COHOMOGENEITY ONE MANIFOLDS AND SELFMAPS OF NONTRIVIAL DEGREE

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Abstract. We construct natural selfmaps of compact cohomgeneity one manifolds and compute their degrees and Lefschetz numbers. On manifolds with simple cohomology rings this yields relations between the order of the Weyl group and the Euler characteristic of a principal orbit. As examples we determine all cohomogeneity one actions on irreducible Riemannian symmetric spaces of compact type that lead to selfmaps of degree $\neq -1, 0, 1$. We derive explicit formulas for new coordinate polynomial selfmaps of the compact matrix groups SU(3), SU(4), and SO($2n$). For SU(3) we determine precisely which integers can be realized as degrees of selfmaps.

1. Introduction

A natural problem in topology is the following: Given a compact oriented manifold M^n, which integers can occur as the degree of maps $M \to M$? In the fundamental case where M is a sphere $\mathbb{S}^n$ all integers can easily be realized since $\mathbb{S}^n$ is a $(n-1)$-fold suspension of $\mathbb{S}^1$. For other manifolds the problem is usually difficult (see [DuWa] for references and a detailed study in the case of $(n-1)$-connected $2n$-manifolds).

In order to construct natural maps of degree $\neq -1, 0, 1$ one might first impose actions of compact Lie groups G on M and then try to find equivariant maps. In the most symmetric case where the action $G \times M \to M$ is transitive, it is well-known and easy to see that every equivariant map is a diffeomorphism and hence has degree ± 1. We deal with the case where the action is of cohomogeneity one, i.e., the principal orbits G/H are of codimension 1 or, equivalently, the orbit space M/G is 1-dimensional and hence a closed interval or a circle. We focus on the much more interesting case where the orbit space M/G is a closed interval.

Any cohomogeneity manifold M with $M/G \approx [0,1]$ can be equiped with a G-invariant Riemannian metric such that the Weyl group is finite or, equivalently, such that the normal geodesics are closed (such metrics are dense in the set of G-invariant metrics). There might, however, be infinitely many G-invariant Riemannian metrics with mutually distinct Weyl groups on the same compact manifold M with a fixed action $G \times M \to M$. The elementary but new construction of this paper depends on the order of the Weyl group: We k-fold the closed normal geodesics of M starting from one of the non-principal orbits. This leads to well-defined maps $\psi_k : M \to M$, $k = j|W|/2 + 1$, for even integers j, and even for all integers j provided a certain condition on the isotropy groups is satisfied. In the following theorem we only deal with the general case. For the exceptional case we refer to section 3.

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Theorem 1. *A compact oriented Riemannian cohomogeneity one manifold M whose orbit space M/G is a closed interval and whose Weyl group W is finite admits equivariant selfmaps $\psi_k : M \to M$ for all $k = j|W|/2+1$, $j \in 2\mathbb{Z}$ with degree*

$$\deg \psi_k = \begin{cases} k & \text{if the codimensions of both non-principal orbits are odd,} \\ +1 & \text{otherwise,} \end{cases}$$

and Lefschetz number

$$L(\psi_k) = \begin{cases} -j\chi(G/H)/2 & \text{if the codimensions of both non-principal orbits are odd,} \\ \chi(M) & \text{otherwise.} \end{cases}$$

Here, $\chi(G/H)$ denotes the Euler characteristic of a principal orbit.

Degree and Lefschetz number of selfmaps are of course not independent. In the case where M^n is a rational homology sphere they are coupled by the simple equation $L(\psi) = 1 + \deg \psi$ for even n or $L(\psi) = 1 - \deg \psi$ for odd n. Hence, we deduce from Theorem 1 that n is odd and $\chi(G/H) = |W|$ (in particular, $\operatorname{rank} G = \operatorname{rank} H$) if the codimensions of both singular orbits are odd[1]. On manifolds M with somewhat more complicated cohomology rings one can still use Theorem 1 to derive restrictions for cohomogeneity one actions $G \times M \to M$ with $\operatorname{rank} G = \operatorname{rank} H$, i.e., $\chi(G/H) > 0$. For such actions the dimensions of all orbits are necessarily even (hence the dimension of M is necessarily odd), and the Weyl groups are necessarily finite. Using Dirichlet's theorem on prime numbers in arithmetic progressions one obtains, for example, the following consequence of Theorem 1:

Corollary 2. *Let $G \times M \to M$ be a cohomogeneity one action with* $\operatorname{rank} G = \operatorname{rank} H$. *If M has the rational cohomology ring of a product*

$$\mathbb{S}^{l_1} \times \mathbb{S}^{l_2} \times \cdots \mathbb{S}^{l_m}, \quad l_1 < l_2 < \ldots < l_m,$$

then $\chi(G/H) = 2^{m-1}|W|$.

