Geometric Quantization

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"Quantization is an art, not a functor." – Folklore

Organization of the seminar

When: Thursdays at 2:00 pm sharp (First talk on 15.4.)

Where: Online

Language: English

Presentation: Participants will give a 90-minutes talk (including 10 minutes of time for questions)

Online talk: Write on a tablet in *real time* (preferred option) or prepare slides using Latex beamer - if you need help with this, ask us. We will also reserve a room in the Mathematikon if the speaker wants to give the talk from there.

Evaluation: Give a presentation, write notes or slides of the talk that will be uploaded to the homepage of the seminar, actively participate during the seminar talks.

Meet us: 1 or 2 weeks before your talk to discuss your plan and to clarify questions.

Please contact the respective organiser of the talk via mail.

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List of Topics

Topic 1: The Mathematical Model of Classical Mechanics (Davide)

We introduce symplectic manifolds (M, ω), the natural setting where classical Hamiltonian systems induced by a smooth function $H : M \to \mathbb{R}$ can be defined. Beyond symplectic vector spaces, the main examples we will consider are cotangent bundles T^*Q of a configuration manifold Q and Kähler manifolds such as S^2 and, more in general, $\mathbb{CP}^n$. The symplectic structure induces a Poisson bracket on the space of observables $C^\infty(M)$ satisfying crucial algebraic properties and determining the dynamics of Hamiltonian systems.

Guidelines:

- State Hamilton equations in $\mathbb{R}^{2n}$, write them via the standard symplectic form. (Section 18.2 in [8], Section 0.4 in [5]).
- Define symplectic manifolds (M, ω) and Hamiltonian systems in general. Give the example of cotangent bundles, Kähler manifolds (in particular S^2 and $\mathbb{CP}^n$) and state Darboux theorem. (Chapter 2 in [5], Chapter 3 in [26], Section 1.7 in [31], Section 1.3 in [17]).
- Introduce the notion of symplectic and Kähler potentials. Observe that ω^n yields a volume form and that no symplectic potential exists if M is closed. [3]
- Give an introduction to linear symplectic spaces (V, ω): example of $W \times W^*$, definition of symplectic orthogonal, isotropic, coisotropic and Lagrangian subspaces, existence of a symplectic basis. Definition of linear symplectic group $\mathrm{Sp}(2n; \mathbb{R})$. (Chaper 1 in [5], Section 1.3 in [31], Section 1.1 in [17])
- Back to symplectic manifolds: definition of symplectomorphisms, Hamiltonian maps are symplectomorphisms (proof using Cartan's magic formula). Introduce the Poisson bracket $\{\cdot, \cdot\}$ and the time evolution of an observable f: $\dot{f} = \{f, H\}$. List the algebraic properties of the bracket. (Chapter 3 in [5], Chapter 1 and 18 in [8], Section 1.8 [31], Section 1.3 [17]).

Topic 2: Symmetries in Classical Mechanics (Davide)

Group actions on (M, ω) by symplectomorphisms describe symmetries of the phase space. If the action comes from a so-called moment map and preserves the Hamiltonian, then Noether's theorem yields the existence of conserved quantities for the system. Fixing the value of the conserved quantities and identifying points by the group action yields a reduced symplectic manifold $(\bar{M}, \bar{\omega})$ where the reduced Hamiltonian system is easier to study.

Guidelines:

- Introduce Lie-groups and their Lie-algebras. Examples of matrix groups. Briefly infinite dimensional examples: diffeomorphisms and vector fields on a manifold. (Chapters 7 and 8 in [24], Chapter 1 in [16], mention [32] for the infinite dimensional case). Introduce the Lie-algebras of Symplectic and Hamiltonian vector fields with Lie bracket, and of Hamiltonian functions with Poisson bracket. Lie-algebras homomorphism from Hamiltonian functions to Hamiltonian vector fields. (Chapter 3 in [5], Section III.5 in [26]).

- Introduce Lie-groups actions (Chapter 7 in [24]). Describe the coadjoint action and the Lie-Poisson structure on the dual of the Lie-algebra $\mathfrak{g}^*$. (Section 2.5 in [5], Chapters 21-22 in [8], Chapter 1 in [21]). Define Poisson manifolds and show that they are foliated by symplectic manifolds. Observe that for $\mathfrak{g}^*$ the leaves are the coadjoint orbits. (Chapter 1 in [10], Chapters 1, 2 and 7 in [42], Chapter III in [26]).

