On the solvability of Pell's equation – an approach using continued fractions

and: the Euler-Muir theorem and equipalindromic numbers

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1. Introduction

In my bachelor's thesis I am going to examine the solvability of *Pell's equation* using the theory of continued fractions and discuss the Euler-Muir theorem.

1.1. What's Pell's equation?

Pell's equation (after *John Pell*[1]) is given as

$$x^2 - dy^2 = 1$$

where d is a fixed positive integer and the question to be examined is whether there exist integer solutions (x, y), that is, it is a so-called *Diophantine equation*.

Obviously, $(x, y) = (1, 0)$ is a trivial solution that we will not discuss any further. In general, we will see that this equation has infinitely many non-trivial solutions if d is not a perfect square.

The *negative Pell's equation* is the related Diophantine equation

$$x^2 - dy^2 = -1$$

where d is a fixed positive integer as well.

This equation does not have a trivial solution except when $d = 1$, where $(x, y) = (0, 1)$ solves the equation. The criteria for the solvability of the negative Pell's equation are a bit too complex for the introduction and will be discussed in section 3.2.

We further note that if (x, y) solves one of the above equations, we see from $n^2 = (-n)^2$ that $(x, -y)$, $(-x, y)$ and $(-x, -y)$ are solutions as well, which is why we will only permit positive values for x, y from now on.

1.2. What's a continued fraction?

A *continued fraction* is a finite or infinite expression

$$[a_0, a_1, a_2, \ldots] := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}};$$

we call the continued fraction *simple* if all a_k are integers.

Every common fraction (every rational number) can be developed into a simple finite cont'd fraction and every irrational number has a simple infinite continued fraction representation as we will see later in this thesis – for example, we have

$$\frac{17}{3} = [5, 1, 2] = 5 + \frac{1}{1 + \frac{1}{2}} \quad \text{and} \quad \sqrt{11} = [3, \overline{3, 6}] = [3, 3, 6, 3, 6, \ldots] = 3 + \frac{1}{3 + \frac{1}{6 + \ldots}}.$$

[1] mistakenly, as Wikipedia states that Pell never analysed the solvability of this equation, a more fitting name would be *Fermat's equation* as Fermat was the first European to discuss its solvability, although this equation was already known to Indian mathematicians *Brahmagupta* in the 7th and *Bhaskara II.* in the 12th century

1.3. What's the link between these topics?

We will prove that the solvability of the (classical or negative) Pell's equation

$$x^2 - dy^2 = \pm 1$$

is linked to the continued fraction representation of $\sqrt{d}$, by means of which we can not only answer the question of solvability, but calculate all non-trivial solutions directly.

As we will see further down in this thesis, the infinite continued fraction representation of $\sqrt{d}$ is periodic if d is not a perfect square – the above example lists

$$\sqrt{11} = [3, \overline{3, 6}] = [3, 3, 6, 3, 6, \ldots] = 3 + \cfrac{1}{3 + \frac{1}{6 + \ldots}}$$

with period length 2 and the solutions of $x^2 - 11y^2 = 1$ can be calculated as follows:

$$\frac{x_1}{y_1} = [3, 3] = \frac{10}{3}, \qquad 10^2 - 11 \cdot 3^2 = 100 - 99 = 1;$$
$$\frac{x_2}{y_2} = [3, 3, 6, 3] = \frac{199}{60}, \qquad 199^2 - 11 \cdot 60^2 = 39601 - 39600 = 1;$$
$$\frac{x_3}{y_3} = [3, 3, 6, 3, 6, 3] = \frac{3970}{1197}, \qquad 3970^2 - 11 \cdot 1197^2 = 15760900 - 15760899 = 1$$

and so on. We should note that, for $\sqrt{d} = [a_0, \overline{a_1, \ldots, a_m}]$ with period length $m \geq 1$, only the convergents $[a_0, a_1, \ldots, a_{km-1}]$ with $k \in \mathbb{N}$ give solutions of the generalised Pell's equation $x^2 - dy^2 = (-1)^m$, for instance we have $[3, 3, 6] = 63/19$, but

$$63^2 - 11 \cdot 19^2 = 3969 - 3971 = -2 \neq 1.$$

In particular, the negative Pell's equation $x^2 - dy^2 = -1$ is only solvable when the period in the continued fraction expansion of $\sqrt{d}$ has odd length – for $d = 11$ we have a period length of 2, so $x^2 - 11y^2 = -1$ has no solutions. On the other hand, we have $\sqrt{13} = [3, \overline{1, 1, 1, 1, 6}]$ with period length 5 and the negative Pell's equation $x^2 - 13y^2 = -1$ is solvable:

$$\frac{x_1}{y_1} = [3, 1, 1, 1, 1] = \frac{18}{5}, \qquad 18^2 - 13 \cdot 5^2 = 324 - 325 = -1.$$

1.4. What's the Euler-Muir theorem?

The period in the continued fraction expansion of $\sqrt{d} = [a_0, \overline{a_1, \ldots, a_m}]$ apart from its last entry is *palindromic* – we have $a_k = a_{m-k}$ for $1 \leq k \leq m/2$:

$$\sqrt{d} = [a_0, \overline{a_1, a_2, \ldots, a_2, a_1, a_m}],$$

furthermore we have $a_m = 2a_0$. The *Euler-Muir theorem* now allows us, given a palindrome $(a_1, a_2, \ldots, a_2, a_1)$, to calculate the three coefficients of a quadratic polynomial

$$f(n) = An^2 + Bn + C,$$

such that for all $n \in \mathbb{N}$ the continued fraction representation of $\sqrt{f(n)}$ contains this palindrome and for all d which have this palindrome in the continued fraction expansion of their square root, there exists an n such that $d = f(n)$:

$$\sqrt{f(n)} = \left[\lfloor \sqrt{f(n)} \rfloor, \overline{a_1, a_2, \ldots, a_2, a_1, 2\lfloor \sqrt{f(n)} \rfloor}\right].$$

The *Euler-Muir polynomial* for the palindrome $(1, 1, 1)$ is $f(n) = 9n^2 - 2n$, and we have

$$\sqrt{f(1)} = \sqrt{7} = [2, \overline{1, 1, 1, 4}], \qquad \sqrt{f(2)} = \sqrt{32} = [5, \overline{1, 1, 1, 10}],$$
$$\sqrt{f(3)} = \sqrt{75} = [8, \overline{1, 1, 1, 16}], \qquad \sqrt{f(4)} = \sqrt{136} = [11, \overline{1, 1, 1, 22}] \text{ and so on.}$$

2. The theory of continued fractions

Definition 1 (cont'd fraction)**.** For $a_0, \dots, a_n \in \mathbb{R}$ with $a_k > 0$ for $k \geq 1$ we inductively define

(i) $[a_0] := a_0$,

(ii) $[a_0, \dots, a_n] := \left[a_0, \dots, a_{n-1} + \frac{1}{a_n}\right]$.

If $a_k \in \mathbb{Z}$ for all k, we call the continued fraction *simple*.

Remark 1.1. We have

$$[a_0, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_n}}}$$

which implies

$$[a_0, \dots, a_n] = a_0 + \frac{1}{[a_1, \dots, a_n]}.$$

Remark 1.2. For $a_0, \dots, a_n$ we have

$$[a_0, \dots, a_n] = [a_0, \dots, a_n - 1, 1].$$

Proof.

$$[a_0, \dots, a_n - 1, 1] = \left[a_0, \dots, a_n - 1 + \frac{1}{1}\right] = [a_0, \dots, a_n].$$

□

Definition 2 (Auxiliary sequences for evaluating continued fractions, [2, equation 7.6])**.** Let $(a_n)_{n\geq 0}$ be a sequence where $a_n \geq 1$ for $n \geq 1$. We define $(h_n)_{n\geq -2}$ and $(k_n)_{n\geq -2}$ by

$$h_n = a_n h_{n-1} + h_{n-2}, \quad k_n = a_n k_{n-1} + k_{n-2} \quad \text{for } n \geq 0, \qquad h_{-1} = k_{-2} = 1, \quad h_{-2} = k_{-1} = 0,$$

or, written out using matrices,

$$\begin{pmatrix} h_n & h_{n-1} \\ k_n & k_{n-1} \end{pmatrix} = \begin{pmatrix} h_{n-1} & h_{n-2} \\ k_{n-1} & k_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } n \geq 0, \qquad \begin{pmatrix} h_{-1} & h_{-2} \\ k_{-1} & k_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 3. *We have $1 = k_0 \leq k_1$ and $k_n < k_{n+1}$ for $n \geq 1$. Also, we have the bound $k_n \geq n$.*

Proof. The k_n are recursively defined by

$$k_n = a_n k_{n-1} + k_{n-2},$$

in particular we have

$$k_0 = a_0 k_{-1} + k_{-2} = k_{-2} = 1, \qquad k_1 = a_1 k_0 + k_{-1} = a_1 \geq 1.$$

The induction base is given by

$$k_2 = a_2 k_1 + k_0 \geq k_1 + k_0 > k_1$$

and using $k_{n-1} > k_{n-2}$ for the induction step we have $k_{n-1} > 0$, so we can conclude

$$\forall\, n \geq 1: \quad k_{n+1} = a_n k_n + k_{n-1} > a_n k_n \geq k_n.$$

The bound $k_n \geq n$ holds for $n \in \{0, 1\}$ and inductively follows for $n \geq 2$ by

$$k_n = a_n k_{n-1} + k_{n-2} \geq 1 \cdot (n-1) + k_0 = (n-1) + 1 = n.$$

□

Theorem 4 ([2, theorem 7.3])**.** *For any $x \in \mathbb{R}_+$ we have*

$$[a_0, \ldots, a_{n-1}, x] = \frac{xh_{n-1} + h_{n-2}}{xk_{n-1} + k_{n-2}}.$$

Proof (by induction). For $n = 0$ we have

$$\frac{xh_{-1} + h_{-2}}{xk_{-1} + k_{-2}} = \frac{x \cdot 1 + 0}{x \cdot 0 + 1} = x = [x].$$

Using the equation for a fixed n as an induction hypothesis, we can conclude

$$[a_0, \ldots, a_n, x] = \left[a_0, \ldots, a_{n-1}, a_n + \frac{1}{x}\right] = \frac{\left(a_n + \frac{1}{x}\right) h_{n-1} + h_{n-2}}{\left(a_n + \frac{1}{x}\right) k_{n-1} + k_{n-2}} = \frac{a_n h_{n-1} + h_{n-2} + \frac{1}{x} h_{n-1}}{a_n k_{n-1} + k_{n-2} + \frac{1}{x} k_{n-1}}$$
$$= \frac{h_n + \frac{1}{x} h_{n-1}}{k_n + \frac{1}{x} k_{n-1}} = \frac{xh_n + h_{n-1}}{xk_n + k_{n-1}}.$$

□

Corollary 5 ([2, theorem 7.4])**.** *With $r_n := [a_0, \ldots, a_n]$ for $n \geq 0$, we have $r_n = h_n / k_n$.*

Proof.

$$r_n = [a_0, \ldots, a_{n-1}, a_n] = \frac{a_n h_{n-1} + h_{n-2}}{a_n k_{n-1} + k_{n-2}} = \frac{h_n}{k_n}.$$

□

Lemma 6 (evaluating a continued fraction using matrix multiplication)**.**
We can evaluate $[a_0, \ldots, a_n]$ using matrix multiplication: for

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \prod_{i=0}^{n} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} \quad \textit{we have} \quad [a_0, \ldots, a_n] = \frac{A}{C}.$$

Proof. On the one hand, the recursive definition 2 implies:

$$\begin{aligned}
\begin{pmatrix} h_n & h_{n-1} \\ k_n & k_{n-1} \end{pmatrix} &= \begin{pmatrix} h_{n-1} & h_{n-2} \\ k_{n-1} & k_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} h_{n-2} & h_{n-3} \\ k_{n-2} & k_{n-3} \end{pmatrix} \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \\
&\ \ \vdots \\
&= \begin{pmatrix} h_{-1} & h_{-2} \\ k_{-1} & k_{-2} \end{pmatrix} \prod_{i=0}^{n} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \prod_{i=0}^{n} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} \\
&= \prod_{i=0}^{n} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\end{aligned}$$

on the other hand we showed in corollary 5 that

$$[a_0, \ldots, a_n] = \frac{h_n}{k_n} = \frac{A}{C}.$$

□

Lemma 7 (reversed continued fractions, [2, exercise 7.5])**.** *If* $a_0 \geq 1$*, we have*

$$\frac{h_n}{h_{n-1}} = [a_n, \dots, a_0] \quad \text{for } n \geq 0, \qquad \frac{k_n}{k_{n-1}} = [a_n, \dots, a_1] \quad \text{for } n \geq 1.$$

Proof. We shall prove these statements using induction. The base case is given by

$$\frac{h_0}{h_{-1}} = \frac{a_0}{1} = [a_0] \qquad \text{and} \qquad \frac{k_1}{k_0} = \frac{a_1}{1} = [a_1],$$

the induction step is given by

$$\frac{h_n}{h_{n-1}} = \frac{a_n h_{n-1} + h_{n-2}}{h_{n-1}} = a_n + \frac{1}{h_{n-1}/h_{n-2}} = a_n + \frac{1}{[a_{n-1}, \dots, a_0]} = [a_n, \dots, a_0]$$

and

$$\frac{k_n}{k_{n-1}} = \frac{a_n k_{n-1} + k_{n-2}}{k_{n-1}} = a_n + \frac{1}{k_{n-1}/k_{n-2}} = a_n + \frac{1}{[a_{n-1}, \dots, a_1]} = [a_n, \dots, a_1].$$

□

Theorem 8 ([2, theorem 7.5])**.** *For* $n \geq 0$ *we have*

$$h_n k_{n-1} - h_{n-1} k_n = (-1)^{n-1}, \qquad h_n k_{n-2} - h_{n-2} k_n = (-1)^n a_n.$$

Proof (by induction). For $n = 0$ we can easily verify these equations:

$$\begin{aligned} h_0 k_{-1} - h_{-1} k_0 = h_0 \cdot 0 - 1 \cdot k_0 = (-1) \cdot 1 = (-1)^{0-1}, \\ h_0 k_{-2} - h_{-2} k_0 = h_0 \cdot 1 - 0 \cdot k_0 = a_0 = (-1)^0 a_0. \end{aligned}$$

Using the first equation for a fixed n as an induction hypothesis, we obtain

$$\begin{aligned} h_{n+1} k_n - h_n k_{n+1} &= (a_{n+1} h_n + h_{n-1}) k_n - h_n (a_{n+1} k_n + k_{n-1}) \\ &= h_{n-1} k_n - h_n k_{n-1} \\ &= (-1) \cdot (h_n k_{n-1} - h_{n-1} k_n) \\ &= (-1) \cdot (-1)^{n-1} = (-1)^n, \\ h_{n+1} k_{n-1} - h_{n-1} k_{n+1} &= (a_{n+1} h_n + h_{n-1}) k_{n-1} - h_{n-1} (a_{n+1} k_n + k_{n-1}) \\ &= a_{n+1} (h_n k_{n-1} - h_{n-1} k_n) \\ &= a_{n+1} (-1)^{n-1} = (-1)^{n+1} a_{n+1}. \end{aligned}$$

□

Corollary 9 ([2, theorem 7.5])**.** *We have*

$$r_n - r_{n-1} = \frac{(-1)^{n-1}}{k_n k_{n-1}} \quad \text{for } n \geq 1, \qquad r_n - r_{n-2} = \frac{(-1)^n a_n}{k_n k_{n-2}} \quad \text{for } n \geq 2.$$

If $a_n \in \mathbb{Z}$ *for all* $n \geq 0$*, we have* $\gcd(h_n, k_n) = \gcd(h_n, h_{n+1}) = \gcd(k_n, k_{n+1}) = 1$.