By Theorem 1 cohomogeneity one actions yield selfmaps of degree $\neq -1, 0, 1$ if and only if the codimensions of both non-principal orbits are odd. Such actions are called *degree generating cohomogeneity one actions* in the following. Degree generating cohomogeneity one actions are scarce in low dimensions. Scanning the classifications [Neu], [Pa], [Hoe] for degree generating cohomogeneity one actions on simply connected compact manifolds in dimensions ≤ 7 yields only a few actions on spheres and products of spheres and two actions on two nontrivial $\mathbb{S}^3$-bundles $M_1^7 \to \mathbb{S}^4$ and $M_2^7 \to \mathbb{CP}^2$ with sections. We briefly discuss these example in section 4 and deduce from the refined version of Theorem 1 that M_1^7 and M_2^7 admit equivariant selfmaps ψ_k for all $k \in \mathbb{Z}$ with degree k and Lefschetz number $2(1-k)$ and $3(1-k)$, respectively. The space M_1^7 provides an example for Corollary 2.

Our first main example appears in dimension 8: the compact Lie group SU(3). Simple selfmaps of SU(3) are already given by the power maps $\rho_k : A \mapsto A^k$. These

[1] Note that $\chi(G/H) = |W|$ implies $W = N(H)/H$. For the linear cohomogeneity one actions on spheres with $\operatorname{rank} G = \operatorname{rank} H$ the identity $W = N(H)/H$ was already observed in [GWZ]. Cohomogeneity one actions on simply connected $\mathbb{Z}_2$-homology spheres were classified by Asoh [As]. Except for the linear cohomogeneity one actions on spheres and the standard actions on the Brieskorn manifolds W_d^{2m-1} with odd d there is only one infinite family of cohomogeneity one $\mathbb{Z}_2$-homology spheres in dimension 7. The symmetric space $M^5 = \mathrm{SU}(3)/\mathrm{SO}(3)$ with the U(2)-action from the left provides an example of a simply connected cohomogeneity one rational homology sphere with $H_2(M^5) = \mathbb{Z}_2$ which hence does not appear in Asoh's classification.

maps are equivariant with respect to conjugation and have degree $\deg \rho_k = k^2$ by a well-known result of Hopf [Ho]. Our construction yields additional selfmaps ψ_k of SU(3) of every odd degree k and Lefschetz number 0 that extend identity and transposition to an infinite family. These maps are equivariant with respect to the cohomogeneity one action

$$\mathrm{SU}(3) \times \mathrm{SU}(3) \to \mathrm{SU}(3), \quad (A, B) \mapsto ABA^{\mathrm{T}}.$$

Explicit formulas are given in section 5.5. Combining the maps ρ_k with the maps ψ_k and a simple argument involving Steenrod squares we prove

Theorem 3. *For any $m \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ there is a selfmap of* SU(3) *with degree $4^m(2\ell + 1)$. For each of these selfmaps the entries of the target matrix are real polynomials in the complex entries of the argument matrix. Other integers do not appear as degrees of selfmaps of* SU(3)*.*

Motivated by this example we determine all degree generating cohomogeneity one actions on irreducible Riemannian symmetric spaces of compact type using Kollross' classification [Ko]. The result is a short list of actions. In particular, among the simply connected irreducible Riemannian symmetric spaces of compact type only compact Lie groups and spheres admit degree generating cohomogeneity one actions (see Table 1 and Table 2 in section 5). We determine the entire set of possible degrees of the induced equivariant selfmaps and derive explicit coordinate polynomial formulas for these maps on the matrix groups SU(4) and SO($2n$).