- Introduce symplectic and Hamiltonian actions, define the moment map. Examples: lift of the action from the base for cotangent bundles (e.g. linear and angular momentum) and coadjoint action. Observe that the moment map is a G-equivariant Poisson morphism. (Chapter 4 in [5], Chapter 22 in [8], Chapter IV in [26], Chapter 4 in [34]).

- State Noether's theorem. Sketch the Weinstein–Marsden symplectic reduction. Examples: linear/angular momentum, complex projective spaces, and S^1-actions on cotangent bundles. (Chapter 23-24 in [8], Chapter 4 in [26], Chapter 6 in [34]).

Topic 3: The Mathematical Model of Quantum Mechanics (Gabriele)

Following Dirac and Schrödinger, observables in quantum mechanics are self-adjoint operators on a Hilbert space of quantum states and their time evolution is given by unitary one-parameter groups. A quantization procedure is then a recipe to associate a quantum observable to a classical one. We identify a list of properties that this recipe should follow, basically yielding an irreducible, self-adjoint representation of a Lie-subalgebra of classical observables on the Hilbert space. Finally, we give some examples of such representations and illustrate some difficulties of the program (no-go theorems to be discussed in detail in Talk 10) which geometric quantization (introduced in Talk 4) will try to solve.

Guidelines:

- Define Hilbert spaces and their projectivizations. Introduce unitary operators (respectively transformations) as automorphisms of the Hilbert space (respectively isomorphisms between Hilbert spaces). Describe strongly continuous, one-parameter unitary groups by self-adjoint operators using Stone's theorem. Give a definition of self-adjointness without dwelving into details (Chapter 3 in [23], Section 10.2 in [15], Sections 2.1-2.3 in [28]).

- Examples: $L^2(X, \mu)$ with multiplication operators (observe that every self-adjoint operator is unitarily equivalent to a multiplication operator) and $L^2(\mathbb{R}^n, \mathrm{d}x^n)$ with the translation operator.

- Write Schrödinger's equation and use it to motivate the definition of quantum observables H as self-adjoint operators. Derive the time evolution of an observable f: $\dot{f} = [f, H]$ and make the parallelism to $\dot{f} = \{f, H\}$ in classical mechanics (Chapter 2 in [11], Sections 3.7-3.8 in [15]).

- State the Dirac "axioms" for quantum mechanics and quantization as an irreducible, selfadjoint representation of a Lie-subalgebra of observables. If the Lie-subalgebra comes from a group of symmetries of the symplectic manifold, then quantization reduces to the theory of irreducible unitary representations of Lie-groups which will be studied in Talk 11. [11]

- Example: Heisenberg algebra, Heisenberg group and representation (Section 5.2 [5]).

- Example (time-permitting): oscillator algebra as extension of the Heisenberg algebra. [40, page 217-218].

- No-go theorem of Groenewold. Problems: how to decide which observables can be quantized? How can we quantize observable for general symplectic manifolds in a coordinate-free way? Geometric quantization will try to answer these questions. (Section 13.4 in [15]).

Topic 4 & 5: Prequantization & Hermitian Line Bundle (Johanna)
The task at hand is to associate a Hilbert space $\mathcal{H}$ to a given symplectic manifold (M, ω) and find a map assigning to each $f \in C^\infty(M, \mathbb{R})$ a self-adjoint operator O_f of $\mathcal{H}$ that satisfies Dirac's wish list. The prequantum Hilbert space will be the space of square integrable sections in a suitable Hermitian line bundle over M. We will therefore look at the general theory of Hermitian line bundles. In the end we will notice that the Hilbert space we constructed is too 'big' in the sense that the representation of a complete set of observables is not irreducible.

