RUHR UNIVERSITÄT BOCHUM RUB

Metastability in the reversible inclusion process

Sander Dommers

Joint work with

Alessandra Bianchi and Cristian Giardinà

Inclusion process

Interacting particle system with N particles

Vertex set S with $|S| < \infty$

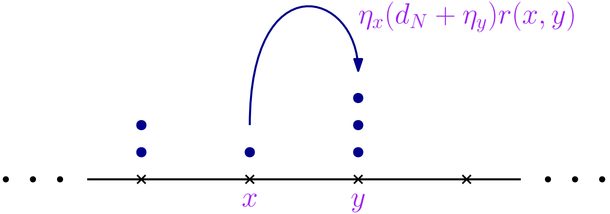
Configuration $\eta = (\eta_x)_{x \in S} \in \{0, \dots, N\}^S$, $\eta_x = \#\text{particles on } x \in S$

Underlying random walk on S with transition rates $r(x, y)$

Inclusion process is continuous time Markov process with generator

$$\mathcal{L}f(\eta) = \sum_{x,y \in S} \eta_x (d_N + \eta_y) r(x, y) \left[f(\eta^{x,y}) - f(\eta) \right]$$

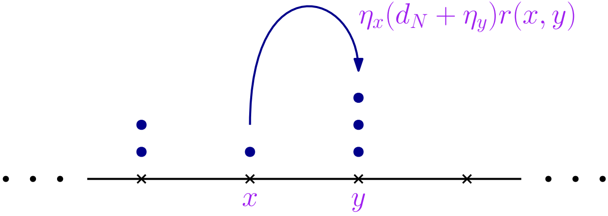
Particle jump rates



Particle jump rates can be split into

$\eta_x \, d_N \, r(x, y)$	independent random walkers	diffusion
$\eta_x \, \eta_y \, r(x, y)$	attractive interaction	inclusion

Particle jump rates



Particle jump rates can be split into

$\eta_x \, d_N \, r(x, y)$	independent random walkers	diffusion
$\eta_x \, \eta_y \, r(x, y)$	attractive interaction	inclusion

Comparison with other processes

$\eta_x \, (1 - \eta_y) \, r(x, y)$	exclusion process
$g(\eta_x) \, r(x, y)$	zero range process

Motivation

Symmetric IP on $\mathbb{Z}$ introduced as dual of Brownian momentum process
Giardinà, Kurchan, Redig, Vafayi, 2007–2010

Natural bosonic counterpart to the (fermionic) exclusion process

Interesting dynamical behavior: condensation / metastability
In symmetric IP: Grosskinsky, Redig, Vafayi 2011, 2013

Motivation

Symmetric IP on $\mathbb{Z}$ introduced as dual of Brownian momentum process
Giardinà, Kurchan, Redig, Vafayi, 2007–2010

Natural bosonic counterpart to the (fermionic) exclusion process

Interesting dynamical behavior: condensation / metastability
In symmetric IP: Grosskinsky, Redig, Vafayi 2011, 2013

Can we analyze this using the martingale approach?
Beltrán, Landim, 2010

Successfully used for reversible zero range process Beltrán, Landim, 2012

Can we generalize results to the reversible IP?

Reversible inclusion process

Random walk $r(\cdot,\cdot)$ reversible w.r.t. some measure $m(\cdot)$

$$m(x)r(x,y) = m(y)r(y,x) \qquad \forall\, x,y \in S$$

Normalized such that

$$\max_{x\in S} m(x) = 1$$

Reversible inclusion process

Random walk $r(\cdot,\cdot)$ reversible w.r.t. some measure $m(\cdot)$

$$m(x)r(x,y) = m(y)r(y,x) \qquad \forall\, x,y \in S$$

Normalized such that

$$\max_{x\in S} m(x) = 1$$

Then, also inclusion process reversible w.r.t. probability measure

$$\mu_N(\eta) = \frac{1}{Z_N} \prod_{x\in S} m(x)^{\eta_x} w_N(\eta_x)$$

where Z_N is a normalization constant and

$$w_N(k) = \frac{\Gamma(d_N + k)}{k!\Gamma(d_N)}$$

Condensation

Let $S_\star = \{x \in S : m(x) = 1\}$ and $\eta^{x,N}$ the configuration η with $\eta_x = N$