Proof. The equations can be obtained by dividing the equations from the previous theorem by $k_n k_{n-1}$ and $k_n k_{n-2}$ respectively. If $a_n \in \mathbb{Z}$ for all $n \geq 0$, we can conclude that $h_n, k_n \in \mathbb{Z}$ for $n \geq 0$, and if $d \in \mathbb{N}$ divides $\gcd(h_n, k_n)$, $\gcd(h_n, h_{n+1})$ or $\gcd(k_n, k_{n+1})$, we can see from 8 that

$$d \mid (h_{n+1} k_n - h_n k_{n+1}) = (-1)^{n-1},$$

so we must have $d = 1$. □

Theorem 10 ([2, thm. 7.6]). *The sequence $(r_n)_{n\in\mathbb{N}}$ converges to a number $\xi \in \mathbb{R}$ and we have*

$$r_0 < r_2 < r_4 < \ldots < \xi < \ldots < r_5 < r_3 < r_1.$$

Proof. Per lemma 3 we have $k_n > 0$ for all $n \geq 0$. For $n \geq 2$ we have $a_n > 0$, so the expression for $r_n - r_{n-2}$ from corollary 9 has the same sign as $(-1)^n$ which gives us

$$r_{n-2} < r_n \quad \text{if } 2 \mid n, \qquad r_n < r_{n-2} \quad \text{if } 2 \nmid n.$$

Furthermore, the expression for $r_n - r_{n-1}$ from corollary 9 has the same sign as $(-1)^{n-1}$, so we have $r_n < r_{n-1}$ for $2 \mid n$ and we can conclude for $l \geq 1$ and $m \geq 0$ that

$$r_{2m} < r_{2m+2l} < r_{2m+2l-1} \leq r_{2l-1}.$$

Therefore, the subsequence $(r_{2n})_{n\in\mathbb{N}}$ is monotonically increasing and bounded above by r_1, so it converges. Analogously, the subsequence $(r_{2n+1})_{n\in\mathbb{N}}$ is monotonically decreasing and bounded below by r_0, so it converges. Using the bound from lemma 3 we see that the expression for $r_n - r_{n-1}$ from corollary 9 tends to zero, so both subsequences as well as the sequence $(r_n)_{n\in\mathbb{N}}$ itself converge to the same limit $\xi \in \mathbb{R}$. As the subsequences are strictly monotonous, this limit is never reached as there would have to be an n such that $r_n = r_{n+2}$. □

Definition 11 (infinite continued fraction, [2, definition 7.1]).
For $(a_n)_{n\in\mathbb{N}}$ with $a_n \geq 1$ for $n \geq 1$ we define the *infinite continued fraction*

$$[a_0, a_1, a_2, \ldots] := \lim_{n\to\infty} r_n$$

and we call r_n the n^{th} *convergent* of $[a_0, a_1, a_2, \ldots]$. If there exist $l \geq 1$ and $N \geq 0$ such that

$$a_n = a_{n+l} \qquad \forall\, n \geq N,$$

we call the continued fraction *periodic* and we write

$$[a_0, a_1, a_2, \ldots] =: [a_0, \ldots, a_{N-1}, \overline{a_N, \ldots, a_{N+l-1}}].$$

As in the finite case, we call the continued fraction *simple* if $a_n \in \mathbb{Z}$ for all $n \geq 0$.

Lemma 12 ([2, theorem 7.15]). *If $x > 1$ with $x = [a_0, a_1, \ldots]$ and if h_n/k_n is the n^{th} convergent of x, the $(n+1)^{st}$ convergent of $1/x = [0, a_0, a_1, \ldots]$ is the reciprocal value k_n/h_n.*

Proof. Evaluating the n^{th} convergent of x using matrix multiplication, we obtain

$$\begin{pmatrix} h_n & h_{n-1} \\ k_n & k_{n-1} \end{pmatrix} = \prod_{i=0}^{n} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}.$$

Evaluating the $(n+1)^{\text{st}}$ convergent of $1/x$ using matrix multiplication, we obtain

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\prod_{i=0}^{n} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_n & h_{n-1} \\ k_n & k_{n-1} \end{pmatrix} = \begin{pmatrix} k_n & k_{n-1} \\ h_n & h_{n-1} \end{pmatrix},$$

so the $(n+1)^{\text{st}}$ convergent of $1/x$ is indeed k_n/h_n. □

From now on, we assume that $a_n \in \mathbb{Z}$ for $n \geq 0$.

Lemma 13 ([2, theorem 7.13]).
If $|\xi b - a| < |\xi k_n - h_n|$ for $a \in \mathbb{Z}$, $b \in \mathbb{N}$ and $n \geq 0$, we have $b \geq k_{n+1}$.

Proof. We assume that $b < k_{n+1}$ and consider the system of linear equations

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad A = \begin{pmatrix} h_n & h_{n+1} \\ k_n & k_{n+1} \end{pmatrix}.$$

Using theorem 8, we see that the determinant of A is

$$h_n k_{n+1} - h_{n+1} k_n = (-1)^{n+1},$$

so the inverse of A is

$$A^{-1} = (-1)^{n+1}\begin{pmatrix} k_{n+1} & -h_{n+1} \\ -k_n & h_n \end{pmatrix}.$$

The system of linear equations therefore has the solution

$$x = (-1)^{n+1}(k_{n+1}a - h_{n+1}b), \qquad y = (-1)^{n+1}(h_n b - k_n a),$$

but we are only interested in the fact that $x, y \in \mathbb{Z}$. We can't have $x = 0$, for this would imply

$$b = k_{n+1}y \quad \overset{b,k_{n+1}\in\mathbb{N}}{\Longrightarrow} \quad y \in \mathbb{N} \quad \Longrightarrow \quad b \geq k_{n+1},$$

contradicting our assumption. On the other hand, we can't have $y = 0$, for this would imply

$$(a,b) = (xh_n, xk_n) \qquad \Longrightarrow \qquad |\xi b - a| = |x| \cdot |\xi k_n - h_n| \geq |\xi k_n - h_n|.$$

If $y < 0$, we see from

$$0 < b = k_n x + k_{n+1} y,$$

that $x > 0$. On the other hand, if $y > 0$, our assumption implies that

$$b < k_{n+1} \leq k_{n+1} y = b - k_n x \qquad \Longrightarrow \qquad x < 0.$$

This proves that x and y have different signs. We have

$$\xi b - a = \xi(k_n x + k_{n+1} y) - (h_n x + h_{n+1} y) = x(\xi k_n - h_n) + y(\xi k_{n+1} - h_{n+1}),$$

and we can conclude from theorem 10 that $\xi k_n - h_n$ and $\xi k_{n+1} - h_{n+1}$ have different signs, as $\xi - r_n$ and $\xi - r_{n+1}$ have different signs. We can thus conclude that $x(\xi k_n - h_n)$ and $y(\xi k_{n+1} - h_{n+1})$ have different signs, but this implies

$$|\xi b - a| = |x(\xi k_n - h_n) + y(\xi k_{n+1} - h_{n+1})| = |x| \cdot |\xi k_n - h_n| + |y| \cdot |\xi k_{n+1} - h_{n+1}| \geq |\xi k_n - h_n|,$$

contradicting our assumption, so we must indeed have $b \geq k_{n+1}$. □

Theorem 14 ([2, theorem 7.14]).
If $|\xi - a/b| < 1/2b^2$ for $a \in \mathbb{Z}$, $b \in \mathbb{N}$, there exists an $n \geq 1$ such that $a/b = r_n = h_n/k_n$.

Proof. As $(k_n)_{n\in\mathbb{N}}$ is strictly monotonically increasing and unbounded, there exists a unique $n \in \mathbb{N}$ such that $k_n \leq b < k_{n+1}$. Lemma 13 then implies that

$$|\xi k_n - h_n| \leq |\xi b - a| < \frac{1}{2b}.$$

We now assume that $a/b \neq h_n/k_n$, so $|ak_n - bh_n| \geq 1$. This yields the contradiction

$$1 \leq |ak_n - bh_n| \leq |\xi bk_n - ak_n| + |\xi bk_n - bh_n| = k_n|\xi b - a| + b|\xi k_n - h_n| < \frac{k_n}{2b} + \frac{b}{2b} \leq \frac{1}{2} + \frac{1}{2} = 1.$$

□

Corollary 15. *ξ is irrational.*

Proof. For $\xi = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ we would have $|\xi q - p| = 0 < 1/2q^2$, so the previous theorem gives us an $n \in \mathbb{N}$ such that $r_n = p/q = \xi$, contradicting theorem 10. □

Lemma 16. *Writing $\overline{x + y\sqrt{d}} := x - y\sqrt{d}$, we have $\overline{\alpha \cdot \beta} = \overline{\alpha} \cdot \overline{\beta}$ and $\overline{\alpha/\beta} = \overline{\alpha}/\overline{\beta}$.*

Proof. We have

$$\begin{aligned}\overline{a + b\sqrt{d}} \cdot \overline{A + B\sqrt{d}} = (a - b\sqrt{d})(A - B\sqrt{d}) &= (aA + bBd) - (aB + Ab)\sqrt{d} \\ &= \overline{(aA + bBd) + (aB + Ab)\sqrt{d}} \\ &= \overline{(a + b\sqrt{d})(A + B\sqrt{d})}\end{aligned}$$

and

$$\overline{1/(a + b\sqrt{d})} = \overline{(a - b\sqrt{d})/(a^2 - b^2 d)} = (a + b\sqrt{d})/(a^2 - b^2 d) = 1/(a - b\sqrt{d}) = 1/\overline{(a + b\sqrt{d})}.$$

□

Theorem 17 ([2, thm. 7.19])**.** *Every simple periodic continued fraction corresponds to a number*

$$[a_0, \ldots, a_{N-1}, \overline{a_N, \ldots, a_{N+l-1}}] = \xi = \frac{a + \sqrt{b}}{c}$$

with $a, c \in \mathbb{Z}$, $b \in \mathbb{N}$ and $c \neq 0$, where b is not a perfect square. Conversely, any such number $(a + \sqrt{b})/c$ can be developed into a simple periodic continued fraction.

Proof (⇒). Writing $\theta := [\overline{a_N, \ldots, a_{N+l-1}}]$, we have

$$\theta = [a_N, \ldots, a_{N+l-1}, \overline{a_N, \ldots, a_{N+l-1}}] = [a_N, \ldots, a_{N+l-1}, \theta]$$

and theorem 4 implies that there exist $H, H', K, K' \in \mathbb{Z}$ with $K \geq 1$ and $K' \geq 0$, such that

$$\theta = \frac{\theta H + H'}{\theta K + K'} \qquad \Longrightarrow \qquad K\theta^2 + K'\theta = H\theta + H' \qquad \Longrightarrow \qquad K\theta^2 + (K' - H)\theta - H' = 0.$$

The quadratic formula yields the solutions

$$\theta \in \left\{ \frac{(H - K') + \sqrt{(K' - H)^2 + 4KH'}}{2K}, \frac{(H - K') - \sqrt{(K' - H)^2 + 4KH'}}{2K} \right\},$$

and both cases can be expressed as

$$\theta = \frac{A + \sqrt{B}}{C}$$

with $A, C \in \mathbb{Z}$, $B \in \mathbb{N}$ and $C \neq 0$. Using theorem 4 again, we get $h, h', k, k' \in \mathbb{Z}$ such that

$$\begin{aligned}\xi = [a_0, \ldots, a_{N-1}, \overline{a_N, \ldots, a_{N+l-1}}] &= [a_0, \ldots, a_{N-1}, \theta] \\ &= \frac{\theta h + h'}{\theta k + k'} \\ &= \frac{(A + \sqrt{B})h + Ch'}{(A + \sqrt{B})k + Ck'} \\ &= \frac{(Ah + Ch') + \sqrt{B}h}{(Ak + Ck') + \sqrt{B}k} \\ &= \frac{((Ah + Ch') + h\sqrt{B})((Ak + Ck') - k\sqrt{B})}{(Ak + Ck')^2 - Bk^2} \\ &= \frac{((Ah + Ch')(Ak + Ck') - Bhk) + C(hk' - h'k)\sqrt{B}}{(Ak + Ck')^2 - Bk^2} \\ &= \frac{((Ah + Ch')(Ak + Ck') - Bhk) + \sqrt{(C(hk' - h'k))^2 B}}{(Ak + Ck')^2 - Bk^2}\end{aligned}$$

and using suitable $a, c \in \mathbb{Z}$, $b \in \mathbb{N}$ and $c \neq 0$, we can write

$$\xi = \frac{a + \sqrt{b}}{c},$$

where b cannot be a perfect square, as ξ would then be rational, contradicting corollary 15. $\square$

Proof ($\Leftarrow$). We define $m_0 := a|c|$, $d = bc^2$ and $q_0 = c|c|$ and show that the algorithm given by

$$a_n = \lfloor \xi_n \rfloor, \quad \xi_n = \frac{m_n + \sqrt{d}}{q_n}, \quad m_{n+1} = a_n q_n - m_n, \quad q_{n+1} = \frac{d - m_{n+1}^2}{q_n} \in \mathbb{Z} \setminus \{0\}$$

computes the continued fraction representation of $(a + \sqrt{b})/c$, i.e. for $n \geq 0$ we must have

$$\frac{a + \sqrt{b}}{c} = [a_0, \dots, a_{n-1}, \xi_n].$$

We obtain the induction base case $n = 0$ by expanding with $|c| = \sqrt{c^2}$:

$$\frac{a + \sqrt{b}}{c} = \frac{a|c| + \sqrt{bc^2}}{c|c|} = \frac{m_0 + \sqrt{d}}{q_0} = \xi_0 = [\xi_0].$$

For the induction step, we obtain

$$\xi_n - a_n = \frac{m_n + \sqrt{d} - a_n q_n}{q_n} = \frac{\sqrt{d} - m_{n+1}}{q_n} = \frac{d - m_{n+1}^2}{q_n(\sqrt{d} + m_{n+1})} = \frac{q_{n+1}}{m_{n+1} + \sqrt{d}} = \frac{1}{\xi_{n+1}},$$

so, using the induction hypothesis, we see that

$$\frac{a + \sqrt{b}}{c} = [a_0, \dots, a_{n-1}, \xi_n] = \left[a_0, \dots, a_{n-1}, a_n + \frac{1}{\xi_{n+1}}\right] = [a_0, \dots, a_{n-1}, a_n, \xi_{n+1}].$$

What's left to prove is that $q_n \in \mathbb{Z} \setminus \{0\}$ for all $n \geq 0$. Due to

$$q_{n+1} = \frac{d - m_{n+1}^2}{q_n} = \frac{d - (a_n q_n - m_n)^2}{q_n} = \frac{d - m_n^2}{q_n} + 2a_n m_n - a_n^2 q_n$$

we have to show that $(d - m_n^2)/q_n$ is an integer for $n \geq 0$.
For $n = 0$ this is clear from the definition as

$$\frac{d - m_0^2}{q_0} = \frac{bc^2 - (a|c|)^2}{c|c|} = \frac{bc^2 - a^2c^2}{c|c|} = \frac{c}{|c|}(b - a^2),$$

for $n \geq 1$ it follows from

$$\frac{d - m_n^2}{q_n} = \frac{d - m_n^2}{q_{n-1}} \cdot \frac{q_{n-1}}{q_n} = q_n \cdot \frac{q_{n-1}}{q_n} = q_{n-1} \in \mathbb{Z},$$

and we always have $q_n \neq 0$, for else we would have $d = m_n^2$, implying that b is a perfect square.
Theorem 4 gives us

$$\xi_0 = \frac{a + \sqrt{b}}{c} = [a_0, \dots, a_{n-1}, \xi_n] = \frac{\xi_n h_{n-1} + h_{n-2}}{\xi_n k_{n-1} + k_{n-2}}$$

and using $\xi_n' := \overline{\xi_n} = (m_n - \sqrt{d})/q_n$, lemma 16 yields the equation

$$\xi_0' = \frac{\xi_n' h_{n-1} + h_{n-2}}{\xi_n' k_{n-1} + k_{n-2}} \qquad \Longrightarrow \qquad \xi_n'(\xi_0' k_{n-1} - h_{n-1}) = h_{n-2} - \xi_0' k_{n-2}$$

$$\Longrightarrow \qquad \xi_n' = \frac{h_{n-2} - \xi_0' k_{n-2}}{\xi_0' k_{n-1} - h_{n-1}} = -\frac{k_{n-2}}{k_{n-1}} \left(\frac{\xi_0' - h_{n-2}/k_{n-2}}{\xi_0' - h_{n-1}/k_{n-1}} \right).$$

For $n \to \infty$ both numerator and denominator of the fraction in brackets tend to $\xi_0' - \xi_0 \neq 0$, so said fraction tends to 1. Hence, there exists an $n_0 \in \mathbb{N}$, such that the fraction in brackets becomes positive for $n \geq n_0$, making ξ_n' negative. On the other hand, for $n \geq 0$ we have

$$\xi_n - a_n = \frac{1}{\xi_{n+1}} \qquad \Longrightarrow \qquad \xi_{n+1} = \frac{1}{\xi_n - a_n} = \frac{1}{\xi_n - \lfloor \xi_n \rfloor} > 1$$

and hence for all $n \geq n_0$:

$$0 < \xi_n - \xi_n' = \frac{m_n + \sqrt{d}}{q_n} - \frac{m_n - \sqrt{d}}{q_n} = \frac{2\sqrt{d}}{q_n} \qquad \Longrightarrow \qquad q_n > 0.$$

From the definition of m_n and q_n, we can thus conclude for $n \geq n_0$:

$$1 \leq q_n \leq q_n q_{n+1} = d - m_{n+1}^2 \leq d, \qquad m_{n+1}^2 < m_{n+1}^2 + q_n q_{n+1} = d, \tag{2.1}$$

but since d is fixed and $m_n, q_n \in \mathbb{Z}$, the pairs (m_n, q_n) can only take a finite number of values, so $\exists\, N \in \mathbb{N}_0, l \in \mathbb{N}$ with $(m_N, q_N) = (m_{N+l}, q_{N+l})$. Since

$$\xi_N = \frac{m_N + \sqrt{d}}{q_N} = \frac{m_{N+l} + \sqrt{d}}{q_{N+l}} = \xi_{N+l}$$

and $(\xi_{n+1}, m_{n+1}, q_{n+1})$ is computed solely from (ξ_n, m_n, q_n), it inductively follows that

$$\forall\, n \geq N : a_n = \lfloor \xi_n \rfloor = \lfloor \xi_{n+l} \rfloor = a_{n+l} \qquad \Longrightarrow \qquad \frac{a + \sqrt{b}}{c} = [a_0, \dots, a_{N-1}, \overline{a_N, \dots, a_{N+l-1}}].$$

□

Theorem 18 (continued fraction representation of square roots, [2, theorem 7.21]).
If $d \in \mathbb{N}$ is not a perfect square, the continued fraction representation of $\sqrt{d}$ takes the form

$$\sqrt{d} = [\lfloor \sqrt{d} \rfloor, \overline{a_1, \dots, a_l, 2\lfloor \sqrt{d} \rfloor}]$$

and we have $(a_1, \dots, a_l) = (a_l, \dots, a_1)$ with $l \in \mathbb{N}_0$, hence $a_k = a_{l-k+1}$ for $1 \leq k \leq l$.

