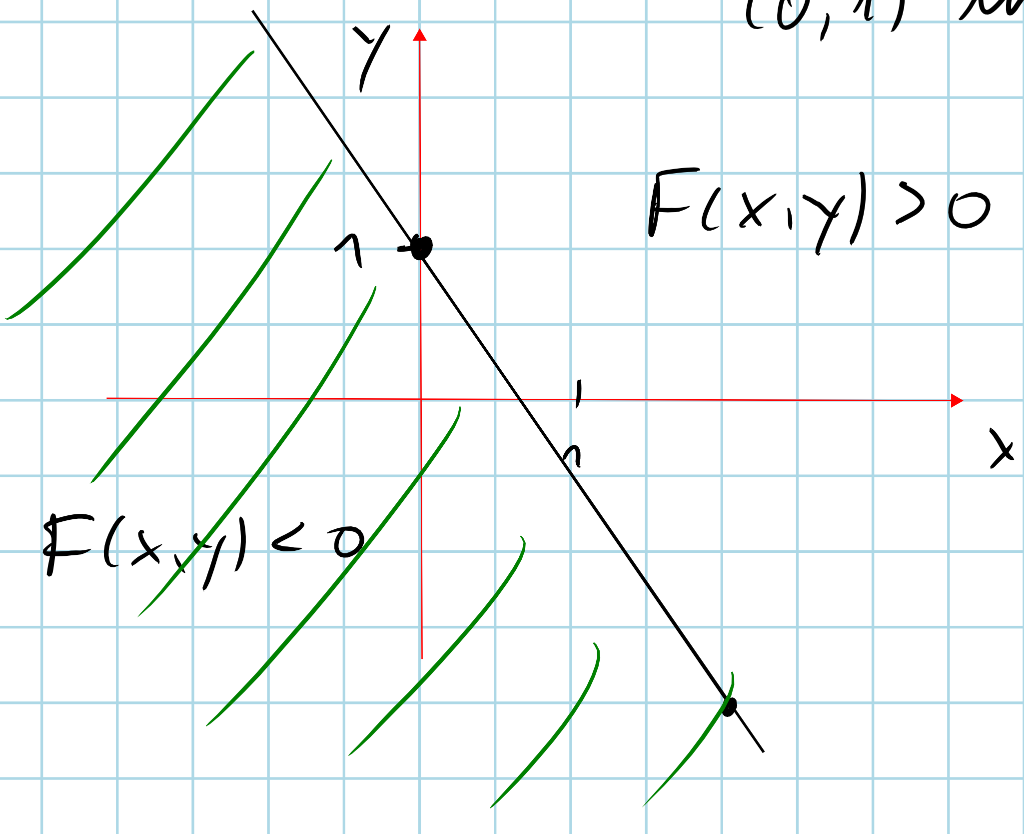


- Lösungen von Ungleichungen

$$\mathbb{L} = \{ (x,y) \mid F(x,y) < 0 \}$$

z.B. $F(x,y) = 3x + 2y - 2$

$F(x,y) = 0$ Gerade durch $(0,1)$ und $(2,-2)$



Trigonometrie

Vollwinkel $360°$

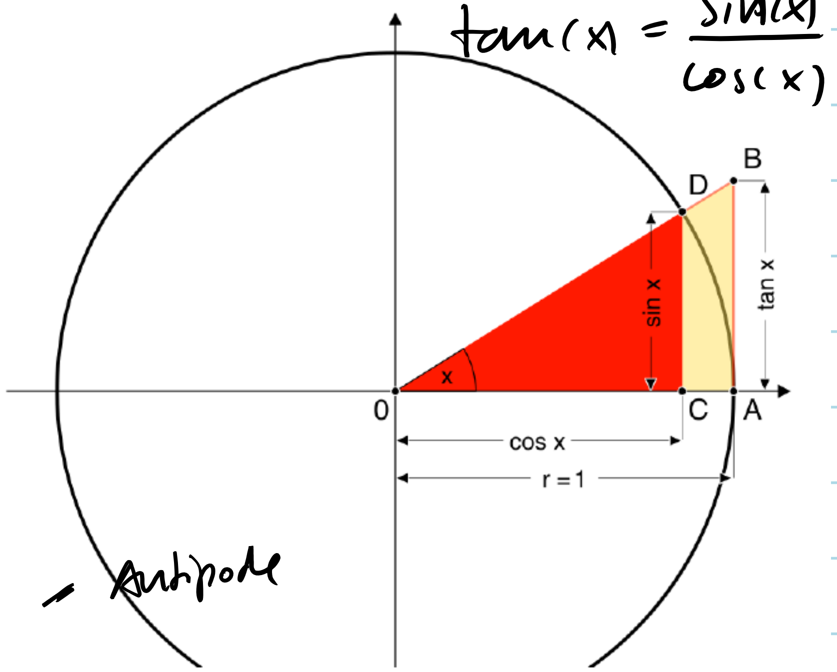
Bogenmaß 2π

$$x = \frac{2\pi}{360°} \alpha$$

Bogenmaß (x), Grad (α)

$D = (\cos(x), \sin(x))$