Proposition

Suppose that $d_N \log N \to 0$ as $N \to \infty$. Then

$$\lim_{N\to\infty} \mu_N(\eta^{x,N}) = \frac{1}{|S_\star|} \qquad \forall\, x \in S_\star$$

Condensation

Let $S_\star = \{x \in S : m(x) = 1\}$ and $\eta^{x,N}$ the configuration η with $\eta_x = N$

Proposition

Suppose that $d_N \log N \to 0$ as $N \to \infty$. Then

$$\lim_{N\to\infty} \mu_N(\eta^{x,N}) = \frac{1}{|S_\star|} \qquad \forall\, x \in S_\star$$

Assumption on d_N such that

$$\frac{N}{d_N} w_N(N) = \frac{1}{d_N \Gamma(d_N)} \frac{\Gamma(N + d_N)}{(N-1)!} = \frac{1}{\Gamma(d_N + 1)} e^{d_N \log N}(1 + o(1)) \to 1$$

(using Stirling's approximation)

Movement of the condensate

RUB

Consider the following process on $S_\star \cup \{0\}$:

$$X_N(t) = \sum_{x \in S_\star} x \, \mathbb{1}_{\{\eta_x(t)=N\}}$$

Theorem (Bianchi, D., Giardinà, 2016)

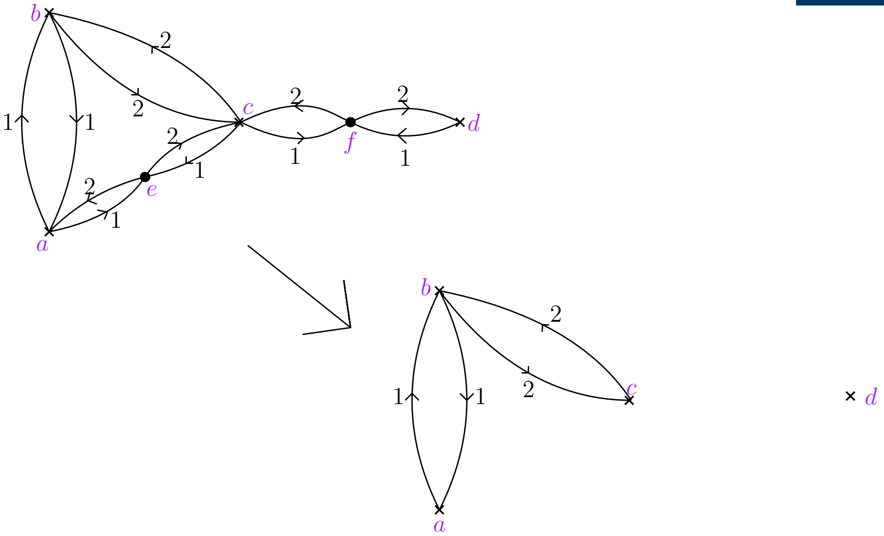
Suppose that $d_N \log N \to 0$ *as* $N \to \infty$ *and that* $\eta_y(0) = N$ *for some* $y \in S_\star$. *Then*

$$X_N(t/d_N) \text{ converges weakly to } x(t) \qquad \text{as } N \to \infty$$

where $x(t)$ *is a Markov process on* $S_\star$ *with* $x(0) = y$ *and transition rates*

$$p(x, y) = r(x, y)$$

Example



Zero range process Beltrán, Landim, 2012

Underlying reversible random walk $r(\cdot,\cdot)$

Transition rates for a particle to move from x to y

$$\left(\frac{\eta_x}{\eta_x - 1}\right)^{\alpha} r(x,y), \qquad \alpha > 1$$

Condensate consists of at least $N - \ell_N$ particles, $\ell_N = o(N)$

At timescale $t\,N^{\alpha+1}$ the condensate moves from $x \in S_\star$ to $y \in S_\star$ at rate

$$p(x,y) = C_\alpha \mathrm{cap}(x,y)$$

where $\mathrm{cap}(x,y)$ is the capacity of the underlying random walk between x and y