We finally note that the degree generating linear cohomogeneity one actions on spheres remain the only examples of degree generating cohomogeneity one actions when one inspects the following classifications: The classification of cohomogeneity one actions on the Stiefel manifolds $V_{n,k}$ with $k < n - 1$ implicitly contained in Kollross' paper [Ko], the (candidate) classification of cohomogeneity one actions on manifolds with positive sectional curvature by Grove, Wilking, and Ziller [GWZ] and the classification of cohomogeneity one actions on simply connected $\mathbb{Z}_2$-homology spheres by Asoh [As]. The classifications of cohomogeneity one actions on simply connected rational homology complex projective spaces by Uchida [Uch], on rational homology quaternionic projective spaces by Iwata [Iw1], and on rational homology octonionic projective planes [Iw2] also do not yield any further degree generating actions.

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2. Construction of the equivariant selfmaps

Before we describe our construction we first summarize the necessary facts about cohomogeneity one manifolds. We refer to the standard sources [Mo], [Bd1], [AA] and the recent paper [GWZ].

Let M be a compact manifold on which a compact Lie group G acts with cohomogeneity one. Then M/G is a circle or a closed interval. In the former case all orbits are principal and M is a G/H-bundle over $\mathbb{S}^1$ with structure group $N(H)/H$. In particular, M has infinite fundamental group. We only consider the case where M/G is a closed interval. The regular part M_0 of M (i.e., the union of principal orbits) projects to the interior of the interval and the two end points of the interval correspond to non-principal orbits N_0 and N_1. The manifold M is obtained by gluing the normal disk bundles over N_0 and N_1 along their common boundary. In particular, $\chi(M) = \chi(N_0) + \chi(N_1) - \chi(G/H)$.

Suppose that M is equipped with an invariant Riemannian metric $\langle \cdot\,, \cdot\rangle$. After rescaling we can assume that M/G is isometric to the unit interval $[0,1]$. A normal geodesic is an (unparametrized) geodesic that passes through all orbits perpendicularly. Through any point $p \in M$ there is at least one normal geodesic and precisely one if p is contained in a principal orbit. The group G acts transitively on the set of normal geodesics. We fix a normal geodesic segment $\gamma : [0,1] \to M$ such that $\gamma(]0,1[)$ projects to the interior of M/G and $p_0 = \gamma(0) \in N_0$ and $p_1 = \gamma(1) \in N_1$. This segment is a shortest curve from N_0 to N_1. It extends to a parametrized normal geodesic $\gamma : \mathbb{R} \to M$. The isotropy groups $G_{\gamma(t)}$ are constant along γ except at the points $\gamma(t)$ with $t \in \mathbb{Z}$, i.e., at points where γ intersects the non-principal orbits. We denote the generic isotropy group along γ by H.

The Weyl group W of $(M, \langle \cdot\,, \cdot\rangle)$ is by definition the subgroup of elements of G that leave γ invariant modulo the subgroup of elements that fix γ pointwise. It is a dihedral subgroup of $N(H)/H$. The geodesic γ is periodic if and only if W is finite. More precisely, the order $|W|$ of W equals the number of times that γ passes through a fixed principal orbit before it closes. To give an example of what might happen on a standard space consider the diagonal SO(3)-action on $\mathbb{S}^2 \times \mathbb{S}^2(\sqrt{\alpha})$. The normal geodesics are closed if and only if α is rational. In this case, the Weyl group is the dihedral group D_{p+q} of order $2(p+q)$ where $\alpha = p/q$ for positive integers p and q with $\gcd(p,q) = 1$. Otherwise, the Weyl group is the infinite dihedral group D_∞.

In the following we assume that the Weyl group of $(M, \langle \cdot\,, \cdot\rangle)$ is finite, i.e., the fixed unit speed normal geodesic γ and all their translated copies $g \cdot \gamma$ are closed with period $|W|$.

Our construction is based on the following elementary fact: Let

$$\psi_k : \mathbb{S}^1 \to \mathbb{S}^1, \quad \lambda \mapsto \lambda^k$$

be the k-th power of $\mathbb{S}^1$. Except in the trivial case where $k = 1$ the fixed points of ψ_k are precisely the $|k-1|$-roots of 1 and ψ_k has the property $\psi_k(\lambda \cdot \lambda_0) = \psi_k(\lambda) \cdot \lambda_0$ for λ_0 with $\lambda_0^{k-1} = 1$. In other words, if we have a circle of any length with a fixed base point p_0 then k-folding the intrinsic distance to p_0 is a well-defined map ψ_k of the circle with $|k-1|$ fixed points and k-folding the distance to any of these fixed points leads to the same map ψ_k.