Proof. Theorem 17 yields a representation

$$\lfloor \sqrt{d} \rfloor + \sqrt{d} = [a_0, \dots, a_{N-1}, \overline{a_N, \dots, a_{N+l}}]$$

and using the notation from said theorem, we have $(a, b, c) = (\lfloor \sqrt{d} \rfloor, d, 1)$. We write

$$\xi_n = \frac{m_n + \sqrt{d}}{q_n}, \qquad \xi_n' = \frac{m_n - \sqrt{d}}{q_n}$$

and using lemma 16, we see that

$$\frac{1}{\xi_{n+1}} = \xi_n - a_n \qquad \Longrightarrow \qquad \frac{1}{\xi_{n+1}'} = \xi_n' - a_n.$$

We now show that $-1 < \xi_n' < 0$ for all $n \geq 0$. For $n = 0$ this is clear from

$$\xi_0' = \lfloor \sqrt{d} \rfloor - \sqrt{d},$$

and inductively it follows from $a_0 = \lfloor \lfloor \sqrt{d} \rfloor + \sqrt{d} \rfloor = 2\lfloor \sqrt{d} \rfloor \geq 2$ and $a_n \geq 1$ for $n \geq 1$ via

$$\frac{1}{\xi_{n+1}'} = \xi_n' - a_n < 0 - 1 = -1 \qquad \Longrightarrow \qquad -1 < \xi_{n+1}' < 0.$$

We now conclude for all $n \geq 0$:

$$a_n = \xi_n' - \frac{1}{\xi_{n+1}'} \quad \Longrightarrow \quad -1 - \frac{1}{\xi_{n+1}'} < a_n < -\frac{1}{\xi_{n+1}'} \quad \Longrightarrow \quad a_n = \left\lfloor -\frac{1}{\xi_{n+1}'} \right\rfloor.$$

Therefore, if there exist indices $j < k$ with $\xi_j = \xi_k$, then $\xi_j' = \xi_k'$ implies that

$$a_{j-1} = \left\lfloor -\frac{1}{\xi_j'} \right\rfloor = \left\lfloor -\frac{1}{\xi_k'} \right\rfloor = a_{k-1} \quad \Longrightarrow \quad \xi_{j-1} = a_{j-1} + \frac{1}{\xi_j} = a_{k-1} + \frac{1}{\xi_k} = \xi_{k-1},$$

and by induction we get $a_n = a_{n+(k-j)}$ for all $n \geq 0$. As $\xi_N = \xi_{N+l+1}$, we get

$$\lfloor\sqrt{d}\rfloor + \sqrt{d} = [\overline{2\lfloor\sqrt{d}\rfloor, a_1, \ldots, a_l}] = [2\lfloor\sqrt{d}\rfloor, \overline{a_1, \ldots, a_l, 2\lfloor\sqrt{d}\rfloor}]$$

and it follows that

$$\sqrt{d} = (\lfloor\sqrt{d}\rfloor + \sqrt{d}) - \lfloor\sqrt{d}\rfloor = [\lfloor\sqrt{d}\rfloor, \overline{a_1, \ldots, a_l, 2\lfloor\sqrt{d}\rfloor}].$$

We now note that by theorem 4, $\xi := \lfloor\sqrt{d}\rfloor + \sqrt{d}$ satisfies the quadratic equation

$$\xi = [a_0, \ldots, a_l, \xi] = \frac{\xi h_l + h_{l-1}}{\xi k_l + k_{l-1}} \quad \Longleftrightarrow \quad k_l\xi^2 + (k_{l-1} - h_l)\xi - h_{l-1} = 0.$$

We now denote the n^{th} convergents of $\theta := [\overline{a_l, \ldots, a_0}]$ by H_n/K_n, where these fractions are reduced to lowest terms and $K_n \in \mathbb{N}$. Lemma 7 now implies that

$$\frac{H_l}{K_l} = [a_l, \ldots, a_0] = \frac{h_l}{h_{l-1}}, \qquad \frac{H_{l-1}}{K_{l-1}} = [a_l, \ldots, a_1] = \frac{k_l}{k_{l-1}},$$

and since H_l/K_l, h_l/h_{l-1}, H_{l-1}/K_{l-1} and k_l/k_{l-1} have positive denominators and are reduced to lowest terms by corollary 9, we have the equalities

$$H_l = h_l, \qquad K_l = h_{l-1}, \qquad H_{l-1} = k_l, \qquad K_{l-1} = k_{l-1}.$$

Theorem 4 now implies that

$$\begin{aligned} & \theta = [a_l, \ldots, a_0, \theta] = \frac{\theta H_l + H_{l-1}}{\theta K_l + K_{l-1}} = \frac{\theta h_l + k_l}{\theta h_{l-1} + k_{l-1}} \\ \Longrightarrow \quad & h_{l-1}\theta^2 + (k_{l-1} - h_l)\theta - k_l = 0 \\ \overset{\cdot(-1),\ /\theta^2}{\Longrightarrow} \quad & k_l\left(\frac{-1}{\theta}\right)^2 + (k_{l-1} - h_l)\left(\frac{-1}{\theta}\right) - h_{l-1} = 0, \end{aligned}$$

so $-1/\theta$ satisfies the same quadratic equation as ξ. But as $-1/\theta < 0 < \xi = \lfloor\sqrt{d}\rfloor + \sqrt{d}$ and the two solutions of a quadratic equation always take the form $(A \pm \sqrt{B})/C$, we conclude that

$$\frac{-1}{\theta} = \lfloor\sqrt{d}\rfloor - \sqrt{d} \quad \Longrightarrow \quad [\overline{a_l, \ldots, a_1, a_0}] = \theta = \frac{1}{\sqrt{d} - \lfloor\sqrt{d}\rfloor}.$$

On the other hand, we have

$$\sqrt{d} - \lfloor\sqrt{d}\rfloor = (\lfloor\sqrt{d}\rfloor + \sqrt{d}) - 2\lfloor\sqrt{d}\rfloor = \xi - a_0 = [0, \overline{a_1, \ldots, a_l, a_0}] = \frac{1}{[\overline{a_1, \ldots, a_l, a_0}]}$$

and hence

$$\frac{1}{\sqrt{d} - \lfloor\sqrt{d}\rfloor} = [\overline{a_1, \ldots, a_l, a_0}].$$

As the continued fraction representation per theorem 17 is unique, it follows that

$$(a_1, \ldots, a_l) = (a_l, \ldots, a_1).$$

□

Theorem 19 ([2, theorem 7.22]). *If $d \in \mathbb{N}$ is not a perfect square,*

$$\sqrt{d} = [\lfloor\sqrt{d}\rfloor, \overline{a_1, \ldots, a_l, 2\lfloor\sqrt{d}\rfloor}]$$

and l minimal, then with q_n as in theorem 17, we have

$$h_n^2 - dk_n^2 = (-1)^{n-1} q_{n+1} \qquad \textit{for } n \geq -1,$$

where $q_{n+1} = 1$ holds if and only if $(l+1) \mid (n+1)$ and we never have $q_{n+1} = -1$.

Proof. With ξ_n, m_n and q_n as in theorem 17, we have

$$\begin{aligned}
\sqrt{d} &= \frac{\xi_{n+1}h_n + h_{n+1}}{\xi_{n+1}k_n + k_{n+1}} \\
&= \frac{(m_{n+1} + \sqrt{d})h_n + q_{n+1}h_{n+1}}{(m_{n+1} + \sqrt{d})k_n + q_{n+1}k_{n+1}} \\
&= \frac{(m_{n+1}h_n + q_{n+1}h_{n+1}) + \sqrt{d}h_n}{(m_{n+1}k_n + q_{n+1}k_{n+1}) + \sqrt{d}k_n} \\
&= \frac{((m_{n+1}h_n + q_{n+1}h_{n+1}) + \sqrt{d}h_n)((m_{n+1}k_n + q_{n+1}k_{n+1}) - \sqrt{d}k_n)}{((m_{n+1}k_n + q_{n+1}k_{n+1}) + \sqrt{d}k_n)((m_{n+1}k_n + q_{n+1}k_{n+1}) - \sqrt{d}k_n)} \\
&= \frac{((m_{n+1}h_n + q_{n+1}h_{n+1})(m_{n+1}k_n + q_{n+1}k_{n+1}) - dh_nk_n) + q_{n+1}(h_{n+1}k_n - h_nk_{n+1})\sqrt{d}}{(m_{n+1}k_n + q_{n+1}k_{n+1})^2 - dk_n^2}.
\end{aligned}$$

As this equation takes the form $A + B\sqrt{d} = A' + B'\sqrt{d}$ with $A, B, A', B' \in \mathbb{Q}$, we must in particular have $A = A'$ and $B = B'$. Comparing coefficients yields two equations

$$(m_{n+1}h_n + q_{n+1}h_{n+1})(m_{n+1}k_n + q_{n+1}k_{n+1}) - dh_nk_n = 0, \tag{I}$$

$$(m_{n+1}k_n + q_{n+1}k_{n+1})^2 - dk_n^2 = q_{n+1}(h_{n+1}k_n - h_nk_{n+1}). \tag{II}$$

For $n = -1$ we can easily verify this claim: with c as in 17, we have

$$h_{-1}^2 - dk_{-1}^2 = 1^2 - d \cdot 0^2 = 1 = 1 \cdot 1 = 1 \cdot c|c| = (-1)^{-2} q_0.$$

For $n \geq 0$ we have $h_n, k_n \neq 0$, and multiplying (I) by k_n/h_n yields

$$0 = \left(m_{n+1}k_n + q_{n+1}\frac{h_{n+1}k_n}{h_n}\right)(m_{n+1}k_n + q_{n+1}k_{n+1}) - dk_n^2.$$

For all A, B, C, the following identity holds:

$$(A + C)(A + B) = (A + B)^2 - (A + B)^2 + (A + C)(A + B) = (A + B)^2 + (A + B)(C - B),$$

using which we can rewrite (I) as

$$\begin{aligned}
0 &= (m_{n+1}k_n + q_{n+1}k_{n+1})^2 - dk_n^2 + (m_{n+1}k_n + q_{n+1}k_{n+1})q_{n+1}\left(\frac{h_{n+1}k_n}{h_n} - k_{n+1}\right) \\
&= (m_{n+1}k_n + q_{n+1}k_{n+1})^2 - dk_n^2 + \frac{m_{n+1}k_n + q_{n+1}k_{n+1}}{h_n}q_{n+1}(h_{n+1}k_n - h_nk_{n+1}).
\end{aligned}$$

Substituting (II) into (I) and reducing $q_{n+1}(h_{n+1}k_n - h_nk_{n+1})$, we obtain

$$0 = 1 + \frac{m_{n+1}k_n + q_{n+1}k_{n+1}}{h_n} \qquad \Longrightarrow \qquad -h_n = m_{n+1}k_n + q_{n+1}k_{n+1}. \tag{$*$}$$

Substituting this result into (II) and applying theorem 8, we obtain

$$\begin{aligned} h_n^2 - dk_n^2 &= (-h_n)^2 - dk_n^2 \\ &\overset{(*)}{=} (m_{n+1}k_n + q_{n+1}k_{n+1})^2 - dk_n^2 \\ &\overset{(II)}{=} q_{n+1}(h_{n+1}k_n - h_n k_{n+1}) \\ &\overset{8}{=} (-1)^n q_{n+1} \end{aligned}$$

and the claim follows.
We can never have $q_{n+1} = -1$, for this would imply $n + 1 > 0$ and hence

$$1 < \xi_{n+1} = -m_{n+1} - \sqrt{d},$$

o.t.o.h. we would have $-1 < \xi'_{n+1} = -m_{n+1} + \sqrt{d} < 0$ as in thm. 18, yielding the contradiction

$$0 < \sqrt{d} < m_{n+1} < -1 - \sqrt{d} < 0.$$

If $(l + 1) \mid (n + 1)$, then periodicity implies that $q_{n+1} = q_0 = 1$. Conversely, $q_{n+1} = 1$ implies that $\xi_{n+1} = m_{n+1} + \sqrt{d}$, and from $-1 < \xi'_{n+1} = m_{n+1} - \sqrt{d} < 0$ we see that $m_{n+1} = \lfloor\sqrt{d}\rfloor$ and hence $\xi_{n+1} = \lfloor\sqrt{d}\rfloor + \sqrt{d} = \xi_0$, and the minimality of l necessitates $(l + 1) \mid (n + 1)$. □

Corollary 20. *We have $h_n^2 - dk_n^2 = \pm 1$ if and only if there exists an $r \in \mathbb{N}_0$ such that $n = r(l + 1) - 1$. In particular, we can only have $h_n^2 - dk_n^2 = -1$ if $2 \mid l$.*

Proof. We have $h_n^2 - dk_n^2 = \pm 1$ if and only if $q_{n+1} = 1$, i.e. for $(l + 1) \mid (n + 1)$.
If $2 \nmid l$, we have

$$h_n^2 - dk_n^2 = (-1)^{n-1} = (-1)^{r(l+1)-2} = 1^{r((l+1)/2)-1} = 1.$$

□

Corollary 21. *For $\sqrt{d} = [\lfloor\sqrt{d}\rfloor, \overline{a_1, \dots, a_l, 2\lfloor\sqrt{d}\rfloor}]$, l minimal, we have $\lfloor\sqrt{d}\rfloor \geq a_k$ for $1 \leq k \leq l$.*

Proof. With N, n_0, a_k, m_k, q_k as in the proof of 17 we have $N = 1$, $n_0 = 0$ and like (2.1) we have

$$m_n^2 < d \quad \Longrightarrow \quad m_n < \sqrt{d} \quad \overset{m_n \in \mathbb{Z}}{\Longrightarrow} \quad m_n \leq \lfloor\sqrt{d}\rfloor \tag{2.2}$$

for $n \geq 1$. On the other hand, we have

$$m_n + m_{n+1} = m_n + (a_n q_n - m_n) = a_n q_n$$

and we showed in (2.1) that

$$1 \leq q_n \leq d, \tag{2.3}$$

which in conjunction with (2.2) implies that

$$a_n q_n \leq 2\lfloor\sqrt{d}\rfloor \quad \Longrightarrow \quad a_n \leq \frac{2\lfloor\sqrt{d}\rfloor}{q_n}.$$

Lastly, we will show that $q_n \geq 2$ for $1 \leq n \leq l$ for this would imply the inequality

$$a_n \leq \lfloor\sqrt{d}\rfloor.$$

Theorem 19 states that $q_n = 1$ can only hold for $(l + 1) \mid n$, which in conjunction with (2.3) proves the claim. □

2.1. The Euler-Muir theorem and equipalindromic numbers

The following section was inspired by the website [4] which describes the Euler-Muir theorem without giving a proof. For proving this theorem, we need the following lemma:

Lemma 22. *For $a_1, \dots, a_l$ with $l \in \mathbb{N}_0$, $a_k \in \mathbb{N}$, $a_k = a_{l-k+1}$ $\forall$ k and A, B, C, D given by*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_l & 1 \\ 1 & 0 \end{pmatrix},$$

we have $B = C$ and $B^2 = AD - (-1)^l$, furthermore $A \geq B$ ($A > B$ for $l \neq 1$) and $A \geq D$.