Proof strategy

If $r(\cdot,\cdot)$ is symmetric ($S = S_\star$), cite Grosskinsky, Redig, Vafayi, 2013
They analyze directly rescaled generator

Otherwise, martingale approach Beltrán, Landim, 2010
Potential theory combined with martingale arguments

Successfully applied to zero range process Beltrán, Landim, 2012

Martingale approach Beltrán, Landim, 2010

To prove the theorem we need to check the following three hypotheses:

(H0) $\lim_{N\to\infty} \frac{1}{d_N} p_N(\eta^{x,N}, \eta^{y,N}) \to p(x,y) = r(x,y)$

where $p_N(\eta^{x,N}, \eta^{y,N})$ rate to go from $\eta^{x,N}$ to $\eta^{y,N}$ in original process

(H1) All states in each metastable set are visited before exiting

(H2) $$\lim_{N\to\infty} \frac{\mu_N(\eta : \nexists y \in S_\star : \eta_y = N)}{\mu_N(\eta^{x,N})} = 0 \qquad \forall x \in S_\star$$

Martingale approach Beltrán, Landim, 2010

RUB

To prove the theorem we need to check the following three hypotheses:

(H0) $\lim_{N\to\infty} \frac{1}{d_N} p_N(\eta^{x,N}, \eta^{y,N}) \to p(x,y) = r(x,y)$

where $p_N(\eta^{x,N}, \eta^{y,N})$ rate to go from $\eta^{x,N}$ to $\eta^{y,N}$ in original process

(H1) All states in each metastable set are visited before exiting Trivial

(H2) $$\lim_{N\to\infty} \frac{\mu_N(\eta : \nexists y \in S_\star : \eta_y = N)}{\mu_N(\eta^{x,N})} = 0 \qquad \forall x \in S_\star$$

Martingale approach Beltrán, Landim, 2010

RUB

To prove the theorem we need to check the following three hypotheses:

(H0) $\lim_{N\to\infty} \frac{1}{d_N} p_N(\eta^{x,N}, \eta^{y,N}) \to p(x,y) = r(x,y)$

where $p_N(\eta^{x,N}, \eta^{y,N})$ rate to go from $\eta^{x,N}$ to $\eta^{y,N}$ in original process

(H1) All states in each metastable set are visited before exiting Trivial

(H2) $$\lim_{N\to\infty} \frac{\mu_N(\eta : \nexists y \in S_\star : \eta_y = N)}{\mu_N(\eta^{x,N})} = 0 \qquad \forall x \in S_\star$$ Easy

Potential theory

Relation between random walks and electric networks
Doyle, Snell, 1984

For reversible dynamics we can define *conductances* (=1/resistance)

$$c(x, y) = \mu(x)p(x, y)$$

If A, B disjoint, let $h_{A,B}$ be the equilibrium potential, i.e., the solution to the Dirichlet problem

$$\begin{cases} \mathcal{L}h_{A,B}(x) = 0 & \text{if } x \notin A \cup B \\ h_{A,B}(x) = 1 & \text{if } x \in A \\ h_{A,B}(x) = 0 & \text{if } x \in B \end{cases}$$

Potential theory

Probabilistic interpretation

$$h_{A,B}(x) = \mathbb{P}_x[\tau_A < \tau_B]$$

where τ_A is the hitting time of A

$$\tau_A = \inf\{t \geq 0 \,:\, x(t) \in A\}$$

Capacities

Important quantity is the *capacity* (=1/effective resistance) between A and B, given by

$$\mathrm{Cap}(A,B) := D(h_{A,B}) := \frac{1}{2}\sum_{x,y} c(x,y)[h_{A,B}(x) - h_{A,B}(y)]^2$$