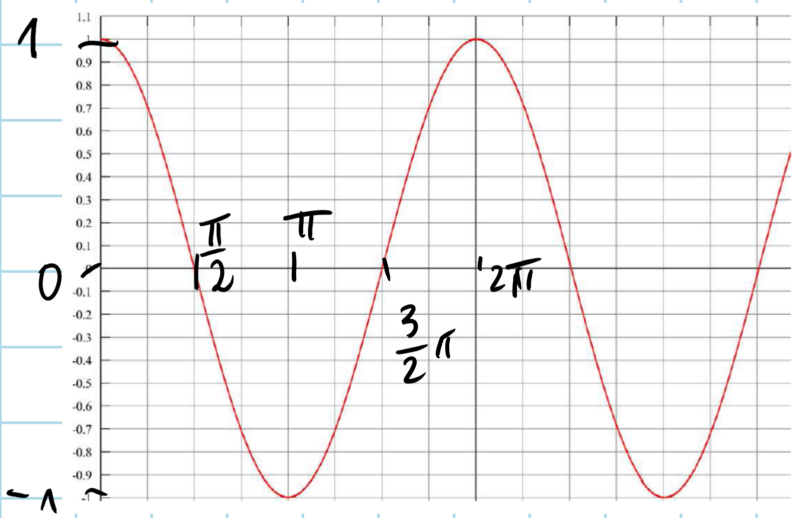
$\tan(x) = \frac{\sin(x)}{\cos(x)}$



$\cos(x) = \frac{\text{Ankatete}}{\text{Hypotenuse}}$ $\sin(x) = \frac{\text{Gegenkat}}{\text{Hypotenuse}}$

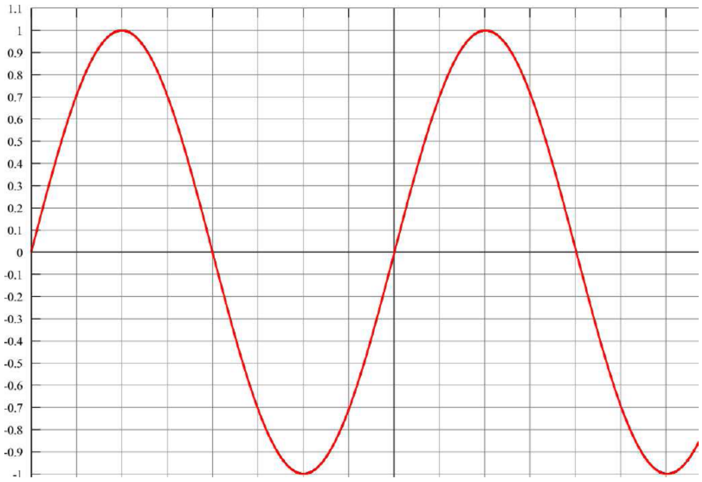
Cosinus

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
sin ~~cos~~ α	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