Proof. We prove the statement using induction over $\lfloor l/2 \rfloor$. If $l = 0$ (empty product), we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B^2 = 0^2 = 0 = 1 - 1 = AD - (-1)^0;$$

if $l = 1$, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B^2 = 1^2 = 1 = a_1 \cdot 0 - (-1) = AD - (-1)^1,$$

in both cases we have $B = C$, $A \geq B$ ($A > B$ for $l = 0$) and $A \geq D$. For

$$\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} = \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{l-1} & 1 \\ 1 & 0 \end{pmatrix}$$

we take $\bar{B}^2 = \bar{A}\bar{D} - (-1)^{l-2}$ and $\bar{B} = \bar{C}$ as the induction hypothesis and conclude

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1^2\bar{A} + 2a_1\bar{B} + \bar{D} & a_1\bar{A} + \bar{B} \\ a_1\bar{A} + \bar{B} & \bar{A} \end{pmatrix}, \qquad B = C,$$

$$B^2 = (a_1\bar{A} + \bar{B})^2 = a_1^2\bar{A}^2 + 2a_1\bar{A}\bar{B} + \bar{B}^2 = (a_1^2\bar{A} + 2a_1\bar{B} + \bar{D})\bar{A} - (-1)^{l-2} = AD - (-1)^l.$$

We further obtain the estimates

$$B = a_1\bar{A} + \bar{B} < a_1^2\bar{A} + 2a_1\bar{B} + \bar{D} = A, \qquad D = \bar{A} \leq a_1^2\bar{A} < a_1^2\bar{A} + 2a_1\bar{B} + \bar{D} = A,$$

and those inequalities hold for $l \geq 2$, as we have $\bar{A} > 0$ and either $\bar{B} > 0$ or $\bar{D} > 0$. □

Theorem 23 (Euler-Muir). *For $a_1, \dots, a_l$ with $l \in \mathbb{N}_0$, $a_k = a_{l-k+1}$ $\forall$ k, we define A, B, D by*

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_l & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$m = (((-1)^{l+1}(A+1)BD + \max a_k) \mod 2A') - \max a_k, \qquad A' = \begin{cases} A & \text{if } 2 \nmid A, \\ A/2 & \text{if } 2 \mid A. \end{cases}$$

If $l = 0$ (empty product) the above matrix multiplication yields the identity matrix. Then the numbers d in $\mathbb{N}^*$, the set of positive integers that are not perfect squares, with

$$\sqrt{d} = [\lfloor\sqrt{d}\rfloor, \overline{a_1, \dots, a_l, 2\lfloor\sqrt{d}\rfloor}]$$

where l is minimal, i.e. $2\lfloor\sqrt{d}\rfloor > \max a_k$[1] *(see corollary 21), are parametrised by*

$$\left\{ (A')^2 n^2 + \left(\frac{2A'}{A}B - mA'\right) n + \frac{D - mB}{A} + \left(\frac{m}{2}\right)^2 : n \in \mathbb{N} \right\},$$

and there exist no such numbers if $2 \nmid BD$. The polynomial from this set description is called the Euler-Muir polynomial *for the palindrome* $(a_1, \dots, a_l)$.

[1] we have $\max \varnothing = -\infty$, so this statement is true for $l = 0$; for further calculation I will use 0 in place of $-\infty$, i.e. $a > \max\{0\} \cup \{a_k : 1 \leq k \leq l\}$, as $a > 0$ and $a_k \geq 1$.

Proof. The cont'd fraction representation of $\sqrt{d}$ with the palindrome $(a_1,\dots,a_l)$ takes the form

$$\begin{aligned} &\sqrt{d} = [\lfloor\sqrt{d}\rfloor, \overline{a_1,\dots,a_l,2\lfloor\sqrt{d}\rfloor}] \\ \iff\quad &\lfloor\sqrt{d}\rfloor + \sqrt{d} = [\overline{2\lfloor\sqrt{d}\rfloor, a_1,\dots,a_l}]. \end{aligned}$$

Let $a \in \mathbb{N}$ and $x = [\overline{a,a_1,\dots,a_l}]$ with $a > a_k$ for all k. Then

$$x = [a,a_1,\dots,a_l,x] = a + \frac{1}{[a_1,\dots,a_l,x]}.$$

Using lemma 6, we evaluate the denominator using matrix multiplication:

$$\left(\prod_{i=1}^{l}\begin{pmatrix} a_i & 1\\ 1 & 0\end{pmatrix}\right)\begin{pmatrix} x & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix} A & B\\ B & D\end{pmatrix}\begin{pmatrix} x & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix} Ax+B & A\\ Bx+D & B\end{pmatrix},$$

so $[a_1,\dots,a_l,x] = (Ax+B)/(Bx+D)$ and we conclude

$$\begin{aligned} & x = a + \frac{Bx+D}{Ax+B} \\ \iff\quad & (Ax+B)x = (Ax+B)a + (Bx+D) \\ \iff\quad & Ax^2 + Bx = Aax + Ba + Bx + D \\ \iff\quad & Ax^2 - Aax - (Ba+D) = 0 \\ \overset{x>0}{\iff}\quad & x = \frac{Aa + \sqrt{(Aa)^2 + 4A(Ba+D)}}{2A} = \frac{a}{2} + \sqrt{\frac{A^2a^2 + 4A(Ba+D)}{4A^2}}. \end{aligned}$$

Obviously, $x = y + \sqrt{z}$ holds for $y, z \in \mathbb{N}$ if and only if the conditions

$$y = \frac{a}{2} \in \mathbb{N} \quad\iff\quad a \in 2\mathbb{N} \quad\iff\quad a \equiv 0 \pmod 2 \quad\wedge\quad a > 0,$$

and

$$z = \frac{A^2a^2 + 4A(Ba+D)}{4A^2} = \left(\frac{a}{2}\right)^2 + \frac{Ba+D}{A} \in \mathbb{N} \quad\overset{a\in 2\mathbb{N}}{\iff}\quad A \mid (Ba+D)$$

are met. Per lemma 22, we have $B^2 \equiv (-1)^{l-1} \pmod A$, so the latter condition yields

$$Ba + D \equiv 0 \pmod A \quad\overset{\cdot(-1)^{l-1}B,\ -(-1)^{l-1}BD}{\iff}\quad a \equiv (-1)^l BD \pmod A,$$

o.t.o.h. we have $B^2 \equiv AD + 1 \pmod 2$, so if $2 \nmid BD \Leftrightarrow B \equiv D \equiv 1 \pmod 2$, we would have

$$1 \equiv A + 1 \pmod 2 \quad\implies\quad 2 \mid A,$$

but $2 \mid a$ would imply $2 \nmid (Ba+D)$ and hence $A \nmid (Ba+D)$, so we must have $2 \mid BD$.
If $2 \nmid A$, the Chinese Remainder Theorem[2] yields the solution

$$a \equiv (-1)^l(A+1)BD \pmod{2A} \quad\iff\quad a \equiv (-1)^l(A+1)BD \pmod{2A'}.$$

If $2 \mid A$, we see from $2 \mid BD$ that the second congruence implies the first, so we have

$$a \equiv (-1)^l BD \pmod A \quad\iff\quad a \equiv (-1)^l(A+1)BD \pmod{2A'}.$$

Furthermore, we must have $y = a/2 = \lfloor\sqrt{z}\rfloor$, as $(a/2)^2 \le z$ is trivial and we have

$$\begin{aligned} & z = \left(\frac{a}{2}\right)^2 + \frac{Ba+D}{A} < \left(\frac{a}{2}+1\right)^2 = \left(\frac{a}{2}\right)^2 + a + 1 \\ \iff\quad & (B-A)a < A - D \qquad (*) \end{aligned}$$

[2] see [1]

and by lemma 22 two cases can arise – we could have $A = B$ which only happens if $l = 1$ and $a_1 = 1$, in which case $A - D = 1 - 0 = 1$, reducing the above inequality to $0 < 1$ which is true, or we have $A > B$ and $(*)$ is equivalent to

$$a > \underbrace{\frac{A - D}{B - A}}_{\leq 0 \text{ since } A \geq D},$$

and this is true as $a \in \mathbb{N}$, so indeed we have $\lfloor \sqrt{z} \rfloor = a/2 = y$.

Now we just need to parametrise these solutions – we want $n \in \mathbb{N}$ to give us

$$2(n-1)A' + m' = 2nA' - 2A' + m' = 2nA' - (2A' - m')$$

where m' is the smallest solution $a > \max a_k$ of the above congruence. We have

$$\max a_k < m' \leq 2A' + \max a_k \qquad \Longleftrightarrow \qquad 0 \leq 2A' - m' + \max a_k < 2A'$$

and hence

$$2A' - m' + \max a_k = ((-1)^{l+1}(A+1)BD + \max a_k) \mod 2A';$$

so, with $m := (((-1)^{l+1}(A+1)BD + \max a_k) \mod 2A') - \max a_k$, we get the parametrisation

$$a = 2nA' - (2A' - m') = 2nA' - m$$

and the possible values for z can be computed using

$$\begin{aligned} f(n) = \left(\frac{a}{2}\right)^2 + \frac{Ba + D}{A} &= \left(nA' - \frac{m}{2}\right)^2 + \frac{B(2nA' - m) + D}{A} \\ &= (A')^2 n^2 - mA'n + \left(\frac{m}{2}\right)^2 + \frac{2A'}{A} Bn + \frac{D - mB}{A} \\ &= (A')^2 n^2 + \left(\frac{2A'}{A} B - mA'\right) n + \frac{D - mB}{A} + \left(\frac{m}{2}\right)^2. \end{aligned}$$

□

Example 23.1. For $(a_1, \ldots, a_l) = (1, 1, 1)$ we have $(A, B, D) = (3, 2, 1)$ and hence $A' = 3$. The smallest solution > 1 of

$$a \equiv (-1)^l (A+1)BD \pmod{2A'}$$

is $m' = 4$ and indeed our formula for m gives us

$$m = (((-1)^{l+1}(A+1)BD + \max a_k) \mod 2A') - \max a_k = 3 - 1 = \boxed{2} = 6 - 4 = 2A' - m',$$

yielding the Euler-Muir polynomial

$$f(n) = 3^2 n^2 + \left(\frac{2 \cdot 3}{3} \cdot 2 - 2 \cdot 3\right) n + \frac{1 - 2 \cdot 2}{3} + \left(\frac{2}{2}\right)^2 = 9n^2 - 2n;$$

indeed we have $f(1) = 7$ with $\sqrt{7} = [2, \overline{1, 1, 1, 4}]$; $f(2) = 32$ with $\sqrt{32} = [5, \overline{1, 1, 1, 10}]$; $f(3) = 75$ with $\sqrt{75} = [8, \overline{1, 1, 1, 16}]$ and so on. We call the numbers $f(n)$ *equipalindromic* as the continued fraction representations of their square roots contain the same palindrome $(a_1, \ldots, a_l)$.

Remark 23.2. A table listing the polynomials for $d \in \mathbb{N}^*$, $d \leq 400$ can be found in the appendix.

Code 23.3 (Python). This code computes the Euler-Muir polynomial given a palindrome.

```
def euler_muir(P): # P is the given palindrome, for example [1, 1, 1] -> (9, -2, 0)
  A, B, C, D = 1, 0, 0, 1
  p = [a for a in P] # create a copy so P doesn't change
  while len(p) > 0:
    a = p.pop()
    A, B, C, D = a * A + C, a * B + D, A, B
  if B * D % 2 == 1: # unsolvable if B * D = 1 (mod 2)
    return
  A_ = A if A % 2 == 1 else A // 2
  M = max([0] + P)
  m = ((-1) ** (len(P) + 1) * (A + 1) * B * D + M) % (2 * A_) - M
  X, Y, Z = A_ ** 2, (2 * A_ // A) * B - m * A_, (D - B * m) // A + (m // 2) ** 2
  return X, Y, Z
```

Code 23.4 (Python). This code computes the palindrome $(a_1, \ldots, a_l)$ of $\sqrt{d}$, cf. proof of 17.

```
def continued_fraction_sqrt_palindrome(d):
  import math
  a_n, m_n, q_n, A = math.floor(math.sqrt(d)), 0, 1, []
  while a_n != 2 * math.floor(math.sqrt(d)):
    A += [a_n]
    m_n = a_n * q_n - m_n
    q_n = (d - m_n ** 2) // q_n
    xi_n = (m_n + math.sqrt(d)) / q_n
    a_n = math.floor(xi_n)
  return A[1:]
```

Lemma 24. *If l is even and $(a_1, \ldots, a_l)$ is a palindrome, i.e. $a_k = a_{l-k+1}\ \forall\ k$ and*

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_l & 1 \\ 1 & 0 \end{pmatrix},$$

then we have $A + BD \equiv 1 \pmod 2$. In particular, we have $2 \nmid A$ if $\sqrt{d} = [\lfloor\sqrt{d}\rfloor, \overline{a_1, \ldots, a_l, 2\lfloor\sqrt{d}\rfloor}]$.

Proof. We prove the statement using induction over $l/2$. For $l = 0$ (empty product) the matrix product yields the identity matrix, i.e. $A = D = 1$ and $B = 0$ and we have

$$A + BD = 1 + 0 \cdot 1 \equiv 0 \pmod 2.$$

For

$$\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{pmatrix} = \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{l-1} & 1 \\ 1 & 0 \end{pmatrix},$$

let the induction hypothesis be $\bar{A} + \bar{B}\bar{D} \equiv 1 \pmod 2$. We have

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1^2\bar{A} + 2a_1\bar{B} + \bar{D} & a_1\bar{A} + \bar{B} \\ a_1\bar{A} + \bar{B} & \bar{A} \end{pmatrix},$$

furthermore lemma 22 implies that $B \equiv AD + 1 \pmod 2$, so we have

$$A + BD \equiv A + (AD + 1)D \equiv A(1 + D) + D \equiv (a_1\bar{A} + \bar{D})(1 + \bar{A}) + \bar{A} \pmod 2.$$

If $\bar{A} \equiv 0 \pmod 2$, the induction hypothesis implies that $\bar{D} \equiv 1 \pmod 2$ and we conclude

$$A + BD \equiv (0 + 1)(1 + 0) + 0 \equiv 1 \pmod 2,$$

if $\bar{A} \equiv 1 \pmod 2$, we conclude

$$A + BD \equiv (a_1 + \bar{D})(1 + 1) + 1 \equiv 1 \pmod 2.$$

Per theorem 23, if $\sqrt{d} = [\lfloor\sqrt{d}\rfloor, \overline{a_1, \ldots, a_l, 2\lfloor\sqrt{d}\rfloor}]$ we have $2 \mid BD$ and hence $2 \nmid A$. □

Definition 25. For the minimal d containing a given palindrome in the continued fraction representation of $\sqrt{d}$, we denote the corresponding Euler-Muir polynomial by $f_d(n)$; we have

$$f_d(1) = d.$$

Lemma 26. *For a odd we have $a^3 \equiv a \pmod{2a}$.*

Proof. We have

$$a^3 \equiv a \pmod{2a} \qquad \Longleftrightarrow \qquad 0 \equiv a^3 - a \equiv a(a^2-1) \pmod{2a}$$

and this is true as $a^2 - 1$ is even and hence in $2\mathbb{Z}$, implying $a(a^2-1) \in 2a\mathbb{Z}$. □

Theorem 27. *For $a \geq 1$ odd, the E.-M. polynomial corresponding to the palindrome (a) is*

$$f_{a^2+2}(n) = a^2n^2 + 2n$$

and for $a \geq 3$ odd, the Euler-Muir polynomial corresponding to the palindrome $(1, a-2, 1)$ is

$$f_{a^2-2}(n) = a^2n^2 - 2n.$$

For $a \geq 2$ even, the Euler-Muir polynomial corresponding to the palindrome (a) is

$$f_{a^2+2}(n) = \left(\frac{a}{2}\right)^2 n^2 + \left(\frac{a^2}{2} + 1\right) n + \left(\frac{a}{2}\right)^2 + 1$$

and for $a \geq 4$ even, the Euler-Muir polynomial corresponding to the palindrome $(1, a-2, 1)$ is

$$f_{a^2-2}(n) = \left(\frac{a}{2}\right)^2 n^2 + \left(\frac{a^2}{2} - 1\right) n + \left(\frac{a}{2}\right)^2 - 1.$$

Proof. We first consider the odd case. For $(a_1, \dots, a_l) = (a)$ we have

$$(A, B, D) = (a, 1, 0), \qquad A' = A = a$$

and the formula for m yields

$$m = (((-1)^{1+1}(a+1) \cdot 1 \cdot 0 + a) \mod 2a) - a = (a \mod 2a) - a = a - a = 0,$$

giving us the Euler-Muir polynomial

$$f(n) = a^2n^2 + \left(\frac{2a}{a} \cdot 1 - 0 \cdot a\right) n + \frac{0 - 0 \cdot 1}{a} + \left(\frac{0}{2}\right)^2 = a^2n^2 + 2n$$

and we have $f(1) = a^2 + 2$. For $(a_1, \dots, a_l) = (1, a-2, 1)$ we compute

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a-2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & a-1 \\ a-1 & a-2 \end{pmatrix}, \quad A' = A = a$$

and the formula for m yields

$$\begin{aligned} m &= (((-1)^{3+1}(a+1) \cdot (a-1) \cdot (a-2) + (a-2)) \mod 2a) - (a-2) \\ &= (((a^2-1)(a-2) + (a-2)) \mod 2a) - (a-2) \\ &= ((a^2(a-2) \mod 2a) - (a-2) \\ &= ((a^3 - 2a^2) \mod 2a) - (a-2) \\ &\overset{26}{=} a - (a-2) \\ &= 2, \end{aligned}$$

giving us the Euler-Muir polynomial

$$f(n) = a^2n^2 + \left(\frac{2a}{a} \cdot (a-1) - 2 \cdot a\right) n + \underbrace{\frac{(a-2) - 2 \cdot (a-1)}{a}}_{=-a/a=-1} + \left(\frac{2}{2}\right)^2 = a^2n^2 - 2n$$

and we have $f(1) = a^2 - 2$.