$D(F)$ is called the *Dirichlet form*

If A and B are disjoint sets, then

$$\mathbb{E}_{\nu_{A,B}}[\tau_B] = \frac{\mu(h_{A,B})}{\mathrm{Cap}(A,B)}$$

Bovier, Eckhoff, Gayrard, Klein, 2001–2004

Variational principles

Capacity can be computed using *Dirichlet principle*

$$\mathrm{Cap}(A,B) = \inf\{D(F) \,:\, F(x) = 1 \;\forall\, x \in A, F(x) = 0 \;\forall\, x \in B\}$$

Minimizer is $F = h_{A,B}$, but get upper bound for any test function F

There also exist variational principles, the *Thomson* and *Berman-Konsowa* principle, where the capacity is expressed as a *supremum* over flows

Capacities in inclusion process

RUB

Capacities satisfy

$$\mathrm{Cap}_N(A,B) = \inf\{D_N(F) : F(\eta) = 1\ \forall\, \eta \in A, F(\xi) = 0\ \forall\, \eta \in B\}$$

where $D_N(F)$ is the Dirichlet form

$$D_N(F) = \frac{1}{2}\sum_{\eta} \mu_N(\eta) \sum_{x,y\in S} \eta_x(d_N + \eta_y) r(x,y)[F(\eta^{x,y}) - F(\eta)]^2$$

Lemma (Beltrán, Landim, 2010)

$$\mu_N(\eta^{x,N}) p_N(\eta^{x,N}, \eta^{y,N})$$
$$= \frac{1}{2}\Big\{\mathrm{Cap}_N\Big(\{\eta^{x,N}\}, \bigcup_{z\in S_\star, z\neq x}\{\eta^{z,N}\}\Big) + \mathrm{Cap}_N\Big(\{\eta^{y,N}\}, \bigcup_{z\in S_\star, z\neq y}\{\eta^{z,N}\}\Big)$$
$$- \mathrm{Cap}_N\Big(\{\eta^{x,N}, \eta^{y,N}\}, \bigcup_{z\in S_\star, z\neq x,y}\{\eta^{z,N}\}\Big)\Big\}$$

Capacities in inclusion process

RUB

Proposition

Let $S_\star^1 \subsetneq S_\star$ *and* $S_\star^2 = S_\star \setminus S_\star^1$. *Then, for* $d_N \log N \to 0$,

$$\lim_{N\to\infty} \frac{1}{d_N} \mathrm{Cap}_N\Big(\bigcup_{x\in S_\star^1} \{\eta^{x,N}\}, \bigcup_{y\in S_\star^2} \{\eta^{y,N}\}\Big) = \frac{1}{|S_\star|} \sum_{x\in S_\star^1} \sum_{y\in S_\star^2} r(x,y)$$

Combining this proposition and the previous lemma indeed gives

$$\lim_{N\to\infty} \frac{1}{d_N} p_N(\eta^{x,N}, \eta^{y,N}) \to r(x,y)$$

Lower bound on Dirichlet form

Fix any function F such that $F(\eta^{x,N}) = 1 \; \forall \, x \in S_\star^1$ and $F(\eta^{y,N}) = 0 \; \forall \, y \in S_\star^2$

Sufficient to show that

$$D_N(F) \geq d_N \frac{1}{|S_\star|} \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x,y)\big(1 + o(1)\big)$$

Lower bound on Dirichlet form

Fix any function F such that $F(\eta^{x,N}) = 1 \; \forall\, x \in S_\star^1$ and $F(\eta^{y,N}) = 0 \; \forall\, y \in S_\star^2$

Sufficient to show that

$$D_N(F) \geq d_N \frac{1}{|S_\star|} \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x,y)\big(1 + o(1)\big)$$

For lower bound we can throw away terms in the Dirichlet form

$$D_N(F) = \frac{1}{2} \sum_{\eta} \mu_N(\eta) \sum_{x,y \in S} \eta_x (d_N + \eta_y) r(x,y) [F(\eta^{x,y}) - F(\eta)]^2$$
$$\geq \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x,y) \sum_{\eta_x + \eta_y = N} \mu_N(\eta) \eta_x (d_N + \eta_y) [F(\eta^{x,y}) - F(\eta)]^2$$