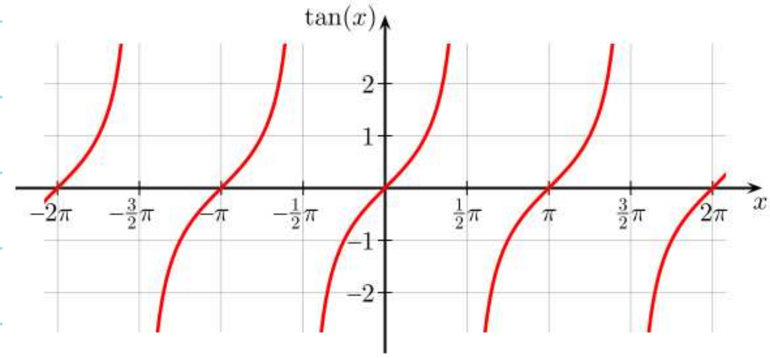


$$\cos(\pi + \alpha) = -\cos\alpha\,, \quad \sin(\pi + \alpha) = -\sin\alpha\,,$$
$$\cos(2\pi + \alpha) = \cos\alpha\,, \quad \sin(2\pi + \alpha) = \sin\alpha\,.$$

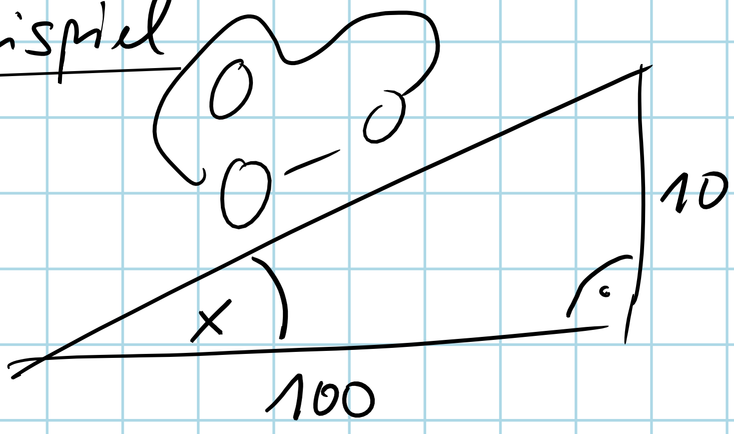
sinus



tangens
nicht definiert
für für $\pi/2 + k\mathbb{Z}$



Beispiel



Umkehrfkt

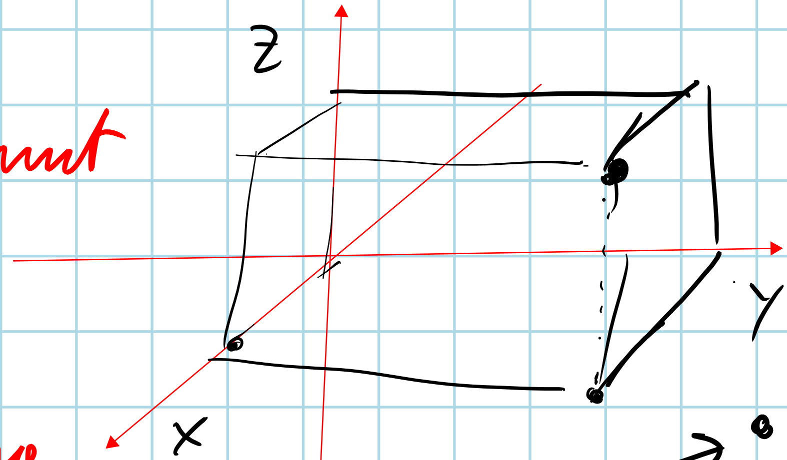
$$\tan(x) = \frac{10}{100} = 0{,}1 \qquad \Big) \tan^{-1} = \arctan$$