Now we consider the even case. For $(a_1, \ldots, a_l) = (a)$ we have

$$(A, B, D) = (a, 1, 0), \qquad A' = \frac{A}{2} = \frac{a}{2}$$

and the formula for m yields

$$m = (((-1)^{1+1}(a+1) \cdot 1 \cdot 0 + a) \mod a) - a = (a \mod a) - a = 0 - a = -a,$$

giving us the Euler-Muir polynomial

$$\begin{aligned} f(n) &= \left(\frac{a}{2}\right)^2 n^2 + \left(\frac{2(a/2)}{a} \cdot 1 - (-a) \cdot \frac{a}{2}\right) n + \frac{0 - (-a) \cdot 1}{a} + \left(\frac{-a}{2}\right)^2 \\ &= \left(\frac{a}{2}\right)^2 n^2 + \left(\frac{a^2}{2} + 1\right) n + \left(\frac{a}{2}\right)^2 + 1 \end{aligned}$$

and we have

$$f(1) = \left(\frac{a}{2}\right)^2 + \left(\frac{a^2}{2} + 1\right) + \left(\frac{a}{2}\right)^2 + 1 = a^2 + 2.$$

For $(a_1, \ldots, a_l) = (1, a-2, 1)$ we compute, as above,

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} = \begin{pmatrix} a & a-1 \\ a-1 & a-2 \end{pmatrix}, \quad A' = \frac{A}{2} = \frac{a}{2}$$

and the formula for m yields

$$\begin{aligned} m &= (((-1)^{3+1}(a+1) \cdot (a-1) \cdot (a-2) + (a-2)) \mod a) - (a-2) \\ &= (((a^2-1)(a-2) + (a-2)) \mod a) - (a-2) \\ &= ((a^2(a-2)) \mod a) - (a-2) \\ &= 0 - (a-2) \\ &= 2 - a, \end{aligned}$$

giving us the Euler-Muir polynomial

$$\begin{aligned} f(n) &= \left(\frac{a}{2}\right)^2 n^2 + \left(\frac{2(a/2)}{a} \cdot (a-1) - (2-a) \cdot \frac{a}{2}\right) n + \frac{(a-2) - (2-a) \cdot (a-1)}{a} + \left(\frac{2-a}{2}\right)^2 \\ &= \left(\frac{a}{2}\right)^2 n^2 + \left((a-1) - \left(a - \frac{a^2}{2}\right)\right) n + \frac{(a-2)a}{a} + \frac{a^2 - 4a + 4}{4} \\ &= \left(\frac{a}{2}\right)^2 n^2 + \left(\frac{a^2}{2} - 1\right) n + (a-2) + \left(\frac{a}{2}\right)^2 - a + 1 \\ &= \left(\frac{a}{2}\right)^2 n^2 + \left(\frac{a^2}{2} - 1\right) n + \left(\frac{a}{2}\right)^2 - 1 \end{aligned}$$

and we have

$$f(1) = \left(\frac{a}{2}\right)^2 + \left(\frac{a^2}{2} - 1\right) + \left(\frac{a}{2}\right)^2 - 1 = a^2 - 2.$$

□

Theorem 28. *For $d = a^2 + 1$ $(a \geq 1)$ we have $l = 0$, for $d = a^2 - 1$ $(a \geq 2)$ we have $l = 1$.*

Proof. We first consider the Euler-Muir polynomial corresponding to the palindrome ().
We have

$$A = D = 1, \qquad B = 0, \qquad A' = A = 1$$

and the formula for m yields

$$m = (((-1)^{0+1}(1+1) \cdot 0 \cdot 1 + 0) \mod 2 - 0 = 0 - 0 = 0,$$

giving us the Euler-Muir polynomial

$$f(n) = 1^2 n^2 + \left(\frac{2 \cdot 1}{1} \cdot 0 - 0 \cdot 1\right) n + \frac{1 - 0 \cdot 0}{1} + \left(\frac{0}{2}\right)^2 = n^2 + 1,$$

and we have

$$f(1) = 2, \qquad f(a) = a^2 + 1,$$

so the numbers $d = a^2 + 1$ have the palindrome () in the cont'd fraction representation of $\sqrt{d}$.
By theorem 27, the Euler-Muir polynomial corresponding to the palindrome (1) is

$$f(n) = n^2 + 2n = (n+1)^2 - 1,$$

and we have

$$f(1) = 3 = 2^2 - 1, \qquad f(a') = (a'+1)^2 - 1,$$

so the numbers $d = a^2 - 1$ have the palindrome (1) in the continued fraction repr. of $\sqrt{d}$. $\square$

3. Pell's equation

We shall now discuss the existence of integer solutions of $x^2 - dy^2 = \pm 1$, where $d \in \mathbb{N}$. As $(-n)^2 = n^2$ for all $n \in \mathbb{Z}$, we assume $x, y \geq 0$ from now on.

3.1. Solvability of $x^2 - dy^2 = \pm 1$

Theorem 29. *If d is a perfect square, $x^2 - dy^2 = \pm 1$ has no solutions other than $(x, y) = (1, 0)$, except when $d = 1$, in which case $(x, y) = (0, 1)$ is a solution.*

Proof. If d is a perfect square, there exists a $k \in \mathbb{N}$ with $k^2 = d$ and we have

$$1 = |\pm 1| = |x^2 - k^2 y^2| = |x + ky| \cdot |x - ky|,$$

so we must have $|x + ky| = |x - ky| = 1$. From $x, y, k \geq 0$ it follows that

$$x \leq x + ky = |x + ky| = 1,$$

so we must have $x \in \{0, 1\}$. If $x = 0$ we have $|ky| = 1$, and hence $k = y = 1$: in particular, we conclude that $d = k^2 = 1$ and we have $x^2 - dy^2 = -1$. If $x = 1$ we have

$$1 = |x + ky| = x + ky = 1 + ky \qquad \Longrightarrow \qquad ky = 0 \qquad \Longrightarrow \qquad y = 0$$

and $x^2 - dy^2 = 1$. □

Theorem 30. *If d is not a perfect square, there are infinitely many solutions of $x^2 - dy^2 = \pm 1$. Furthermore: if $x^2 - dy^2 = \pm 1$, there exists an n with $(x, y) = (h_n, k_n)$, where h_n, k_n are as in theorem 14 for $\xi = \sqrt{d}$.*

Proof. On the one hand, for d not a perfect square, $\sqrt{d}$ takes the form

$$[\lfloor \sqrt{d} \rfloor, \overline{a_1, \ldots, a_l, 2\lfloor \sqrt{d} \rfloor}]$$

and by corollary 20, for $n = r(l + 1) - 1$ where $r \geq 1$, we have

$$h_n^2 - dk_n^2 = (-1)^{n-1}$$

and due to $k_n < k_{n+1}$ for $n \geq 1$ this gives us infinitely many solutions.[1]
In particular, an n such that $h_n^2 - dk_n^2 = -1$ can only exist if l is even.[2]

On the other hand, let's assume that $x^2 - dy^2 = \pm 1$. We shall consider the positive case first: we have

$$\frac{x}{y} - \sqrt{d} = \frac{x - y\sqrt{d}}{y} = \frac{x^2 - dy^2}{y(x + y\sqrt{d})} = \frac{1}{y(x + y\sqrt{d})} > 0.$$

Furthermore, we have the estimate

$$\frac{x}{y} - \sqrt{d} = \frac{1}{y(x + y\sqrt{d})} < \frac{\sqrt{d}}{y(x + y\sqrt{d})} = \frac{\sqrt{d}}{y^2(x/y + \sqrt{d})} = \frac{1}{y^2(x/(y\sqrt{d}) + 1)}.$$

From $x/y - \sqrt{d} > 0 \;\Leftrightarrow\; x/y > \sqrt{d} \;\Leftrightarrow\; x/(y\sqrt{d}) > 1$ we can now conclude that

$$\left| \sqrt{d} - \frac{x}{y} \right| = \frac{x}{y} - \sqrt{d} < \frac{1}{y^2(x/(y\sqrt{d}) + 1)} < \frac{1}{y^2(1 + 1)} = \frac{1}{2y^2}$$

and by theorem 14, there exists an $n \in \mathbb{N}$ such that $(x, y) = (h_n, k_n)$.

[1] lemma 3
[2] for l odd, $l + 1$ is even, whereby n is always odd

In the negative case, we shall argue analogously. By rearranging, we obtain

$$x^2 - dy^2 = -1 \quad \overset{/(-d)}{\Longleftrightarrow} \quad y^2 - \frac{1}{d}x^2 = \frac{1}{d}$$

and we have

$$\frac{y}{x} - \frac{1}{\sqrt{d}} = \frac{y - x(1/\sqrt{d})}{x} = \frac{y^2 - (1/d)x^2}{x(y + x(1/\sqrt{d}))} = \frac{1/d}{x(y + x(1/\sqrt{d}))} > 0.$$

Furthermore, we have the estimate

$$\frac{y}{x} - \frac{1}{\sqrt{d}} = \frac{1/d}{x(y + x(1/\sqrt{d}))} < \frac{1/\sqrt{d}}{x(y + x(1/\sqrt{d}))} = \frac{1/\sqrt{d}}{x^2(y/x + 1/\sqrt{d})} = \frac{1}{x^2((y\sqrt{d})/x + 1)}.$$

From $y/x - 1/\sqrt{d} > 0 \;\Leftrightarrow\; y/x > 1/\sqrt{d} \;\Leftrightarrow\; (y\sqrt{d})/x > 1$ we can now conclude that

$$\left|\frac{1}{\sqrt{d}} - \frac{y}{x}\right| = \frac{y}{x} - \frac{1}{\sqrt{d}} < \frac{1}{x^2((y\sqrt{d})/x + 1)} < \frac{1}{x^2(1+1)} = \frac{1}{2x^2}.$$

By 14, there exists an $n' \in \mathbb{N}$ such that y/x is the n'^{th} convergent of $1/\sqrt{d}$. By lemma 12 we can thus conclude that x/y is the $(n'-1)^{\text{st}}$ convergent of $\sqrt{d}$, so there indeed exists an $n = n' - 1$ such that $(x, y) = (h_n, k_n)$. □

Example 30.1. We consider the case $d = 7$. Using the algorithm from theorem 17, we obtain

$$\sqrt{7} = [2, \overline{1, 1, 1, 4}].$$

Here l is odd, so only the positive Pell's equation is solvable.

We now evaluate the third convergent of $\sqrt{d}$:

$$[2,1,1,1] = \left[2, 1, 1 + \frac{1}{1}\right] = [2,1,2] = \left[2, 1 + \frac{1}{2}\right] = \left[2, \frac{3}{2}\right] = 2 + \frac{2}{3} = \frac{8}{3}$$

and indeed we have

$$8^2 - 7 \cdot 3^2 = 64 - 63 = 1.$$

Lemma 31 ([2, exercise 7.8.5])**.**
If $x^2 - dy^2 = \pm 1$ and $X + Y\sqrt{d} = (x + y\sqrt{d})^n$, we have $X^2 - dY^2 = (\pm 1)^n$.

Proof.

$$X^2 - dY^2 = (X + Y\sqrt{d})(X - Y\sqrt{d}) = (x + y\sqrt{d})^n(x - y\sqrt{d})^n = (x^2 - dy^2)^n = (\pm 1)^n.$$

□

Theorem 32. *If d is not a perfect square, there exist infinitely many solutions of $x^2 - dy^2 = 1$ with $k \mid y$ for any $k \in \mathbb{Z}$.*

Proof. $D := dk^2$ is not a perfect square, so there exist infinitely many solutions (X, Y) of

$$X^2 - DY^2 = 1.$$

For any of these solutions, (X, kY) solves $x^2 - dy^2 = 1$ and we have $k \mid kY$. □

3.2. Solvability of $x^2 - dy^2 = -1$

Theorem 33 ([2, exercise 7.8.3])**.** *If $d \equiv 3 \pmod 4$, then $x^2 - dy^2 = -1$ has no solutions.*

Proof. Let's assume that $x^2 - dy^2 = -1$ is solvable. Taking the equation modulo 4, we get

$$-1 \equiv x^2 - 3y^2 \equiv x^2 + y^2.$$

But as $\{a^2 \mod 4 : a \in \mathbb{Z}\} = \{0, 1\}$, we conclude that $(x^2 + y^2) \mod 4 \in \{0, 1, 2\}$. □

Theorem 34 ([2, exercise 7.8.11])**.** *If d is divisible by a prime p with $p \equiv 3 \pmod 4$, then $x^2 - dy^2 = -1$ has no solutions.*

Proof. Let $p > 2$ be a prime factor of d. If $x^2 - dy^2 = -1$ is solvable, we have $x^2 \equiv -1 \pmod p$. We now consider the product

$$P = 1 \cdot 2 \cdot \ldots \cdot (p-2) \cdot (p-1)$$

modulo p. As p is prime, $\mathbb{Z}/p\mathbb{Z}$ is a field, so any $a \in \{2, \ldots, p-2\}$ has an inverse $a^{-1} \not\equiv a$, and we can group the factors $2 \cdot \ldots \cdot (p-2)$ together as $(p-3)/2$ pairs of elements whose product is 1. Consequently, we have

$$P \equiv 1 \cdot 1 \cdot \ldots \cdot 1 \cdot (p-1) \equiv -1 \pmod p.$$

We first note that the equation $ab \equiv -1$ has a unique solution $b \equiv -a^{-1}$ for any a. If there exists an x with $x^2 \equiv -1 \pmod p$, on the one hand we have $(-x)^2 \equiv -1 \pmod p$, where $x \not\equiv -x$ as p is odd, on the other hand $\mathbb{Z}/p\mathbb{Z}$ is a field and $x^2 + 1 \equiv 0 \pmod p$ only permits two solutions, so $\{1, \ldots, p-1\}$ consists exactly of the elements $\pm x$ as well as $(p-3)/2$ pairs of elements whose product is -1. We have $x \cdot (-x) \equiv -x^2 \equiv 1 \pmod p$, consequently

$$-1 \equiv P \equiv (-1)^{(p-3)/2} \pmod p.$$

Therefore, $(p-3)/2$ must be odd, so there exists a $k \in \mathbb{Z}$ with

$$\frac{p-3}{2} = 2k - 1 \qquad \Longrightarrow \qquad p = 4k + 1 \qquad \Longrightarrow \qquad p \equiv 1 \pmod 4.$$

Consequently, if $x^2 - dy^2 = -1$ is solvable, d cannot have a prime factor $p \equiv 3 \pmod 4$. □

Theorem 35 ([2, exercise 7.8.12])**.** *If $p \equiv 1 \pmod 4$ is a prime, then $x^2 - py^2 = -1$ is solvable.*

Proof. As p is not a perfect square, $x^2 - py^2 = 1$ is solvable. Taking the equation modulo 4 yields

$$x^2 - y^2 \equiv 1,$$

and since squares modulo 4 are always $\in \{0, 1\}$, we must have $x^2 \equiv 1$ and $y^2 \equiv 0$, so x is odd and y even, thus we have $2 \mid \gcd(x+1, x-1)$. On the other hand we have

$$\gcd(x+1, x-1) \mid (x+1) - (x-1) = 2,$$

yielding $\gcd(x+1, x-1) = 2$. Let (x_0, y_0) now be a solution of $x^2 - py^2 = 1$ with $y_0 > 0$ minimal. We have

$$(x_0 + 1)(x_0 - 1) = x_0^2 - 1 = py_0^2.$$

As p is prime, exactly one of the numbers $x_0 \pm 1$ is divisible by p. As $\gcd(x+1, x-1) = 2$, the factors on the left-hand side are coprime integers:

$$\frac{x_0 \pm 1}{2p} \cdot \frac{x_0 \mp 1}{2} = \left(\frac{y_0}{2}\right)^2$$

and as their product is a perfect square, those numbers must be perfect squares as well:

$$\frac{x_0 \pm 1}{2p} = u^2 \quad \Leftrightarrow \quad x_0 \pm 1 = 2pu^2, \qquad \frac{x_0 \mp 1}{2} = v^2 \quad \Leftrightarrow \quad x_0 \mp 1 = 2v^2, \qquad u, v > 0.$$

If $x_0 - 1 = 2pu^2$ and $x_0 + 1 = 2v^2$, we would have

$$1 = \frac{(x_0+1)-(x_0-1)}{2} = \frac{2v^2 - 2pu^2}{2} = v^2 - pu^2,$$

but $u^2 \leq (y_0/2)^2 < y_0^2$ implies that $0 < u < y_0$ contradicts the minimality of y_0. Therefore, $x_0 + 1 = 2pu^2$ and $x_0 - 1 = 2v^2$. It follows that

$$-1 = \frac{(x_0-1)-(x_0+1)}{2} = \frac{2v^2 - 2pu^2}{2} = v^2 - pu^2,$$

so $(x, y) = (v, u)$ solves $x^2 - py^2 = -1$. □

Theorem 36 ([3, p. 171])**.** *If d is not a perfect square, $a_1, \ldots, a_l$ with $a_k = a_{l-k}$ $\forall$ k and l minimal, such that*

$$\sqrt{d} = [\lfloor\sqrt{d}\rfloor, \overline{a_1, \ldots, a_l, 2\lfloor\sqrt{d}\rfloor}],$$

and A, B, D defined by

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_l & 1 \\ 1 & 0 \end{pmatrix},$$

then $x^2 - dy^2 = -1$ is solvable if and only if (i) $4 \nmid d$, (ii) d has no prime factors $\equiv 3 \pmod 4$, and (iii) A is odd.