If condensates jumps from x to y all particles will move from x to y

Lower bound on Dirichlet form (continued)

RUB

Fix $x \in S^1_\star, y \in S^2_\star$. If $\eta_x + \eta_y = N$ it is sufficient to know how many particles are on x

$$\sum_{\eta_x+\eta_y=N} \mu_N(\eta)\eta_x(d_N + \eta_y)[F(\eta^{x,y}) - F(\eta)]^2$$

$$= \sum_{k=1}^{N} \frac{w_N(k)w_N(N-k)}{Z_N} k(d_N + N - k)[G(k-1) - G(k)]^2$$

where $G(k) = F(\eta_x = k, \eta_y = N - k)$ and where we used $m(x) = m(y) = 1$ since $x \in S^1_\star, y \in S^2_\star$

Lower bound on Dirichlet form (continued)

Fix $x \in S_\star^1, y \in S_\star^2$. If $\eta_x + \eta_y = N$ it is sufficient to know how many particles are on x

$$\sum_{\eta_x+\eta_y=N} \mu_N(\eta)\eta_x(d_N + \eta_y)[F(\eta^{x,y}) - F(\eta)]^2$$

$$= \sum_{k=1}^{N} \frac{w_N(k)w_N(N-k)}{Z_N} k(d_N + N - k)[G(k-1) - G(k)]^2$$

$$\geq \frac{d_N}{|S_\star|}(1 + o(1))$$

Lower bound follows from capacity of linear chain
and asymptotics of w_N and Z_N

Lower bound on Dirichlet form (conclusion)

Hence, indeed,

$$\frac{1}{d_N} D_N(F) \geq \frac{1}{|S_\star|} \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x,y)(1+o(1))$$

Taking infimum and limit on both sides indeed proves that

$$\lim_{N \to \infty} \frac{1}{d_N} \mathrm{Cap}_N\Big(\bigcup_{x \in S_\star^1} \{\eta^{x,N}\}, \bigcup_{y \in S_\star^2} \{\eta^{y,N}\} \Big) \geq \frac{1}{|S_\star|} \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x,y)$$

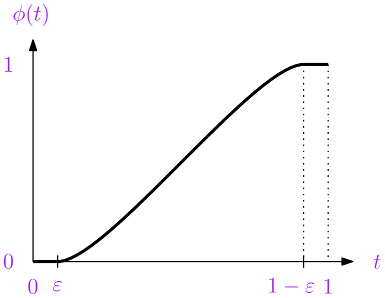
Upper bound on Dirichlet form

Need to construct test function $F(\eta)$

Good guess inside tubes $\eta_x + \eta_y = N$: $F(\eta) \approx \eta_x / N$

In fact better to choose smooth monotone function $\phi(t), t \in [0,1]$ with

$\phi(t) = 1 - \phi(1-t) \; \forall \, t \in [0,1]$
$\phi(t) = 0$ if $t \leq \varepsilon$



and set $F(\eta) = \phi\left(\eta_x / N\right)$

For general η we set

$$F(\eta) = \sum_{x \in S_\star^1} \phi\left(\eta_x / N\right)$$

Observations for upper bound on $D_N(F)$

$$D_N(F) = \frac{1}{2}\sum_{\eta}\mu_N(\eta)\sum_{x,y\in S}\eta_x(d_N+\eta_y)r(x,y)[F(\eta^{x,y})-F(\eta)]^2$$

$$F(\eta) = \sum_{x\in S_\star^1}\phi(\eta_x/N)$$

By construction particles moving from $x \in S_\star^1$ to $y \in S_\star^2$ give correct contribution

If numbers of particles on sites in $S_\star^1$ don't change, or if particles move between sites in $S_\star^1$, F is constant

Unlikely to be in config. with particles on three sites / sites not in $S_\star$

Unlikely for a particle to escape from a tube

Capacities in inclusion process (conclusion)