$$x = \arctan(0{,}1) \approx 0{,}1 \approx 6^\circ$$

entsprechend $\arccos$, $\arcsin$

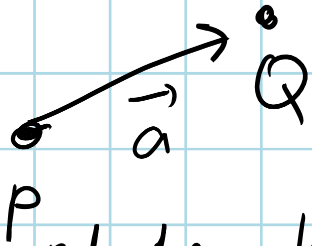
1.4 Vektoren im 3-dimensionalen Raum

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$$

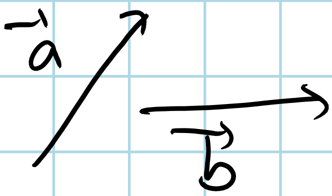


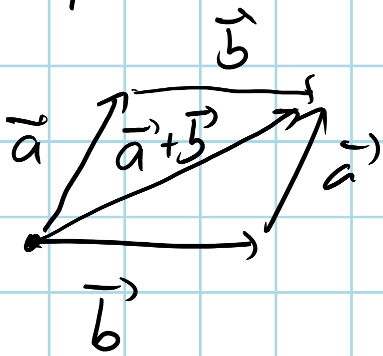
Vektoren sind bestimmt durch ihre Länge und ihre Richtung

$\vec{a} = \overrightarrow{PQ} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ Koordinaten des Punktes des im Nullpunkt verschobenen Vektor

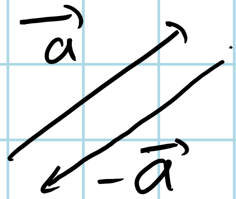


Addition von Vektoren



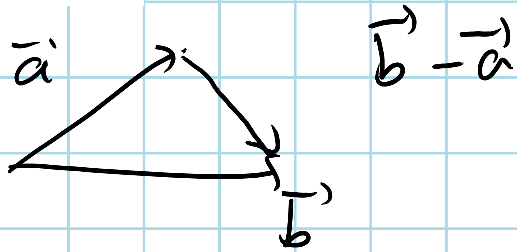


$\vec{0}$ = Nullvektor



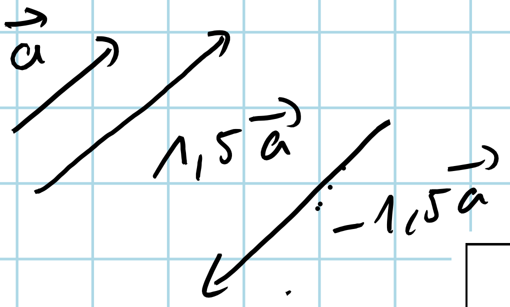
$$\vec{a} + \vec{0} = \vec{a}$$
$$\vec{a} + (-\vec{a}) = \vec{0}$$
$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{Kommutativgesetz}$$

$$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} \qquad \text{Assoziativgesetz.}$$



Skalare Vielfache:

$$\alpha \cdot \vec{a} = \begin{cases} \text{Vektor mit gleicher Richtung} \\ \text{Länge } \alpha \cdot |\vec{a}| & \alpha > 0 \\ \text{Vektor mit entgegengesetzter R.} \\ \text{Länge } |\alpha| \cdot |\vec{a}| & \alpha < 0 \end{cases}$$

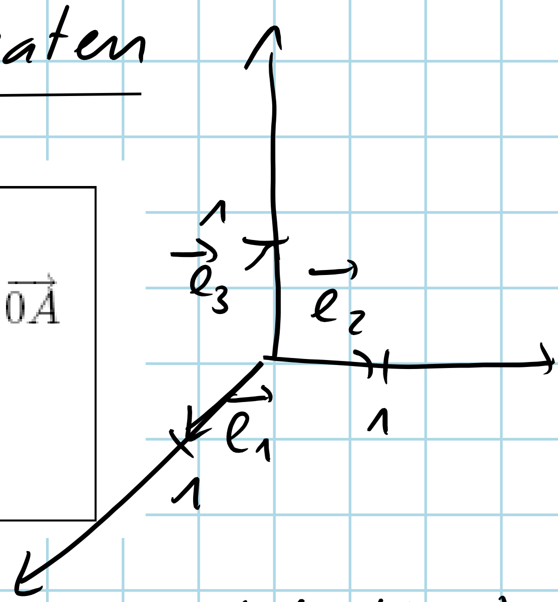


$$\alpha(\beta\vec{a}) = (\alpha\beta)\vec{a}$$
$$\alpha(\vec{a}+\vec{b}) = (\alpha\vec{a}) + (\alpha\vec{b})$$
$$(\alpha+\beta)\vec{a} = (\alpha\vec{a}) + (\beta\cancel{\vec{b}}\,\vec{a}).$$