Guidelines:

- Look at the local situation, i.e. consider $(M, \omega) = (\mathbb{R}^{2n}, \mathrm{d}p \wedge \mathrm{d}q)$. Observe that the prequantum operator is given by $O_f := i\hbar\nabla_{X_f} + f$ where ∇ denotes a Hermitian connection of the trivial Hermitian line bundle $\mathbb{C} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $\mathrm{d}p \wedge \mathrm{d}q$ is the associated curvature form. (chapter 22.2 in [15]).

- Introduce complex line bundles, define hermitian structures, connections and the curvature form.

- For a general symplectic manifold we therefore need to answer the question: Is ω the curvature of some Hermitian line bundle? The answer is the Bohr-Sommerfeld quantization condition

$$\omega \text{ is a curvature form } \iff \frac{1}{2\pi\hbar}\int_S \omega \in \mathbb{Z}.$$

  (chapter 8.3 [44]).

- We can see this by proving a canonical isomorphism between the Picard group $L(M)$ of complex line bundles over M and the second cohomology group $H^2(M,\mathbb{Z})$. (chapter 3.1 & 3.2 in [11]).

- The isomorphism tells us how to canonically associate a complex line bundle to a quantizable symplectic manifold, but in order to define the prequantum operators we need to choose a Hermitian structure and connection. This choice can be parametrized (up to equivalence) by $H^1(M,S^1)$ (chapter 3.6 in [11]), i.e. a circle-valued one-form.

- Define the pre-quantum Hilbert space as the space of square-integrable sections in the Hermitian line bundle and give an expression for the pre-quantum operators. Show that the operators are indeed self-adjoint and that the commutation relation is satisfied. (chapter 23.3 [15]).

- As an example for pre-quantization look at the harmonic oscillator. Observe that our quantization procedure fails, as it produces the wrong spectrum in this case. (chapter 22.3 [15])

- Explore the problems with the prequantization further (chapter 22.3 [15]) and see in the example that the representation is not irreducible. (Prop. 22.13 in [15]).

Topic 6: Lagrangian Distributions and Polarizations (Johanna)
The prequantum Hilbert space is considered to be too 'big'. We therefore need to reduce the number of variables the states depend on from $2n$ to n by choosing a polarization. That is an integrable Lagrangian distribution of $TM^{\mathbb{C}}$ and we define the polarized prequantum Hilbert space to consist of sections of L that are covariantly constant along the polarization. When choosing a polarization we also restrict the set of quantizable classical observables as they need to preserve the polarization. Different choices of polarization therefore might lead to different representations.

Guidelines:

- Recall why we need polarization and why this means choosing a certain type of distribution (chapter 5.1 in [11]).

- Introduce complex, Lagrangian, integrable distributions and state Frobenius theorem. Continue explaining two important classes of polarizations: Kähler and real polarizations (chapter 5.2 in [11]).

- Introduce polarized sections of the Hermitian line bundle and argue that these are in general not square integrable and that this can not be solved by just integrating over the space of integral manifolds as we don't have a measure there (chapter 5.3.1 in [11]).

- If time permits show that these problems do not occur for Kähler polarizations (chapter 5.3.2 in [11]). As an example one could discuss the harmonic oscillator (Prop. 22.14 in [15]).

- Explain why the choice of a polarization also restricts the set of quantizable classical observables and therefore determines the representation (chapter 5.4 in [11]).

- Look at examples of polarizations (chapter 5.5 in [11]).

Topic 7: Half-form correction for real polarizations (Gabriele)

If P is a real polarization with non-compact leaves, polarized sections will not be square-integrable with respect to the Liouville measure and the quantum Hilbert space is trivial. The way out is to integrate the sections on the quotient $\Xi := M/P$ instead that on M. To do that we twist the hermitian line bundle L by the square root of the complexified canonical bundle of the polarization $\delta_P^{\mathbb{C}}$ (sections of $\delta_P^{\mathbb{C}}$ are so-called half-forms, namely square roots of complex-valued n-forms on Q) and consider the quantizable observables acting on the polarized sections of $L \otimes \delta_P^{\mathbb{C}}$ which has a natural structure of Hilbert space.

Guidelines:

- Define the square root of a line bundle L as a pair (K, i) where K is a line bundle and $i : K^2 \to L$ is an isomorphism [36]. Recall that the first Chern class yields an isomorphism between the Picard group and $H^2(M; \mathbb{Z})$ [15, 23.6.3] and [44, Appendix A]. Use this to discuss the existence and uniqueness of square roots. A Hermitian line bundle (L, ∇, h) induces a Hermitian structure and connection on its square root [15, Proposition 23.41].