Combining the lower and upper bound indeed this proposition follows

Proposition

Let $S_\star^1 \subsetneq S_\star$ *and* $S_\star^2 = S_\star \setminus S_\star^1$*. Then, for* $d_N \log N \to 0$,

$$\lim_{N\to\infty} \frac{1}{d_N} \mathrm{Cap}_N\Big(\bigcup_{x\in S_\star^1} \{\eta^{x,N}\}, \bigcup_{y\in S_\star^2} \{\eta^{y,N}\} \Big) = \frac{1}{|S_\star|} \sum_{x\in S_\star^1} \sum_{y\in S_\star^2} r(x,y)$$

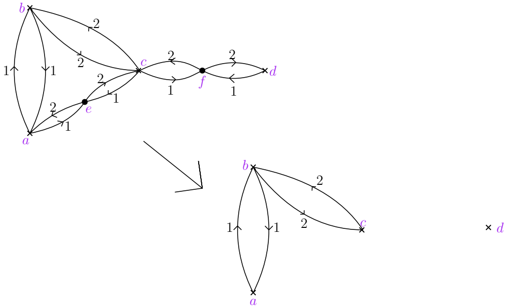
And the transition rates indeed satisfy

$$\lim_{N\to\infty} \frac{1}{d_N} p_N(\eta^{x,N}, \eta^{y,N}) \to r(x,y)$$

proving the theorem

Longer timescales

If induced random walk on $S_\star$ is not connected, condensate jumps on longer timescales



We focus on simple case where the graph is a line

$$S = \{1, \dots, L\} \qquad S_\star = \{1, L\} \qquad r(x, y) \neq 0 \text{ iff } |x - y| = 1$$

Second timescale

RUB

For $L = 3$ jumps occur at rate d_N^2/N

Theorem (Bianchi, D., Giardinà, 2016)

Suppose that $d_N \log N \to 0$ as $N \to \infty$, d_N decays subexponentially and that $\eta_y(0) = N$ for some $y \in S_\star$. Then, for $L = 3$,

$$X_N(tN/d_N^2) \text{ converges weakly to } x(t) \qquad \text{as } N \to \infty$$

where $x(t)$ is a Markov process on $S_\star$ with $x(0) = y$ and transition rates

$$p(1,3) = p(3,1) = \left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)}\right)^{-1} \frac{1}{1 - m_\star(2)}$$

Third timescale

For $L \geq 4$ jumps occur at rate d_N^3/N^2

Theorem (Bianchi, D., Giardinà, 2016)

Suppose that $d_N \log N \to 0$ *as* $N \to \infty$, d_N *decays subexponentially. Then, for* $L \geq 4$, *there exist constants* $0 < C_1 \leq C_2 < \infty$ *such that*

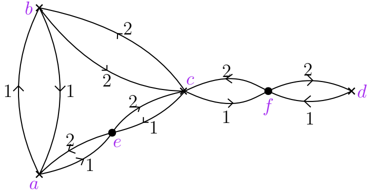
$$C_1 \leq \liminf_{N\to\infty} \frac{d_N^3}{N^2} \mathbb{E}_{\eta^{1,N}}[\tau_{\eta^{L,N}}] \leq \limsup_{N\to\infty} \frac{d_N^3}{N^2} \mathbb{E}_{\eta^{1,N}}[\tau_{\eta^{L,N}}] \leq C_2$$

Conjectured transition rates of time-rescaled process

$$p(1,L) = p(L,1) = 3 \left(\sum_{\ell=2}^{L-2} \frac{(1-m_\star(\ell))(1-m_\star(\ell+1))}{m_\star(\ell) r(\ell,\ell+1)} \right)^{-1}$$

Open problems / future work

Complete picture in case vertices in $S_\star$ are not connected
Conjecture: Only these 3 timescales



Compute relaxation time

Compute thermodynamic limit
Zero-range process: Armendáriz, Grosskinsky, Loulakis, 2015

Study formation of the condensate
Studied for SIP in Grosskinsky, Redig, Vafayi, 2013

Study behavior for non-reversible dynamics
e.g. (T)ASIP on $\mathbb{Z}/L\mathbb{Z}$. Heuristics: Cao, Chleboun, Grosskinsky, 2014