Betrag (Länge)

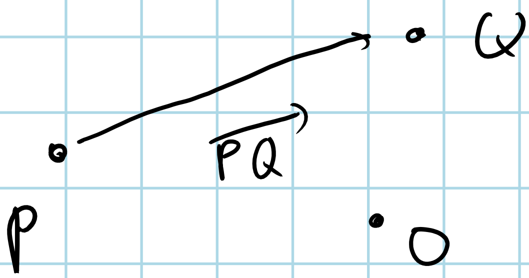
$$|\alpha\vec{a}| = |\alpha||\vec{a}| \qquad \text{insb. } |-\vec{a}| = |\vec{a}|$$
$$|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}| \quad \text{DREIECKSUNGLEICHUNG.}$$

zur Notation in Koordinaten



$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \iff \vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = \overrightarrow{0A}$$

mit $A = (a_1, a_2, a_3)$.



$P = (1, 4, 1)$

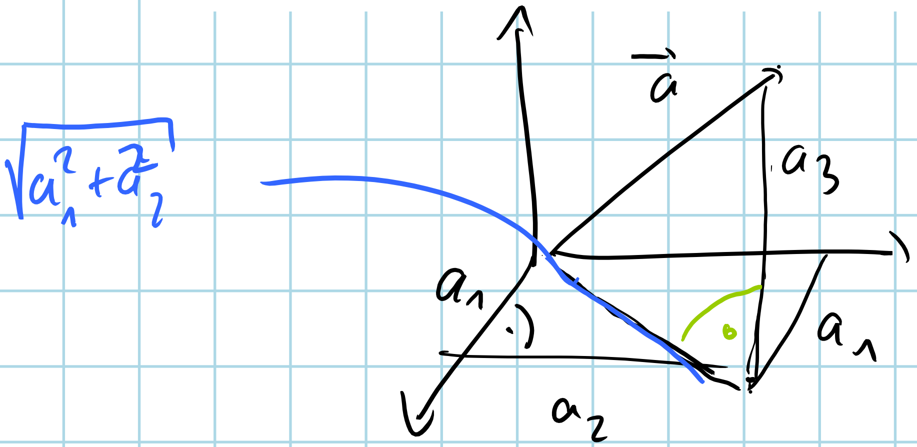
$Q = (8, 8, 4)$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 8 \\ 8 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 8-1 \\ 8-4 \\ 4-1 \end{pmatrix}$$

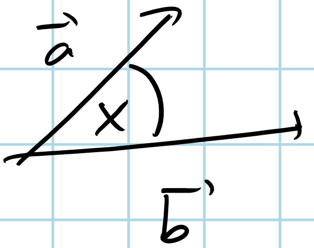
$$= \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}$$

$$\overrightarrow{PQ} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix} \quad \text{falls } P = (p_1, p_2, p_3),\ Q = (q_1, q_2, q_3).$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$
$$\alpha \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \alpha a_1 \\ \alpha a_2 \\ \alpha a_3 \end{pmatrix}$$
$$\left| \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$



