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Pseudoholomorphic Curves

in Hamiltonian Dynamics

The theory of pseudoholomorphic curves is used in many places in symplectic topology. There are several flavours depending on the applications one has in mind. In this workshop we focus on holomorphic planes as covered in [FK] & [AH] and thus work towards Symplectic Field Theory (SFT), but the techniques do not differ too much from considering closed holomorphic curves/ Gromov-Witten theory [McDS12] or Hamiltonian perturbations/ Floer theory [S99].

1. First Part – Foundations

We split the first part of the workshop into 9 topics. For every topic there will be a 90 min slot, that should partly be filled by a talk prepared by one of the participants and partly by discussing questions and ideally working through some examples and details. Since none of us are experts you are not expected to be able to answer all questions about your topic, but rather guide the other members of your small group through the topic and stimulate the discussion. If possible it would be nice to include examples and questions to discuss. We will follow the book by Urs Frauenfelder and Otto van Koert [FK]. As additional references, the book by Casim Abbas and Helmut Hofer [AH] and the lecture notes by Chris Wendl [W16] will be useful.

Topic 1: Pseudoholomorphic planes
Introduce: almost complex structures, holomorphic curves/planes, Hofer-energy. Discuss also the asymptotics of finite energy planes. This is covered in [FK, ch.13.1-13.4]. You can in addition look at [W16, lec. 2] and [AH, ch.4].

Topic 2: Indices
In order to determine dimensions of the moduli spaces we will need to understand indices. This is [FK, ch. 10]. Have a second look at [FS18, ch.2] to select, which results are important.

Topic 3: Spectral flow
First of all we will study a one-to-one correspondence between paths of linear symplectic matrices starting at the identity and paths of symmetric matrices. We will make use of this to define a linear operator and prove that this operator is a Fredholm operator. Afterward we will introduce the so-called spectral flow. We will finish with a discussion of winding numbers of eigenvalues. Cover the results of [FK, ch.11].

Topic 4: Asymptotic formula
We would like to understand the asymptotic behavior of pseudoholomorphic half-cylinders which converge exponentially to a Reeb orbit. In the second part of the talk we will associate to the asymptotic orbit a winding number and will use this number to introduce fast finite energy planes. Cover [FK, ch.13.5-13.6] for details have also a look at [S08] and [Hy12].

Topic 5: Intersection theory
We study the intersection theory of punctured pseudoholomorphic curves in 4- dimensional symplectic cobordisms. Using the asymptotic results from topic 4, we first study the local intersection properties of such curves at the punctures. Finally we will introduce Siefring's intersection number. Cover [FK, ch.14.1-14.3] for details have also a look at [S11, ch.3].

Topic 6: Intersection theory
We will continue with Siefring's inequality, which gives topological controls on the intersection number of two curves. After some computations we will discuss some applications of Siefring's intersection theory to fast finite energy planes. Cover [FK, ch.14.4-14.5] for details have also a look at [S11, ch.4].

Topic 7: Moduli space of fast finite energy planes
In order to determine the dimension of the Moduli space, we have to discuss first Cauchy Riemann operators and their Fredholm indices. Furthermore we will prove an equality betwenn the first Chern number and the difference of two Conley Zehnder indices; which we will use to define the Normal Conley Zehnder index. Cover [FK, ch.15.1-15.3] for details have look in the references there in.

Topic 8: Moduli space of fast finite energy planes
The main ingredient to prove that a Moduli space is a smooth manifold is philosophically always an implicit function theorem. Afterwards we will determine the dimension and will construct some local charts of the moduli space of fast finite energy planes. Finally we will achieve some automatic transversality results. Cover [FK, ch.15.4-15.7], for details also have a look in [Hy12].

Topic 9: SFT compactness
This is a first step into the topic of SFT compactness. For the sake of this, we will discuss weak SFT-compactness in the special case of fast finite energy planes. Cover [FK, ch.16].

2. Second Part – Favourite Papers

In the second part of the workshop we want to discuss papers that participants find interesting or want to understand better. The papers should relate in some way to

pseudoholomorphic curves and Hamiltonian dynamics, but this does not necessarily have to mean SFT. With the background we have worked through we should for example also be able to understand topics in Floer or Gromov-Witten theory. Here is a list of papers to find some inspiration.

- F. Bourgeois & A. Oancea: Symplectic homology, autonomous Hamiltonians, and Morse-Bott moduli spaces [BO09.1]
- F. Bourgeois & A. Oancea: An exact sequence for contact- and symplectic homology [BO09.2]
- B. Bramham: Pseudo-rotations with sufficiently Liouvillean rotation number are C^0-rigid [B15.1]
- B. Bramham: Periodic approximations of irrational pseudo-rotations using pseudoholomorphic curves [B15.2]
- K. Cieliebak, U. Frauenfelder & A. Oancea: Rabinowitz Floer homology and symplectic homology [CFO10]
- A. Cioba, & C. Wendl, Unknotted Reeb orbits and nicely embedded holomorphic curves [CW20]
- U. Frauenfelder & O. Van Koert: Contact geometry and holomorphic curves in the restricted three-body problem. See for the planar case [FK]. Maybe you also have a look on the recent survey paper [M22].
- U. Frauenfelder & A. Moreno: On GIT quotients of the symplectic group, stability and bifurcations of symmetric orbits [FM21].
- U. Hryniewicz: Fast finite-energy planes in symplectizations and applications [Hy12].
- U. Hryniewicz, & P.A.S. Salomão: P.A.S. On the existence of disk-like global sections for Reeb flows on the tight 3-sphere [HyS11].
- K. Niederkrüger, & C. Wendl: Weak symplectic fillings and holomorphic curves. [NW11]
- C. Wendl, Lectures on symplectic field theory. Just look at the cool stuff we didn't yet discuss. [W16]

References

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[CW20] Cioba, A., & Wendl, C. (2020). Unknotted Reeb orbits and nicely embedded holomorphic curves. Journal of Symplectic Geometry, 18(1), 57-122.

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[HyS11] Hryniewicz, U. Salomão, P.A.S. On the existence of disk-like global sections for Reeb flows on the tight 3-sphere. Duke Mathematical Journal, 2011

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