Lemma 2.1. *The assignment $g \cdot \gamma(t) \mapsto g \cdot \gamma(kt)$ leads to a well-defined smooth map $\psi_k : M \to M$ with $k = j|W|/2 + 1$ for all $j \in 2\mathbb{Z}$, and even for all $j \in \mathbb{Z}$ if the isotropy group at $\gamma(t_0)$ is a subgroup of the isotropy group at $\gamma(t_0 + |W|/2)$ for one and hence all odd integers t_0.*

Proof. We first check that the assignment $\psi_k : g \cdot \gamma(t) \mapsto g \cdot \gamma(kt)$ yields a well-defined map on the unparametrized closed normal geodesic $\gamma(\mathbb{R})$. Suppose that $g \cdot \gamma(\mathbb{R}) = \gamma(\mathbb{R})$. Then $g \cdot \gamma(t) = \gamma(2t_0 \pm t)$ with $t_0 \in \mathbb{Z}$. A simple computation shows $\psi_k(g \cdot \gamma(t)) = g \cdot \psi_k(\gamma(t))$. The geometric reason for this equivariance of ψ_k under the Weyl group is that all the points $\gamma(2t_0)$ are fixed points of $\gamma(t) \mapsto \gamma(kt)$ (here the special form of k is essential) and that the k-folding of a circle is the same from every fixed point. Now ψ_k is well-defined on $\gamma(\mathbb{R})$ and hence on the regular part M_0 of M since through any point in M_0 there is only one (unparametrized) normal geodesic. On N_0 the assignment $g \cdot \gamma(t) \mapsto g \cdot \gamma(kt)$ obviously leads to the identity. On N_1 we also get the identity provided $j \in 2\mathbb{Z}$. If the isotropy group at $\gamma(t_0)$ is a subgroup of the isotropy group at $\gamma(t_0 + |W|/2)$ for one odd integer t_0 then by the action of the Weyl group this is true for all odd integers. If j is an odd integer and $k = j|W|/2 + 1$ then

$$g \cdot \gamma(t_0) \mapsto g \cdot \gamma(kt_0) = g \cdot \gamma(t_0 + |W|/2).$$

This actually defines an equivariant diffeomorphism $N_1 \to N_1$ or an equivariant map $N_1 \to N_0$ depending on whether $|W|/2$ is even or odd. All in all we have shown that there is a commutative diagram

$$\begin{array}{ccc} \nu N_0 & \xrightarrow{\times k} & \nu N_0 \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ M & \xrightarrow{\psi_k} & M \end{array}$$

where νN_0 denotes the normal bundle of N_0. This implies smoothness of ψ_k. □
□

3. Degree and Lefschetz number

Let M be a compact Riemannian cohomogeneity one manifold with finite Weyl group W such that the orbit space M/G is isometric to the interval $[0, 1]$. The manifold M is orientable if and only if the principal orbits are orientable and none of the non-principal orbits is exceptional and orientable (see [Mo]). In this section we assume that M is orientable and equipped with a fixed orientation.

Theorem 3.1. *The Lefschetz number of $\psi_k : M \to M$, $k = j|W|/2 + 1$, is*

$$L(\psi_k) = \begin{cases} -j\chi(G/H)/2 & \textit{if } \operatorname{codim} N_0 \textit{ and } \operatorname{codim} N_1 \textit{ are both odd,} \\ \chi(M) & \textit{otherwise,} \end{cases}$$

if j is even and

$$L(\psi_k) = \begin{cases} -j\chi(G/H)/2 & \textit{if } \operatorname{rank} G = \operatorname{rank} H, \\ \chi(N_0) & \textit{otherwise,} \end{cases}$$

if j is odd provided that ψ_k is well-defined in this case.