Winkel zwischen Vektoren



$x = \sphericalangle(\vec{a}, \vec{b})$

$0 \leq x \leq \pi$

$x = \frac{\pi}{2} \iff \vec{a} \perp \vec{b}$

senkrecht

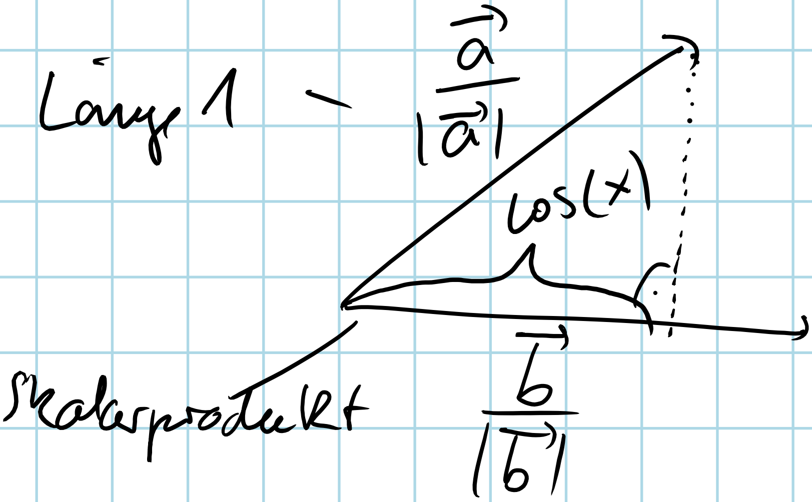
einfache Regeln

$$\angle(\vec{a}, \vec{b}) = \angle(\vec{b}, \vec{a}),$$
$$\angle(\vec{a}, t\vec{a}) = 0 \quad \text{, falls } t > 0,$$
$$\angle(\vec{a}, t\vec{a}) = \pi \quad \text{, falls } t < 0,$$
$$\angle(-\vec{a}, \vec{b}) = \pi - \angle(\vec{a}, \vec{b}).$$

Skalarprodukt

$$\underbrace{\vec{a}\cdot\vec{b}}_{\text{Skalar}} = \begin{cases} |\vec{a}||\vec{b}|\cos(\angle(\vec{a},\vec{b})) & \text{, falls } \vec{a}\neq 0 \text{ und } \vec{b}\neq 0,\\ 0 & \text{, falls } \vec{a}=0 \text{ oder } \vec{b}=0. \end{cases}$$

$$\frac{\vec{a}}{|\vec{a}|}\cdot\frac{\vec{b}}{|\vec{b}|} = \cos(\angle\overbrace{(\vec{a},\vec{b})}^{x})$$



Projektion des normierten Vektors auf den anderen

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} \qquad \text{Kommutativgesetz}$$

$$(\alpha\vec{a})\cdot\vec{b} = \vec{a}\cdot(\alpha\vec{b}) = \alpha(\vec{a}\cdot\vec{b})$$

$$(\vec{a}+\vec{b})\cdot\vec{c} = (\vec{a}\cdot\vec{c}) + (\vec{b}\cdot\vec{c}) \qquad \text{Distributivgesetz}$$

$$\vec{a}\cdot\vec{b} = 0 \iff \vec{a}\perp\vec{b} \qquad \text{Orthogonalitätstest}$$

$$\Rightarrow \quad |\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}.$$

Im Koordinaten

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

folgt aus den Rechenregeln:

$$(a_1 \vec{e}_1 + a_2 \vec{e}_2 + a_3 \vec{e}_3) \cdot (b_1 \vec{e}_1 + b_2 \vec{e}_2 + b_3 \vec{e}_3)$$

$$= a_1 b_1 \underbrace{\vec{e}_1 \cdot \vec{e}_1}_{|\vec{e}_1|^2 = 1} + a_2 b_2 \underbrace{\vec{e}_2 \cdot \vec{e}_2}_{=1} + a_3 b_3 \underbrace{\vec{e}_3 \vec{e}_3}_{=1}$$

Beispiel

$$\sphericalangle \underbrace{\left(\begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right)}_{x} = ?$$

$$\cos(x) = \frac{\begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{7^2+4^2+3^2}\,\sqrt{1^2+1^2}} = \frac{7 \cdot 1 + 4 \cdot 0 + 3(-1)}{\sqrt{49+16+9}\,\sqrt{2}}$$

$$= 0{,}33 \qquad \Big| \underbrace{\arccos}_{\cos^{-1}}$$

$$x = \arccos(0{,}33) \approx 1{,}2.$$