- Define the complexified canonical bundle of a real polarization when Ξ is an orientable manifold and its square root $\delta_P^{\mathbb{C}}$ [15, 23.6.1-3].

- Define the quantum Hilbert space of square-integrable polarized sections of $L \otimes \delta_P^{\mathbb{C}}$ [15, 23.6.4].

- Define the quantizable observables and show that they yield symmetric operators of the quantum Hilbert space [15, 23.6.5].

- Compute the action of the quantum observables explicitly in the case of cotangent bundles [15, Example 23.45, 23.48]. For $\mathbb{R}^{2n}$ compare it with the action for the quantization without half-forms [15, Formula (22.15)].

Topic 8: Half-form correction for Kähler polarizations and metalinear structures (Gabriele)

We carry out the half-form correction for Kähler polarizations using the square root of the canonical bundle $\mathcal{K}$. As an application we compute the correct spectrum of the harmonic oscillator. The existence of the square root is related to metalinear structures on the Kähler manifold which arise from the metalinear group $Ml(n, \mathbb{C})$.

Guidelines:

- Define the canonical bundle $\mathcal{K}_P$ of a complex polarization P. If time permits give the example of $\mathbb{CP}^n$ with computation of the Chern class [30, Theorem 14.10].
- Define the metalinear/metaplectic group and metalinear/metaplectic structures on M [36]. Relate them to the square roots δ_P of the canonical bundle.
- Define the Hilbert product structure on the polarized sections of $L \otimes \delta_P$ and the action of the observables [15, Section 23.7], [44, Section 10.4].
- Look at the example of $\mathbb{C}^n$ [15, Section 23.7] and, time permitting, of D^2 and S^2 [15, Example 23.30] [7, Section 6.4].

Topic 9: Pairings of polarizations and the BKS construction (Gabriele)

From what we have seen, geometric quantization is far from being canonic. One obvious reason is the choice of a polarization that determines the subalgebra of quantizable observables. In this talk we try to relate the quantum systems obtained from two different polarizations contructing a pairing between them. Already for the simplest symplectic manifold $\mathbb{R}^{2n}$ we get many interesting examples: (i) the Fourier transform is the pairing between the momentum and position polarization; (ii) the Segal–Bargmann transform is the pairing between the position and the standard Kähler polarization; (iii) pairings between different Kähler polarizations become unitary after applying the half-form correction and give rise to a representation of the metaplectic group on the quantum Hilbert space. Pairings can also be used to quantize observables which do not preserve the polarization via the BKS construction. Applying this idea on cotangent bundles, the kinetic energy is quantized to the classical Schrödinger operator $-\Delta + \frac{1}{6}\mathrm{scal}$.

Guidelines:

- Describe the pairing construction for two compatible polarizations. Mention that the pairing map might not be unitary up to a constant [38, Chapter 5] and [15, Section 23.8].

- Apply the pairing construction for two real polarizations. In $\mathbb{R}^{2n}$ for the position and momentum polarizations we get the Fourier transform [44, Section 9.5] and [Carosso].

- Apply the pairing construction for a real and a Kähler polarization. In $\mathbb{R}^{2n}$ for the position and complex polarization we get the Segal–Bargmann transform [44, Section 9.5]

- Describe the properties of pairings between two Kähler polarizations on a symplectic vector space $(V, \omega_V) \cong (\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ [44, Section 9.9]. Define the projective representation of the symplectic group on the quantum Hilbert space. Show that it comes from a unitary representation of the metaplectic group [44, Section 10.2].

- Define the BKS construction and explain how this can be used to extend the space of quantizable observables. Show that the operator corresponding to the kinetic energy in $\mathbb{R}^{2n}$ is (up to a constant) the Laplacian [Carosso], [44, Section 9.7], [38, Section 6.3, 7.1-2].

