Nicomachus' Theorem

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November 8, 2022

Abstract

We prove the identity $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$, which is attributed to the antique philosopher and mathematician Nicomachus of Gerasa (c. 60 – c . 120 BCE).

1 Preliminaries

It is assumed that the reader is familiar with the concept of *proof by induction*. See for example [Knu]. Furthermore, we will need the following formulæ, each of which can be easily shown by induction. They hold for all $n \in \mathbb{N}$, respectively all $M, N \in \mathbb{N}$ with $M < N$.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \,. \tag{1}$$

$$\sum_{k=1}^{n} 2k - 1 = n^2 \,. \tag{2}$$

$$\sum_{k=M+1}^{N} f(k) = \sum_{k=1}^{N} f(k) - \sum_{k=1}^{M} f(k) \,. \tag{3}$$

$$\sum_{k=M+1}^{N} f(k) = \sum_{k=1}^{N-M} f(k+M) \,. \tag{4}$$

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2 Main result

Lemma 1. *For each $n \in \mathbb{N}$ we have*

$$S(n) := \sum_{n(n-1)/2+1}^{n(n+1)/2} 2k - 1 = n^3 .$$

Proof. We apply equation (4) to $S(n)$ with

$$N = \frac{n(n+1)}{2} \quad \text{and} \quad M = \frac{n(n-1)}{2} . \tag{5}$$

We then have $N - M = n$, and therefore

$$\begin{aligned} S(n) &= \sum_{k=1}^{n} 2(k+M) = \sum_{k=1}^{n} \big(2k - 1 + n(n-1)\big) \\ &= \left(\sum_{k=1}^{n} 2k - 1\right) + \left(\sum_{k=1}^{n} n(n-1)\right) \\ &= \left(\sum_{k=1}^{n} 2k - 1\right) + n \cdot n(n-1) . \end{aligned}$$

Now, by (2), the first summand equals n^2, and the second $n^3 - n^2$. Therefore, we obtain

$$S(n) = n^2 + n^3 - n^2 = n^3 . \qquad \square$$

By using (3) and the (5) we can write $S(n)$ as a telescope sum:

$$S(n) = \left(\sum_{k=1}^{n(n+1)/2} 2k - 1\right) - \left(\sum_{k=1}^{n(n-1)} 2k - 1\right) . \tag{6}$$

Now we can easily prove:

Lemma 2. *We have*

$$\sum_{j=1}^{n} S(j) = \sum_{k=1}^{n(n+1)/2} 2k - 1 .$$

Proof. We prove the statement by induction over n. For $n = 1$ the assertion is trivially true. For $n + 1$ we can apply the induction hypothesis as follows:

$$\sum_{j=1}^{n+1} S(j) = S(n+1) + \sum_{j=1}^{n} S(j) = S(n+1) + \sum_{k=1}^{n(n+1)/2} 2k - 1 .$$

Using (6) on $S(n+1)$ we obtain

$$\sum_{j=1}^{n+1} S(j) = \left(\sum_{k=1}^{(n+1)(n+2)/2} 2k-1\right) - \left(\sum_{k=1}^{n(n+1)/2} 2k-1\right) + \left(\sum_{k=1}^{n(n+1)/2} 2k-1\right)$$
$$= \sum_{k=1}^{(n+1)(n+2)/2)} 2k-1\,. \qquad \square$$

We can now prove the main result:

Theorem (Nicomachus)**.** *For all $n \in \mathbb{N}$ we have*

$$\sum_{j=1}^{n} j^3 = \left(\sum_{j=1}^{n} j\right)^2 .$$

Proof. We only need to gather up everything we have got so far:

$$\begin{aligned} \sum_{j=1}^{n} j^3 &= \sum_{j=1}^{n} S(j) && \text{(Lemma 1)} \\ &= \sum_{k=1}^{n(n+1)/2} 2k-1 && \text{(Lemma 2)} \\ &= \left(\frac{n(n+1}{2}\right)^2 && \text{(Eq. (2))} \\ &= \left(\sum_{j=1}^{n} j\right)^2 && \text{(Eq. (1))} \end{aligned} \qquad \square$$

References

[Knu] Donald E. Knuth, *The Art of Computer Programming, Vol. 1*, Addison-Wesley, 1997