Proof. Let P and Q be the numerator and denominator of the l^{th} convergent of $\sqrt{d}$:

$$\frac{P}{Q} = \lfloor\sqrt{d}\rfloor + \frac{1}{[a_1, \ldots, a_l]} = \lfloor\sqrt{d}\rfloor + \frac{B}{A} = \frac{\lfloor\sqrt{d}\rfloor A + B}{A}.$$

By corollary 9 this fraction is in lowest terms as A and B are coprime, in particular, $Q = A$.
If $x^2 - dy^2 = -1$ is solvable, then theorem 19 implies that

$$P^2 - dQ^2 = -1.$$

On the one hand d then has no prime factors $\equiv 3 \pmod 4$ by theorem 34, on the other hand we must have $4 \nmid d$ and $A \equiv Q \equiv 1 \pmod 2$, for otherwise we would have $P^2 \equiv -1 \pmod 4$.
Conversely, let A be odd, $4 \nmid d$, and d have no prime factors $\equiv 3 \pmod 4$. On the one hand, theorem 19 gives us

$$P^2 - dQ^2 \in \{\pm 1\}.$$

On the other hand, $Q = A$ is odd and $d \mod 4$ has to be either 1 or 2, so

$$1 \equiv P^2 - dQ^2 \equiv P^2 - d \pmod 4$$

is not satisfiable, hence we must have $P^2 - dQ^2 = -1$. □

Theorem 37. *For $d > 2$ of the form $d = a^2 \pm 2$ for $a \in \mathbb{N}$, $x^2 - dy^2 = -1$ has no solutions.*

Proof. This follows from corollary 20 and theorem 27 as $l \in \{1, 3\}$ is odd. □

Theorem 38. *For numbers $d > 1$ of the form $d = a^2 + 1$ for $a \in \mathbb{N}$, $x^2 - dy^2 = -1$ is solvable. For numbers $d > 2$ of the form $d = a^2 - 1$ for $a \in \mathbb{N}$, $x^2 - dy^2 = -1$ has no solutions.*

Proof. This follows from 20 and 28 as l is even in the first and odd in the second case. □

3.3. Applications

Theorem 39 ([2, exercise 7.8.5])**.**
A number is both a triangular number and a perfect square, i.e.

$$n^2 = \sum_{i=1}^{m} i,$$

if and only if $\left(\sqrt{m+1}, \frac{n}{\sqrt{m+1}}\right)$ *solves* $x^2 - 2y^2 = 1$ *or* $\left(\sqrt{m}, \frac{n}{\sqrt{m}}\right)$ *solves* $x^2 - 2y^2 = -1$.

Proof. From the formula for the arithmetic series, we obtain

$$n^2 = \sum_{i=1}^{m} i = \frac{m(m+1)}{2} \qquad \Longleftrightarrow \qquad 2n^2 = m(m+1).$$

Either m or $m+1$ is even, so there exist A, B with $\{m, m+1\} = \{2A, B\}$ and $m(m+1) = 2AB$. As m and $m+1$ are coprime, so are A and B. On the other hand, as $n^2 = AB$, we can conclude that A and B are perfect squares. If $m = 2A$, then $m+1 = B$ is a perfect square and for $k := \sqrt{m+1}$ we have

$$2n^2 = m(m+1) = (k^2-1)k^2 \qquad \Longleftrightarrow \qquad 2\left(\frac{n}{k}\right)^2 = k^2 - 1 \qquad \Longleftrightarrow \qquad k^2 - 2\left(\frac{n}{k}\right)^2 = 1,$$

if $m+1 = 2A$, then $m = B$ is a perfect square and for $k := \sqrt{m}$ we have

$$2n^2 = m(m+1) = k^2(k^2+1) \qquad \Longleftrightarrow \qquad 2\left(\frac{n}{k}\right)^2 = k^2 + 1 \qquad \Longleftrightarrow \qquad k^2 - 2\left(\frac{n}{k}\right)^2 = -1.$$

Conversely, *any* solution of $x^2 - 2y^2 = \pm 1$ describes a number which is both a triangular number and a perfect square since

$$\begin{aligned} x^2 - 2y^2 = \pm 1 \qquad &\Longleftrightarrow \qquad 2y^2 = x^2 \mp 1 \\ &\Longleftrightarrow \qquad (xy)^2 = \frac{x^2(x^2 \mp 1)}{2} = \begin{cases} \sum_{i=1}^{x^2-1} i & \text{if } x^2 - 2y^2 = 1, \\ \sum_{i=1}^{x^2} i & \text{if } x^2 - 2y^2 = -1. \end{cases} \end{aligned}$$

□

Theorem 40 ([2, exercise 7.8.6])**.** *A perfect square* m^2 *is the sum of two consecutive perfect squares* $n^2 + (n+1)^2$ *if and only if* $(2n+1, m)$ *solves* $x^2 - 2y^2 = -1$.

Proof. We have

$$2m^2 = 2(n^2 + (n+1)^2) = 4n^2 + 4n + 2 = (2n+1)^2 + 1 \qquad \Longleftrightarrow \qquad (2n+1)^2 - 2m^2 = -1.$$

Conversely, *any* solution of $x^2 - 2y^2 = -1$ describes a perfect square which is the sum of two consecutive perfect squares as x^2 must be odd, so x is odd as well, i.e. there exists an n such that $x = 2n+1$. □

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A. Euler-Muir polynomials

These are the Euler-Muir polynomials $f_d(n)$ for *primitive* $d \leq 400$, i.e. for d minimal such that the cont'd fraction representation of $\sqrt{d}$ contains this palindrome. For generating the content of a LaTeX table we can use the following Python code referencing the functions from 23.3 and 23.4:

```
primitives = {}
for d in range(2, 401):
  try:
    palindrome = continued_fraction_sqrt_palindrome(d)
    if palindrome not in [P for D, P in primitives.items()]:
      primitives[d] = palindrome
  except: # fails for perfect squares
    pass
euler_muir_poly = {}
for d in primitives:
  palindrome = primitives[d]
  euler_muir_poly[d] = (palindrome, euler_muir(palindrome))
table_latex = ''
for d in euler_muir_poly:
  palindrome, coeff = euler_muir_poly[d]
  spaced_coeff = ['{:,}'.format(w).replace(',', '\\,') for w in coeff] # separate thousands
  poly_latex = f'{spaced_coeff[0] if coeff[0] != 1 else ""}n^2'
  if coeff[1] > 0:
    poly_latex += f'+{spaced_coeff[1] if coeff[1] != 1 else ""}n'
  elif coeff[1] < 0:
    poly_latex += f'{spaced_coeff[1] if coeff[1] != 1 else ""}n'
  if coeff[2] > 0:
    poly_latex += f'+{spaced_coeff[2]}'
  table_latex += f'${d}$ & ${str(palindrome).replace("[", "(").replace("]", ")")}$ & ${poly_latex}$ \\\\\n'
table_latex = table_latex.rstrip(' \\\\\n')
with open('em_primitive.tex', 'w') as f:
  print(table_latex, file=f)
```

d	palindrome of $\sqrt{d}$	$f_d(n)$
2	()	$n^2 + 1$
3	(1)	$n^2 + 2n$
6	(2)	$n^2 + 3n + 2$
7	(1, 1, 1)	$9n^2 - 2n$
11	(3)	$9n^2 + 2n$
13	(1, 1, 1, 1)	$25n^2 - 14n + 2$
14	(1, 2, 1)	$4n^2 + 7n + 3$
18	(4)	$4n^2 + 9n + 5$
19	(2, 1, 3, 1, 2)	$1\,521n^2 - 2\,702n + 1\,200$
21	(1, 1, 2, 1, 1)	$36n^2 - 17n + 2$
22	(1, 2, 4, 2, 1)	$441n^2 - 685n + 266$
23	(1, 3, 1)	$25n^2 - 2n$
27	(5)	$25n^2 + 2n$
28	(3, 2, 3)	$144n^2 - 161n + 45$
29	(2, 1, 1, 2)	$169n^2 - 198n + 58$
31	(1, 1, 3, 5, 3, 1, 1)	$74\,529n^2 - 146\,018n + 71\,520$
34	(1, 4, 1)	$9n^2 + 17n + 8$
38	(6)	$9n^2 + 19n + 10$
41	(2, 2)	$25n^2 + 14n + 2$
43	(1, 1, 3, 1, 5, 1, 3, 1, 1)	$281\,961n^2 - 556\,958n + 275\,040$
44	(1, 1, 1, 2, 1, 1, 1)	$225n^2 - 251n + 70$
45	(1, 2, 2, 2, 1)	$144n^2 - 127n + 28$
46	(1, 3, 1, 1, 2, 6, 2, 1, 1, 3, 1)	$3\,218\,436n^2 - 6\,412\,537n + 3\,194\,147$
47	(1, 5, 1)	$49n^2 - 2n$
51	(7)	$49n^2 + 2n$
52	(4, 1, 2, 1, 4)	$2\,025n^2 - 3\,401n + 1\,428$
53	(3, 1, 1, 3)	$625n^2 - 886n + 314$
54	(2, 1, 6, 1, 2)	$1\,089n^2 - 1\,693n + 658$
55	(2, 2, 2)	$36n^2 + 17n + 2$
57	(1, 1, 4, 1, 1)	$100n^2 - 49n + 6$
58	(1, 1, 1, 1, 1, 1)	$169n^2 - 140n + 29$
59	(1, 2, 7, 2, 1)	$4\,761n^2 - 8\,462n + 3\,760$
61	(1, 4, 3, 1, 2, 2, 1, 3, 4, 1)	$14\,478\,025n^2 - 28\,896\,614n + 14\,418\,650$
62	(1, 6, 1)	$16n^2 + 31n + 15$
66	(8)	$16n^2 + 33n + 17$
67	(5, 2, 1, 1, 7, 1, 1, 2, 5)	$35\,605\,089n^2 - 71\,112\,494n + 35\,507\,472$
69	(3, 3, 1, 4, 1, 3, 3)	$219\,024n^2 - 430\,273n + 211\,318$
70	(2, 1, 2, 1, 2)	$225n^2 - 199n + 44$
71	(2, 2, 1, 7, 1, 2, 2)	$170\,569n^2 - 334\,178n + 163\,680$
73	(1, 1, 5, 5, 1, 1)	$15\,625n^2 - 29\,114n + 13\,562$
76	(1, 2, 1, 1, 5, 4, 5, 1, 1, 2, 1)	$10\,989\,225n^2 - 21\,920\,651n + 10\,931\,502$
77	(1, 3, 2, 3, 1)	$400n^2 - 449n + 126$
79	(1, 7, 1)	$81n^2 - 2n$
83	(9)	$81n^2 + 2n$
85	(4, 1, 1, 4)	$1\,681n^2 - 2\,606n + 1\,010$
86	(3, 1, 1, 1, 8, 1, 1, 1, 3)	$314\,721n^2 - 619\,037n + 304\,402$
88	(2, 1, 1, 1, 2)	$441n^2 - 488n + 135$
89	(2, 3, 3, 2)	$2\,809n^2 - 4\,618n + 1\,898$
91	(1, 1, 5, 1, 5, 1, 1)	$27\,225n^2 - 51\,302n + 24\,168$
92	(1, 1, 2, 4, 2, 1, 1)	$3\,600n^2 - 6\,049n + 2\,541$
93	(1, 1, 1, 4, 6, 4, 1, 1, 1)	$396\,900n^2 - 781\,649n + 384\,842$
94	(1, 2, 3, 1, 1, 5, 1, 8, 1, 5, 1, 1, 3, 2, 1)	$12\,217\,323\,024n^2 - 24\,432\,502\,753n + 12\,215\,179\,823$
97	(1, 5, 1, 1, 1, 1, 1, 1, 5, 1)	$323\,761n^2 - 636\,314n + 312\,650$
98	(1, 8, 1)	$25n^2 + 49n + 24$
102	(10)	$25n^2 + 51n + 26$
103	(6, 1, 2, 1, 1, 9, 1, 1, 2, 1, 6)	$502\,611\,561n^2 - 1\,004\,768\,066n + 502\,156\,608$

d	palindrome of $\sqrt{d}$	$f_d(n)$
106	(3, 2, 1, 1, 1, 1, 2, 3)	$151\,321n^2 - 294\,632n + 143\,417$
107	(2, 1, 9, 1, 2)	$8\,649n^2 - 15\,374n + 6\,832$
108	(2, 1, 1, 4, 1, 1, 2)	$4\,225n^2 - 7\,099n + 2\,982$
109	(2, 3, 1, 2, 4, 1, 6, 6, 1, 4, 2, 1, 3, 2)	$725\,094\,825\,625n^2 - 1\,450\,171\,870\,886n + 725\,077\,045\,370$
111	(1, 1, 6, 1, 1)	$196n^2 - 97n + 12$
113	(1, 1, 1, 2, 2, 1, 1, 1)	$5\,329n^2 - 9\,106n + 3\,890$
114	(1, 2, 10, 2, 1)	$2\,304n^2 - 3\,583n + 1\,393$
115	(1, 2, 1, 1, 1, 1, 1, 2, 1)	$11\,025n^2 - 19\,798n + 8\,888$
116	(1, 3, 2, 1, 4, 1, 2, 3, 1)	$207\,025n^2 - 404\,249n + 197\,340$
117	(1, 4, 2, 4, 1)	$900n^2 - 1\,151n + 368$
118	(1, 6, 3, 2, 10, 2, 3, 6, 1)	$199\,572\,129n^2 - 398\,837\,341n + 199\,265\,330$
119	(1, 9, 1)	$121n^2 - 2n$
123	(11)	$121n^2 + 2n$
124	(7, 2, 1, 1, 1, 3, 1, 4, 1, 3, 1, 1, 1, 2, 7)	$43\,047\,950\,400n^2 - 86\,091\,280\,001n + 43\,043\,329\,725$
125	(5, 1, 1, 5)	$3\,721n^2 - 6\,078n + 2\,482$
126	(4, 2, 4)	$400n^2 - 351n + 77$
127	(3, 1, 2, 2, 7, 11, 7, 2, 2, 1, 3)	$176\,211\,050\,625n^2 - 352\,412\,640\,002n + 176\,201\,589\,504$
128	(3, 5, 3)	$2\,601n^2 - 4\,048n + 1\,575$
129	(2, 1, 3, 1, 6, 1, 3, 1, 2)	$550\,564n^2 - 1\,084\,273n + 533\,838$
131	(2, 4, 11, 4, 2)	$859\,329n^2 - 1\,697\,438n + 838\,240$
133	(1, 1, 7, 5, 1, 1, 1, 2, 1, 1, 1, 5, 7, 1, 1)	$12\,595\,572\,900n^2 - 25\,188\,557\,201n + 12\,592\,984\,434$
134	(1, 1, 2, 1, 3, 1, 10, 1, 3, 1, 2, 1, 1)	$39\,727\,809n^2 - 79\,309\,693n + 39\,582\,018$
135	(1, 1, 1, 1, 1, 1, 1)	$441n^2 - 394n + 88$
137	(1, 2, 2, 1, 1, 2, 2, 1)	$22\,201n^2 - 40\,914n + 18\,850$
139	(1, 3, 1, 3, 7, 1, 1, 2, 11, 2, 1, 1, 7, 3, 1, 3, 1)	$43\,280\,991\,011\,241n^2 - 86\,561\,826\,895\,982n + 43\,280\,835\,884\,880$
142	(1, 10, 1)	$36n^2 + 71n + 35$
146	(12)	$36n^2 + 73n + 37$
149	(4, 1, 5, 3, 3, 5, 1, 4)	$86\,583\,025n^2 - 172\,938\,886n + 86\,356\,010$
151	(3, 2, 7, 1, 3, 4, 1, 1, 1, 11, 1, 1, 1, 4, 3, 1, 7, 2, 3)	$19\,778\,116\,875\,204\,249n^2 - 39\,556\,230\,294\,112\,418n + 19\,778\,113\,418\,908\,320$
153	(2, 1, 2, 2, 2, 1, 2)	$7\,744n^2 - 13\,311n + 5\,720$
154	(2, 2, 3, 1, 2, 1, 3, 2, 2)	$736\,164n^2 - 1\,451\,033n + 715\,023$
155	(2, 4, 2)	$100n^2 + 49n + 6$
157	(1, 1, 7, 1, 5, 2, 1, 1, 1, 1, 2, 5, 1, 7, 1, 1)	$148\,722\,066\,025n^2 - 297\,434\,467\,814n + 148\,712\,401\,946$
158	(1, 1, 3, 12, 3, 1, 1)	$94\,864n^2 - 181\,985n + 87\,279$
159	(1, 1, 1, 1, 3, 1, 1, 1, 1)	$11\,025n^2 - 19\,402n + 8\,536$
160	(1, 1, 1, 5, 1, 1, 1)	$3\,249n^2 - 5\,056n + 1\,967$
161	(1, 2, 4, 1, 2, 1, 4, 2, 1)	$215\,296n^2 - 418\,817n + 203\,682$
162	(1, 2, 1, 2, 12, 2, 1, 2, 1)	$592\,900n^2 - 1\,166\,199n + 573\,461$
163	(1, 3, 3, 2, 1, 1, 7, 1, 11, 1, 7, 1, 1, 2, 3, 3, 1)	$25\,191\,716\,148\,225n^2 - 50\,383\,304\,136\,398n + 25\,191\,587\,988\,336$
164	(1, 4, 6, 4, 1)	$6\,400n^2 - 10\,751n + 4\,515$
165	(1, 5, 2, 5, 1)	$1\,764n^2 - 2\,449n + 850$
166	(1, 7, 1, 1, 1, 2, 4, 1, 3, 2, 12, 2, 3, 1, 4, 2, 1, 1, 1, 7, 1)	$4\,357\,032\,433\,168\,041n^2 - 8\,714\,063\,165\,433\,517n + 4\,357\,030\,732\,265\,642$
167	(1, 11, 1)	$169n^2 - 2n$
171	(13)	$169n^2 + 2n$
172	(8, 1, 2, 2, 1, 1, 3, 6, 3, 1, 1, 2, 2, 1, 8)	$854\,646\,629\,841n^2 - 1\,709\,269\,011\,035n + 854\,622\,381\,366$
173	(6, 1, 1, 6)	$7\,225n^2 - 12\,214n + 5\,162$
174	(5, 4, 5)	$3\,025n^2 - 4\,599n + 1\,748$
175	(4, 2, 1, 2, 4)	$23\,409n^2 - 42\,770n + 19\,536$
176	(3, 1, 3)	$225n^2 - 52n + 3$
177	(3, 3, 2, 8, 2, 3, 3)	$5\,503\,716n^2 - 10\,945\,009n + 5\,441\,470$
178	(2, 1, 12, 1, 2)	$3\,600n^2 - 5\,599n + 2\,177$
179	(2, 1, 1, 1, 3, 5, 13, 5, 3, 1, 1, 1, 2)	$98\,088\,602\,481n^2 - 196\,168\,824\,542n + 98\,080\,222\,240$
181	(2, 4, 1, 8, 6, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 6, 8, 1, 4, 2)	$6\,822\,224\,927\,691\,121n^2 - 13\,644\,447\,632\,930\,702n + 6\,822\,222\,705\,239\,762$
183	(1, 1, 8, 1, 1)	$324n^2 - 161n + 20$
184	(1, 1, 3, 2, 1, 2, 1, 2, 3, 1, 1)	$804\,609n^2 - 1\,584\,883n + 780\,458$
186	(1, 1, 1, 3, 4, 3, 1, 1, 1)	$75\,625n^2 - 143\,749n + 68\,310$