Topic 10: Groenewold-Van Hove Problem for $\mathbb{R}^{2n}$ (Steffen)
In this talk we investigate the Groenewold-Van Hove problem for $\mathbb{R}^{2n}$ following mainly [13] which yields to a weak and a strong no-go theorem in mathematical physics. Our starting point is the space $\mathcal{P}(2n)$ of polynomials on $\mathbb{R}^{2n}$ equipped with the Poisson bracket $\{\cdot,\cdot\}$ which contains the Heisenberg algebra $\mathfrak{h}_n$, the symplectic algebra $\mathfrak{sp}(2n,\mathbb{R})$ as well as the extended symplectic algebra $\mathfrak{hsp}(2n,\mathbb{R})$. We are then ready to define a general concept of quantization of subalgebras of $\mathcal{P}(2n)$ showing the importance of the unitary dual of the Heisenberg group (Schrödinger representations). Starting from construction of the Schrödinger representation we will show that there exists an (extended metaplectic) quantization of the expanded symplectic algebra $\mathfrak{hsp}(2n,\mathbb{R})$ which cannot be expanded beyond by von Neumann rules. This is called the *weak No-Go theorem*. Finally, since any quantization of $\mathfrak{hsp}(2n,\mathbb{R})$ is unitary equivalent to the extended metaplectic quantization the strong no-go-theorem follows, i.e. there exists no quantization of $\mathcal{P}(2n)$.
Guidelines:

- Recall the definitions of the Heisenberg group and Heisenberg algebra (cf. [12, Section 1.2]) and explain their physical significance.

- Classify the unitary irreducible representations of the Heisenberg group. Therefore introduce the Schrödinger representations of the Heisenberg group (see [12, Section 1.3]) and state the Stone-Von Neumann Theorem (see [12, Section 1.5]) without proof.

- Define the concept of qunatization given in [13] and relate it to the concept of geometric quantization. Explain furthermore the relation to the Schrödinger representation and the derived Schrödinger representation.

- State and proof the weak no-go theorem starting from introducing the extended metaplectic quantization, its uniqueness and the Von Neumann rules.

- State and proof finally the strong no-go theorem by following the path given in Section 4 in [13]. Prove in the process only the very important results coming from functional analysis.

Topic 11: Unitary Representations via The Orbit Method (Steffen)
As an application of geometric quantization we want to consider Kirillov's orbit method in the case of connected, simply connected, nilpotent Lie groups as the Heisenberg group. The method was historically proposed by Kirillov in [19] to describe the unitary dual $\widehat{G}$ of a general reductive Lie group and indeed geometric quantization has its origins in this method. Kirillov's orbit method is a heuristic method in representation theory establishing a correspondence between unitary representations of certain Lie groups and geometric objects called coadjoint orbits. In this sense the method tries to complete the circle: we understand geometric objects with group actions in terms of representations, and we understand representations in terms of geometric objects with group actions. The goal of this talk is to illustrate the connection between geometric quantization and the orbit method and give a complete description of the unitary dual $\widehat{G}$ of a simply connected nilpotent Lie group G. As an example we will consider the Heisenberg group $\mathrm{Heis}(\mathbb{R})$ and the (non-nilpotent) Lie group $SU(2)$.
Guidelines:

- We start our journey with the geometry of coadjoint orbits. Introduce coadjoint orbits and show that these orbits have a symplectic structure ([21, Section 1.1,1.2] and [21, Section 2.1]). Give as examples the coadjoint orbits of $\mathrm{SL}(2, \mathbb{R}), \mathrm{Heis}(\mathbb{R})$ and $\mathrm{SU}(2)$. Moreover, prove Theorem 1.1 in [20, Section 1.3].

- Explain both views of the orbit method: geometric quantization and induced representations (cf. [27, Section 6], [20, Section 1.3], [21]). Relate both points of view (cf. [18, Section 7]).

- Apply the orbit method to the Heisenberg group and the compact group $\mathrm{SU}(2)$. State for which types of groups the orbit method works ([21]).

- Try finally to explain from a mathematical and physical point of view why the orbit method works and finally the merits and demerits of the orbit method ([21], Introduction). Moreover, if there is still time give an outlook to the general compact group case.