d	palindrome of $\sqrt{d}$	$f_d(n)$
187	(1, 2, 13, 2, 1)	$15\,129n^2 - 26\,894n + 11\,952$
188	(1, 2, 2, 6, 2, 2, 1)	$28\,224n^2 - 51\,841n + 23\,805$
190	(1, 3, 1, 1, 1, 2, 2, 2, 1, 1, 1, 3, 1)	$3\,560\,769n^2 - 7\,069\,517n + 3\,508\,938$
191	(1, 4, 1, 1, 3, 2, 2, 13, 2, 2, 3, 1, 1, 4, 1)	$423\,518\,513\,089n^2 - 847\,019\,038\,178n + 423\,500\,525\,280$
193	(1, 8, 3, 2, 1, 3, 3, 1, 2, 3, 8, 1)	$16\,125\,190\,225n^2 - 32\,246\,852\,186n + 16\,121\,662\,154$
194	(1, 12, 1)	$49n^2 + 97n + 48$
198	(14)	$49n^2 + 99n + 50$
199	(9, 2, 1, 2, 2, 5, 4, 1, 1, 13, 1, 1, 4, 5, 2, 2, 1, 2, 9)	$1\,329\,593\,714\,709\,849\,801n^2 - 2\,659\,187\,396\,887\,306\,562n + 1\,329\,593\,682\,177\,456\,960$
201	(5, 1, 1, 1, 2, 1, 8, 1, 2, 1, 1, 1, 5)	$330\,003\,556n^2 - 659\,492\,017n + 329\,488\,662$
202	(4, 1, 2, 2, 1, 4)	$48\,841n^2 - 91\,400n + 42\,761$
204	(3, 1, 1, 6, 1, 1, 3)	$30\,625n^2 - 56\,251n + 25\,830$
205	(3, 6, 1, 4, 1, 6, 3)	$1\,920\,996n^2 - 3\,802\,303n + 1\,881\,512$
206	(2, 1, 5, 14, 5, 1, 2)	$4\,301\,476n^2 - 8\,543\,417n + 4\,242\,147$
207	(2, 1, 1, 2, 1, 1, 2)	$1\,600n^2 - 2\,049n + 656$
208	(2, 2, 1, 2, 2)	$2\,025n^2 - 2\,752n + 935$
209	(2, 5, 3, 2, 3, 5, 2)	$2\,592\,100n^2 - 5\,137\,649n + 2\,545\,758$
211	(1, 1, 9, 5, 1, 2, 2, 1, 1, 4, 3, 1, 13, 1, 3, 4, 1, 1, 2, 2, 1, 5, 9, 1, 1)	$367\,209\,276\,445\,894\,854\,609n^2 - 734\,418\,552\,335\,080\,961\,918n + 367\,209\,275\,889\,186\,107\,520$
212	(1, 1, 3, 1, 1, 1, 6, 1, 1, 1, 3, 1, 1)	$5\,175\,625n^2 - 10\,285\,001n + 5\,109\,588$
213	(1, 1, 2, 6, 1, 8, 1, 6, 2, 1, 1)	$44\,355\,600n^2 - 88\,516\,801n + 44\,161\,414$
214	(1, 1, 1, 2, 3, 1, 4, 9, 1, 1, 5, 3, 14, 3, 5, 1, 1, 9, 4, 1, 3, 2, 1, 1, 1)	$564\,864\,956\,791\,065\,679\,329n^2 - 1\,129\,729\,912\,886\,772\,168\,733n + 564\,864\,956\,095\,706\,489\,618$
216	(1, 2, 3, 2, 1)	$1\,089n^2 - 1\,208n + 335$
217	(1, 2, 1, 2, 1, 1, 9, 4, 9, 1, 1, 2, 1, 2, 1)	$17\,023\,986\,576n^2 - 34\,044\,129\,089n + 17\,020\,142\,730$
218	(1, 3, 3, 1)	$289n^2 - 76n + 5$
221	(1, 6, 2, 6, 1)	$3\,136n^2 - 4\,607n + 1\,692$
223	(1, 13, 1)	$225n^2 - 2n$
227	(15)	$225n^2 + 2n$
229	(7, 1, 1, 7)	$12\,769n^2 - 22\,118n + 9\,578$
232	(4, 3, 7, 3, 4)	$1\,656\,369n^2 - 3\,273\,532n + 1\,617\,395$
233	(3, 1, 3, 1, 1, 1, 1, 3, 1, 3)	$2\,301\,289n^2 - 4\,556\,266n + 2\,255\,210$
234	(3, 2, 1, 2, 1, 2, 3)	$28\,900n^2 - 52\,599n + 23\,933$
236	(2, 1, 3, 5, 1, 6, 1, 5, 3, 1, 2)	$334\,341\,225n^2 - 668\,120\,651n + 333\,779\,662$
237	(2, 1, 1, 7, 10, 7, 1, 1, 2)	$54\,908\,100n^2 - 109\,588\,049n + 54\,680\,186$
238	(2, 2, 1, 14, 1, 2, 2)	$142\,884n^2 - 274\,105n + 131\,459$
239	(2, 5, 1, 2, 4, 15, 4, 2, 1, 5, 2)	$160\,583\,731\,441n^2 - 321\,155\,072\,642n + 160\,571\,341\,440$
241	(1, 1, 9, 1, 5, 3, 3, 1, 1, 3, 3, 5, 1, 9, 1, 1)	$20\,923\,534\,350\,625n^2 - 41\,846\,926\,679\,114n + 20\,923\,392\,328\,730$
242	(1, 1, 3, 1, 14, 1, 3, 1, 1)	$396\,900n^2 - 774\,199n + 377\,541$
243	(1, 1, 2, 3, 15, 3, 2, 1, 1)	$20\,295\,025n^2 - 40\,449\,598n + 20\,154\,816$
244	(1, 1, 1, 1, 1, 2, 1, 5, 1, 1, 9, 1, 6, 1, 9, 1, 1, 5, 1, 2, 1, 1, 1, 1, 1)	$3\,196\,601\,416\,865\,025n^2 - 6\,393\,201\,067\,411\,001n + 3\,196\,599\,650\,546\,220$
245	(1, 1, 1, 7, 6, 7, 1, 1, 1)	$2\,742\,336n^2 - 5\,432\,831n + 2\,690\,740$
246	(1, 2, 5, 1, 14, 1, 5, 2, 1)	$8\,014\,561n^2 - 15\,940\,317n + 7\,926\,002$
247	(1, 2, 1, 1, 9, 1, 9, 1, 1, 2, 1)	$29\,452\,329n^2 - 58\,734\,074n + 29\,281\,992$
249	(1, 3, 1, 1, 5, 1, 3, 10, 3, 1, 5, 1, 1, 3, 1)	$73\,461\,597\,444n^2 - 146\,914\,641\,073n + 73\,453\,043\,878$
250	(1, 4, 3, 3, 4, 1)	$78\,961n^2 - 149\,036n + 70\,325$
251	(1, 5, 2, 1, 2, 2, 15, 2, 2, 1, 2, 5, 1)	$53\,804\,049\,849n^2 - 107\,600\,749\,918n + 53\,796\,700\,320$
253	(1, 9, 1, 1, 1, 2, 1, 7, 4, 2, 2, 2, 4, 7, 1, 2, 1, 1, 1, 9, 1)	$10\,262\,117\,490\,452\,100n^2 - 20\,524\,231\,758\,286\,801n + 10\,262\,114\,267\,834\,954$
254	(1, 14, 1)	$64n^2 + 127n + 63$
258	(16)	$64n^2 + 129n + 65$
259	(10, 1, 2, 3, 4, 3, 2, 1, 10)	$692\,847\,684n^2 - 1\,384\,848\,143n + 692\,000\,718$
261	(6, 2, 3, 7, 1, 3, 1, 2, 1, 3, 1, 7, 3, 2, 6)	$35\,354\,202\,483\,600n^2 - 70\,708\,212\,847\,999n + 35\,354\,010\,364\,660$
262	(5, 2, 1, 2, 1, 10, 16, 10, 1, 2, 1, 2, 5)	$10\,516\,134\,493\,881n^2 - 21\,032\,164\,007\,245n + 10\,516\,029\,513\,626$
263	(4, 1, 1, 1, 1, 15, 1, 1, 1, 1, 4)	$73\,599\,241n^2 - 146\,920\,226n + 73\,321\,248$
265	(3, 1, 1, 2, 2, 1, 1, 3)	$139\,129n^2 - 266\,114n + 127\,250$
266	(3, 4, 3)	$441n^2 - 197n + 22$
267	(2, 1, 15, 1, 2)	$21\,609n^2 - 38\,414n + 17\,072$
268	(2, 1, 2, 3, 3, 1, 3, 1, 10, 8, 10, 1, 3, 1, 3, 3, 2, 1, 2)	$21\,234\,349\,584\,091\,449n^2 - 42\,468\,694\,397\,100\,971n + 21\,234\,344\,813\,009\,790$
270	(2, 3, 6, 3, 2)	$25\,921n^2 - 46\,551n + 20\,900$

d	palindrome of $\sqrt{d}$	$f_d(n)$
271	(2, 6, 10, 1, 4, 1, 1, 2, 1, 2, 1, 15, 1, 2, 1, 2, 1, 1, 4, 1, 10, 6, 2)	$49\,631\,722\,586\,790\,660\,369n^2 - 99\,263\,444\,941\,631\,353\,538n + 49\,631\,722\,354\,840\,693\,440$
273	(1, 1, 10, 1, 1)	$484n^2 - 241n + 30$
274	(1, 1, 4, 4, 1, 1)	$7\,225n^2 - 11\,636n + 4\,685$
276	(1, 1, 1, 1, 2, 2, 2, 1, 1, 1, 1)	$54\,756n^2 - 101\,737n + 47\,257$
277	(1, 1, 1, 4, 10, 1, 7, 2, 2, 3, 3, 2, 2, 7, 1, 10, 4, 1, 1, 1)	$287\,274\,501\,442\,203\,025n^2 - 574\,548\,985\,043\,437\,814n + 287\,274\,483\,601\,235\,066$
278	(1, 2, 16, 2, 1)	$5\,625n^2 - 8\,749n + 3\,402$
279	(1, 2, 2, 1, 2, 2, 1)	$8\,281n^2 - 13\,522n + 5\,520$
280	(1, 2, 1, 2, 1)	$225n^2 + 52n + 3$
281	(1, 3, 4, 1, 1, 6, 6, 1, 1, 4, 3, 1)	$4\,025\,268\,025n^2 - 8\,048\,408\,986n + 4\,023\,141\,242$
282	(1, 3, 1, 4, 1, 3, 1)	$4\,900n^2 - 7\,449n + 2\,831$
283	(1, 4, 1, 1, 1, 3, 10, 1, 15, 1, 10, 3, 1, 1, 1, 4, 1)	$67\,560\,854\,250\,681n^2 - 135\,121\,431\,953\,198n + 67\,560\,577\,702\,800$
284	(1, 5, 1, 3, 2, 1, 4, 8, 4, 1, 2, 3, 1, 5, 1)	$516\,414\,704\,400n^2 - 1\,032\,805\,188\,001n + 516\,390\,483\,885$
285	(1, 7, 2, 7, 1)	$5\,184n^2 - 7\,937n + 3\,038$
286	(1, 10, 3, 3, 2, 3, 3, 10, 1)	$275\,925\,321n^2 - 551\,288\,807n + 275\,363\,772$
287	(1, 15, 1)	$289n^2 - 2n$
291	(17)	$289n^2 + 2n$
292	(11, 2, 1, 3, 8, 3, 1, 2, 11)	$4\,455\,562\,500n^2 - 8\,908\,843\,751n + 4\,453\,281\,543$
293	(8, 1, 1, 8)	$21\,025n^2 - 37\,086n + 16\,354$
294	(6, 1, 4, 1, 6)	$19\,600n^2 - 34\,399n + 15\,093$
295	(5, 1, 2, 3, 2, 6, 2, 3, 2, 1, 5)	$3\,475\,102\,500n^2 - 6\,948\,180\,001n + 3\,473\,077\,796$
296	(4, 1, 7, 1, 4)	$46\,225n^2 - 85\,052n + 39\,123$
297	(4, 3, 1, 1, 2, 1, 1, 3, 4)	$1\,988\,100n^2 - 3\,927\,601n + 1\,939\,798$
298	(3, 1, 4, 5, 1, 1, 5, 4, 1, 3)	$562\,875\,625n^2 - 1\,124\,932\,136n + 562\,056\,809$
300	(3, 8, 3)	$1\,521n^2 - 1\,691n + 470$
301	(2, 1, 6, 3, 1, 2, 2, 1, 1, 8, 11, 2, 4, 2, 11, 8, 1, 1, 2, 2, 1, 3, 6, 1, 2)	$28\,749\,425\,043\,692\,036\,541\,456n^2 - 57\,498\,850\,081\,500\,680\,545\,217n + 28\,749\,425\,037\,808\,644\,004\,062$
302	(2, 1, 1, 1, 4, 2, 1, 16, 1, 2, 4, 1, 1, 1, 2)	$15\,140\,318\,116n^2 - 30\,276\,359\,609n + 15\,136\,041\,795$
303	(2, 2, 5, 2, 2)	$21\,025n^2 - 37\,002n + 16\,280$
304	(2, 3, 2, 1, 1, 1, 1, 1, 2, 3, 2)	$10\,989\,225n^2 - 21\,862\,852n + 10\,873\,931$
305	(2, 6, 2)	$196n^2 + 97n + 12$
307	(1, 1, 11, 5, 1, 3, 17, 3, 1, 5, 11, 1, 1)	$25\,529\,100\,232\,689n^2 - 51\,058\,023\,406\,814n + 25\,528\,923\,174\,432$
309	(1, 1, 2, 1, 2, 4, 1, 1, 1, 8, 6, 1, 10, 1, 6, 8, 1, 1, 1, 4, 2, 1, 2, 1, 1)	$3\,334\,943\,334\,131\,329\,284n^2 - 6\,669\,886\,604\,059\,933\,073n + 3\,334\,943\,269\,928\,604\,098$
310	(1, 1, 1, 1, 5, 3, 1, 2, 1, 3, 5, 1, 1, 1, 1)	$580\,906\,404n^2 - 1\,160\,964\,089n + 580\,057\,995$
311	(1, 1, 1, 2, 1, 6, 3, 17, 3, 6, 1, 2, 1, 1, 1)	$916\,609\,015\,609n^2 - 1\,833\,184\,263\,458n + 916\,575\,248\,160$
313	(1, 2, 4, 11, 1, 1, 3, 2, 2, 3, 1, 1, 11, 4, 2, 1)	$51\,418\,723\,369\,225n^2 - 102\,837\,193\,013\,714n + 51\,418\,469\,644\,802$
314	(1, 2, 1, 1, 2, 1)	$625n^2 - 364n + 53$
316	(1, 3, 2, 8, 2, 3, 1)	$129\,600n^2 - 246\,401n + 117\,117$
317	(1, 4, 8, 1, 2, 2, 1, 8, 4, 1)	$392\,238\,025n^2 - 783\,770\,814n + 391\,533\,106$
319	(1, 6, 5, 1, 4, 3, 1, 3, 4, 1, 5, 6, 1)	$521\,805\,414\,321n^2 - 1\,043\,585\,025\,082n + 521\,779\,611\,080$
322	(1, 16, 1)	$81n^2 + 161n + 80$
326	(18)	$81n^2 + 163n + 82$
329	(7, 4, 2, 1, 1, 4, 1, 1, 2, 4, 7)	$4\,291\,298\,064n^2 - 8\,580\,219\,713n + 4\,288\,921\,978$
331	(5, 5, 1, 6, 2, 3, 1, 1, 2, 1, 2, 1, 11, 2, 1, 1, 17, 1, 1, 2, 11, 1, 2, 1, 2, 1, 1, 3, 2, 6, 1, 5, 5)	$23\,442\,630\,035\,977\,813\,320\,534\,892\,329n^2 - 46\,885\,260\,071\,950\,055\,461\,466\,896\,718n + 23\,442\,630\,035\,972\,242\,140\,932\,004\,720$
332	(4, 1, 1, 8, 1, 1, 4)	$136\,161n^2 - 258\,875n + 123\,046$
334	(3, 1, 1, 1, 2, 5, 1, 2, 2, 11, 1, 3, 7, 18, 7, 3, 1, 11, 2, 2, 1, 5, 2, 1, 1, 1, 3)	$3\,047\,154\,270\,780\,318\,840\,142\,884n^2 - 6\,094\,308\,541\,496\,833\,306\,566\,073n + 3\,047\,154\,270\,716\,514\,466\,423\,523$
335	(3, 3, 3)	$1\,089n^2 - 970n + 216$
337	(2, 1, 3, 1, 11, 2, 4, 1, 3, 3, 1, 4, 2, 11, 1, 3, 1, 2)	$3\,062\,033\,164\,880\,881n^2 - 6\,124\,064\,298\,107\,090n + 3\,062\,031\,133\,226\,546$
339	(2, 2, 2, 1, 17, 1, 2, 2, 2)	$28\,313\,041n^2 - 56\,430\,142n + 28\,117\,440$
340	(2, 3, 1, 1, 1, 1, 8, 1, 1, 1, 1, 3, 2)	$60\,047\,001n^2 - 119\,808\,233n + 59\,761\,572$
341	(2, 6, 1, 8, 2, 1, 2, 1, 2, 8, 1, 6, 2)	$82\,788\,552\,900n^2 - 165\,566\,479\,249n + 82\,777\,926\,690$
343	(1, 1, 11, 1, 5, 3, 1, 17, 1, 3, 5, 1, 11, 1, 1)	$49\,708\,972\,110\,681n^2 - 99\,417\,683\,068\,706n + 49\,708\,710\,958\,368$
344	(1, 1, 4, 1, 3, 1, 4, 1, 1)	$314\,721n^2 - 608\,632n + 294\,255$
345	(1, 1, 2, 1, 6, 1, 2, 1, 1)	$33\,124n^2 - 59\,487n + 26\,708$
347	(1, 1, 1, 2, 4, 1, 17, 1, 4, 2, 1, 1, 1)	$1\,186\,320\,249n^2 - 2\,371\,357\,294n + 1\,185\,037\,392$
348	(1, 1, 1, 8, 1, 1, 1)	$1\,764n^2 - 1\,961n + 545$
349	(1, 2, 7, 7, 2, 1)	$243\,049n^2 - 467\,678n + 224\,978$
351	(1, 2, 1, 3, 2, 2, 2, 3, 1, 2, 1)	$2\,775\,556n^2 - 5\,488\,687n + 2\,713\,482$

d	palindrome of $\sqrt{d}$	$f_d(n)$
352	(1, 3, 5, 9, 5, 3, 1)	$17\,114\,769n^2 - 34\,074\,304n + 16\,959\,887$
353	(1, 3, 1, 2, 1, 1, 1, 1, 1, 1, 2, 1, 3, 1)	$14\,386\,849n^2 - 28\,631\,170n + 14\,244\,674$
354	(1, 4, 2, 2, 18, 2, 2, 4, 1)	$47\,032\,164n^2 - 93\,806\,263n + 46\,774\,453$
355	(1, 5, 3, 3, 1, 6, 1, 3, 3, 5, 1)	$642\,014\,244n^2 - 1\,283\,073\,679n + 641\,059\,790$
356	(1, 6, 1, 1, 2, 1, 8, 1, 2, 1, 1, 6, 1)	$175\,562\,500n^2 - 350\,624\,999n + 175\,062\,855$
357	(1, 8, 2, 8, 1)	$8\,100n^2 - 12\,799n + 5\,056$
358	(1, 11, 1, 1, 1, 3, 1, 1, 4, 1, 5, 2, 18, 2, 5, 1, 4, 1, 1, 3, 1, 1, 1, 11, 1)	$21\,774\,041\,770\,465\,247\,769n^2 - 43\,548\,083\,364\,350\,689\,741n + 21\,774\,041\,593\,885\,442\,330$
359	(1, 17, 1)	$361n^2 - 2n$
363	(19)	$361n^2 + 2n$
364	(12, 1, 2, 3, 1, 8, 1, 3, 2, 1, 12)	$16\,862\,321\,025n^2 - 33\,719\,687\,099n + 16\,857\,366\,438$
365	(9, 1, 1, 9)	$32\,761n^2 - 58\,606n + 26\,210$
366	(7, 1, 1, 1, 2, 12, 2, 1, 1, 1, 7)	$563\,065\,441n^2 - 1\,125\,222\,957n + 562\,157\,882$
367	(6, 2, 1, 3, 1, 1, 2, 1, 12, 19, 12, 1, 2, 1, 1, 3, 1, 2, 6)	$985\,722\,701\,380\,761\,969n^2 - 1\,971\,445\,364\,721\,532\,802n + 985\,722\,663\,340\,771\,200$
368	(5, 2, 5)	$900n^2 - 649n + 117$
369	(4, 1, 3, 2, 7, 4, 7, 2, 3, 1, 4)	$47\,768\,473\,600n^2 - 95\,528\,550\,399n + 47\,760\,077\,168$
370	(4, 4)	$289n^2 + 76n + 5$
371	(3, 1, 4, 1, 3)	$1\,936n^2 - 2\,177n + 612$
372	(3, 2, 12, 2, 3)	$99\,225n^2 - 186\,299n + 87\,446$
373	(3, 5, 5, 3)	$70\,225n^2 - 130\,214n + 60\,362$
374	(2, 1, 18, 1, 2)	$7\,569n^2 - 11\,773n + 4\,578$
375	(2, 1, 2, 1, 5, 1, 2, 1, 2)	$609\,961n^2 - 1\,189\,674n + 580\,088$
376	(2, 1, 1, 3, 1, 2, 2, 4, 2, 2, 1, 3, 1, 1, 2)	$3\,054\,330\,756n^2 - 6\,106\,518\,217n + 3\,052\,187\,837$
378	(2, 3, 1, 4, 1, 3, 2)	$50\,625n^2 - 92\,501n + 42\,254$
379	(2, 7, 3, 2, 2, 6, 12, 1, 4, 1, 1, 1, 3, 4, 19, 4, 3, 1, 1, 1, 4, 1, 12, 6, 2, 2, 3, 7, 2)	$441\,885\,449\,870\,527\,916\,227\,060\,542\,681n^2 - 883\,770\,899\,741\,029\,950\,059\,680\,003\,982n + 441\,885\,449\,870\,502\,033\,832\,619\,461\,680$
381	(1, 1, 12, 1, 1)	$676n^2 - 337n + 42$
382	(1, 1, 5, 12, 1, 5, 1, 1, 2, 3, 1, 18, 1, 3, 2, 1, 1, 5, 1, 12, 5, 1, 1)	$17\,817\,071\,467\,345\,290\,000n^2 - 35\,634\,142\,769\,692\,140\,001n + 17\,817\,071\,302\,346\,850\,383$
383	(1, 1, 3, 19, 3, 1, 1)	$919\,681n^2 - 1\,801\,826n + 882\,528$
384	(1, 1, 2, 9, 2, 1, 1)	$60\,025n^2 - 110\,448n + 50\,807$
385	(1, 1, 1, 1, 1, 3, 1, 2, 1, 3, 1, 1, 1, 1, 1)	$5\,963\,364n^2 - 11\,830\,897n + 5\,867\,918$
386	(1, 1, 1, 4, 1, 18, 1, 4, 1, 1, 1)	$8\,059\,921n^2 - 16\,008\,287n + 7\,948\,752$
387	(1, 2, 19, 2, 1)	$31\,329n^2 - 55\,694n + 24\,752$
388	(1, 2, 3, 4, 12, 1, 8, 1, 12, 4, 3, 2, 1)	$2\,541\,913\,658\,244n^2 - 5\,083\,764\,506\,855n + 2\,541\,850\,848\,999$
389	(1, 2, 1, 1, 1, 1, 2, 1)	$4\,225n^2 - 5\,886n + 2\,050$
391	(1, 3, 2, 2, 1, 1, 2, 19, 2, 1, 1, 2, 2, 3, 1)	$137\,739\,703\,689n^2 - 275\,464\,730\,018n + 137\,725\,026\,720$
393	(1, 4, 1, 2, 4, 1, 1, 1, 1, 12, 1, 1, 1, 1, 4, 2, 1, 4, 1)	$1\,371\,760\,973\,284n^2 - 2\,743\,475\,509\,425n + 1\,371\,714\,536\,534$
394	(1, 5, 1, 1, 1, 3, 1, 3, 5, 2, 2, 5, 3, 1, 3, 1, 1, 1, 5, 1)	$396\,048\,726\,346\,729n^2 - 792\,096\,662\,647\,388n + 396\,047\,936\,301\,053$
397	(1, 12, 3, 4, 9, 1, 2, 1, 2, 1, 1, 2, 1, 2, 1, 9, 4, 3, 12, 1)	$1\,056\,324\,667\,563\,199\,225n^2 - 2\,112\,649\,294\,169\,792\,486n + 1\,056\,324\,626\,606\,593\,658$
398	(1, 18, 1)	$100n^2 + 199n + 99$

Finally, we list the non-primitive $d \leq 400$ and their representation $f_D(n)$ where D is primitive.

5	$f_2(2)$	30	$f_6(4)$	50	$f_2(7)$	78	$f_{34}(2)$	101	$f_2(10)$	136	$f_7(4)$	152	$f_{11}(4)$	195	$f_3(13)$	224	$f_3(14)$	255	$f_3(15)$	299	$f_{28}(2)$	325	$f_2(18)$	350	$f_{45}(2)$	399	$f_3(19)$
8	$f_3(2)$	32	$f_7(2)$	56	$f_6(6)$	80	$f_3(8)$	104	$f_{27}(2)$	138	$f_{14}(5)$	156	$f_6(11)$	197	$f_2(14)$	226	$f_2(15)$	257	$f_2(16)$	306	$f_6(16)$	327	$f_{146}(2)$	360	$f_3(18)$		
10	$f_2(3)$	33	$f_{14}(2)$	60	$f_{14}(3)$	82	$f_2(9)$	105	$f_{18}(4)$	140	$f_{34}(3)$	168	$f_3(12)$	200	$f_{51}(2)$	228	$f_{102}(2)$	260	$f_{66}(3)$	308	$f_{57}(2)$	328	$f_{83}(2)$	362	$f_2(19)$		
12	$f_6(2)$	35	$f_3(5)$	63	$f_3(7)$	84	$f_{38}(2)$	110	$f_6(9)$	141	$f_{62}(2)$	170	$f_2(13)$	203	$f_{18}(6)$	230	$f_{38}(4)$	264	$f_{18}(7)$	312	$f_7(6)$	330	$f_{38}(5)$	377	$f_{55}(3)$		
15	$f_3(3)$	37	$f_2(6)$	65	$f_2(8)$	87	$f_{11}(3)$	112	$f_{21}(2)$	143	$f_3(11)$	180	$f_{55}(2)$	210	$f_6(13)$	231	$f_{27}(3)$	269	$f_{41}(3)$	315	$f_{14}(8)$	333	$f_{18}(8)$	380	$f_6(18)$		
17	$f_2(4)$	39	$f_{18}(2)$	68	$f_{18}(3)$	90	$f_6(8)$	120	$f_3(10)$	145	$f_2(12)$	182	$f_6(12)$	215	$f_7(5)$	235	$f_{11}(5)$	272	$f_6(15)$	318	$f_{34}(5)$	336	$f_{11}(6)$	390	$f_{14}(9)$		
20	$f_6(3)$	40	$f_{11}(2)$	72	$f_6(7)$	95	$f_{14}(4)$	122	$f_2(11)$	147	$f_{66}(2)$	185	$f_{13}(3)$	219	$f_{23}(3)$	240	$f_6(14)$	275	$f_{21}(3)$	320	$f_{79}(2)$	338	$f_{29}(2)$	392	$f_{23}(4)$		
24	$f_3(4)$	42	$f_6(5)$	74	$f_{13}(2)$	96	$f_{23}(2)$	130	$f_{41}(2)$	148	$f_{38}(3)$	189	$f_{14}(6)$	220	$f_{34}(4)$	248	$f_{14}(7)$	288	$f_3(16)$	321	$f_{142}(2)$	342	$f_6(17)$	395	$f_{62}(4)$		
26	$f_2(5)$	48	$f_3(6)$	75	$f_7(3)$	99	$f_3(9)$	132	$f_6(10)$	150	$f_{18}(5)$	192	$f_{47}(2)$	222	$f_{98}(2)$	252	$f_{62}(3)$	290	$f_2(17)$	323	$f_3(17)$	346	$f_{13}(4)$	396	$f_{98}(3)$		

The Python code generating this LaTeX table is a bit more complex due to the column layout:

```
non_primitives = {}
for d in euler_muir_poly:
  def f_d(n):
    _, coeff = euler_muir_poly[d]
    return coeff[0] * n ** 2 + coeff[1] * n + coeff[2]
  n = 2
  while f_d(n) <= 400:
    non_primitives[f_d(n)] = (d, n)
    n += 1
non_primitives = dict(sorted(non_primitives.items(), key=lambda x: x[0])) # sort
import math
np_item_list = list(non_primitives.items())
N = len(np_item_list)
cols = 14
rows = math.ceil(N / cols)
table_latex = f'\\begin{{tabular}}{{*{{{cols}}}{{|c|c|}}}}\n\\hline\n'
for r in range(0, rows):
  for c in range(0, cols):
    i = r + rows * c
    if i < N:
      np_item = np_item_list[i]
      d, D, n = np_item[0], np_item[1][0], np_item[1][1]
      table_latex += f'${d}$ & $f_{{{D}}}({n})$'
      if c < cols - 1 and r + rows * (c + 1) < N:
        table_latex += ' & '
  table_latex += ' \\\\\n'
  if r == rows - 1:
    if rows * cols == N:
      table_latex += '\\hline\n'
    else:
      table_latex += f'\\cline{{1-{2 * cols - 2}}}\n'
  if i == N - 1 and rows * cols != N:
    table_latex += f'\\cline{{{2 * cols - 1}-{2 * cols}}}\n'
table_latex += '\\end{tabular}'
with open('em_other.tex', 'w') as f:
  print(table_latex, file=f)
```