Topic 12: Conformal Quantum Mechanics (Steffen)
The conformal group of a $1 + 0$ dimensional quantum field theory is given by the Lie group $\mathrm{SL}(2, \mathbb{R})$. Therefore, we can use representations of $\mathrm{SL}(2, \mathbb{R})$ to describe the conformal symmetries of such systems. We start our journey by describing the (unitary) representation theory of $\mathrm{SL}(2, \mathbb{R})$. The irreducible unitary representations of $\mathrm{SL}(2, \mathbb{R})$,

up to unitary equivalence, are the discrete series $\mathscr{D}_n^+$ $(n \geq 2)$, limits of discrete series representation $\mathscr{D}_1^{\pm}$, principal series $\mathscr{P}^{+,iy}$ for $y \in \mathbb{R}$ and $\mathscr{P}^{,iy}$ for $y \in \mathbb{R} \setminus \{0\}$ such as the complementary series $\mathscr{C}^u$ for $0 < u < 1$. We will use this fact to study the DFF model, a conformal invariant quantum mechanical model introduced by de Alfaro, Furlan and Fubini in 1976. The Hamiltonian of this model is given by

$$H = \frac{p^2}{2m} + \frac{g^2}{2x^2}.$$

We will use the irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$ and show in the DFF model that for all values of the coupling constant g one can find the relevant quantum mechanical system exhibiting exact $\mathrm{SL}(2,\mathbb{R})$ conformal symmetry. Its Hilbert space of states spans an irreducible unitary representation of the $\mathrm{SL}(2,\mathbb{R})$ group (or its universal cover). We finally indicate how the resulting theory emerges form the geometric quantization of the Hamiltonian dynamics on the relevant coadjoint orbits. *Guidelines:*

- Define conformal transformations and the conformal group of the semi-Riemannian manifold $\mathbb{R}^{p,q}$ ([37, Section 1.2]), but only sketch the idea of conformal Killing vector fields and conformal compactification. Now restrict to the case $d = 1 + 0$ and show that the conformal group is given by $\mathrm{SL}(2,\mathbb{R})$.
- State Bargmann's Theorem (cf. Theorem 16.3 in [22]) of the unitary dual of $\mathrm{SL}(2,\mathbb{R})$. Explain shortly each type of unitary irreducible representation in a specific realization ([22, Section 2.5]).
- Explain the coadjoint geometry of $\mathrm{SL}(2,\mathbb{R})$ and relate the coadjoint orbits to representations of $\mathrm{SL}(2,\mathbb{R})$ ((([6, Section 8.3]) or [35]).
- As a toy model of conformal quantum mechanics we look at the DFF model named after de Alfaro, Fubini and Furlan (see [9]). Construct for the DFF model fully conformally invariant quantum mechanics for all values of the coupling constant following [2]. Here its enough to explain the discrete series case in full detail.

What to do now?

Get acquainted with the general idea behind geometric quantization: Have a quick look to the summary of all talks above and read the introduction to [Baykara].

Study the references of your talk and clarify questions: Ask us if you need help, more references or copies of books. In the guidelines of the talks we suggested some of the references in the list. However, useful information about every topic can be found in almost all the items in the reference list so please have a look at several places.

Make an appointement with the person tutoring your talk: 1 or 2 weeks before your talk to discuss your plan of the talk and to clarify open questions

What to do after the seminar?

There is a Summer School *Deformation quantization and convergence* at the University of Freiburg and online. The mini-courses look extremely interesting. All information can be found at http://home.mathematik.uni-freiburg.de/GEOQUANT2021/school

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Further readings

- S. Twareque Ali, Miroslav Englis, Quantization Methods: A Guide for Physicists and Analysts, preprint available at https://arxiv.org/pdf/math-ph/0405065.pdf

- John Baez, Geometric Quantization, a short pedagogical introduction available at https://math.ucr.edu/home/baez/quantization.html and a nice blog entry https://johncarlosbaez.wordpress.com/2018/12/01/geometric-quantization-part-1

- Zihni Kaan Baykara, Geometric Quantization, preprint available online at the website http://math.uchicago.edu/ may/REU2019/REUPapers/Baykara.pdf

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Geometric quantization is not the only possible quantization procedure. For other important approaches that we do not pursue in this seminar, you can consult

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