Proof. We perturb the map $\psi_k : M \to M$ to a map with only finitely many fixed points and compute the Lefschetz number as the sum of fixed point indices. For this pertubation we use the map $l_g : M \to M$, $p \mapsto g \cdot p$ where $g \in G$ is a general element, i.e., the closure of $\{g^m \mid m \in \mathbb{Z}\}$ is a maximal torus of G. By a classical theorem of Hopf and Samelson [HS], the restriction of l_g to any of the orbits $G \cdot p$ has precisely $\chi(G/G_p)$ fixed points and each of these fixed points has fixed point index one, i.e., $\det(\mathbb{1} - A) > 0$ where A is the derivative of l_g at the fixed point (note that our sign convention is different from that in [HS]). We now consider the composition $l_g \circ \psi_k$ for a general element $g \in G$ sufficiently close to $1 \in G$. The equivariant map ψ_k maps orbits to orbits. It is clear from the construction of ψ_k that only finitely many orbits $V_1, \ldots, V_m$ are mapped to themselves, i.e., $\psi_k(V_i) = V_i$. On each of these orbits, ψ_k is an equivariant diffeomorphism. Hence, the restriction of ψ_k to V_i is either the identity or does not have fixed points. The latter property is preserved under small perturbations of ψ_k. Therefore, the map $l_g \circ \psi_k$ can only have fixed points in the orbits V_i on which ψ_k restricts to the identity, and the number of fixed points in V_i is $\chi(V_i)$. Let p be one of these fixed points in the orbit V. In order to compute the fixed point index at p we need the derivative of l_g and ψ_k at p. Both derivatives clearly preserve both tangent and normal space to V at p. The derivative of ψ_k is multiplication with k on the normal space and the identity on the tangent space. The derivative of l_g is close to the identity on the normal space and given by a matrix A with $\det(\mathbb{1} - A) > 0$ on the tangent space by the above menitoned result of Hopf. Hence, if B denotes the derivative of $l_g \circ \psi_k$ at p then the fixed point index at p is given by

$$\det(\mathbb{1} - B) = (\operatorname{sgn}(1-k))^{\operatorname{codim} V} = (-\operatorname{sgn} j)^{\operatorname{codim} V}.$$

The final step is now to sum these fixed point indices. We first consider the case where j is even. Then ψ_k restricts to the identity on both N_0 and N_1. Since ψ_k has $|k-1|$ fixed points on $\gamma([0,|W|])$, we have $|k-1|/|W| - 1 = |j|/2 - 1$ fixed points on $\gamma(]0,1[)$. These fixed points correspond precisely to the orbits on which ψ_k restricts to the identity. Hence we get

$$\begin{aligned} L(\psi_k) = L(l_g \circ \psi_k) = (-\operatorname{sgn} j)^{\operatorname{codim} N_0}\chi(N_0) + (-\operatorname{sgn} j)^{\operatorname{codim} N_0}\chi(N_0) \\ + (|j|/2 - 1)(-\operatorname{sgn} j)\,\chi(G/H). \end{aligned}$$

Now suppose j is odd. Then ψ_k restricts to the identity on N_0 but not on N_1. Since ψ_k has $|k-1|$ fixed points on $\gamma([0,|W|])$, we have $|k-1|/|W| - 1/2 = (|j|-1)/2$ fixed points on $\gamma(]0,1[)$. Hence,

$$L(\psi_k) = L(l_g \circ \psi_k) = (-\operatorname{sgn} j)^{\operatorname{codim} N_0}\chi(N_0) + \frac{|j|-1}{2}(-\operatorname{sgn} j)\,\chi(G/H).$$

It is straightforward to simplify these two formulas for $L(\psi_k)$ using the facts that the Euler characteristic of an odd dimensional manifold vanishes, that $\chi(M) = \chi(N_0) + \chi(N_1) - \chi(G/H)$, and that $\chi(G/H) > 0$ implies that $\operatorname{rank} G = \operatorname{rank} H$ and hence that all orbits have even dimension. In the case where the codimensions of both singular orbits are odd one also has to use the fact that $\chi(N_0) = \chi(N_1)$ since both fibrations $G/H \to N_i$ have fibers that are even dimensional spheres. □ □

The degree of ψ_k is the sum of the oriented preimages of a regular value. Since the two non-principal orbits are mapped to themselves or one to the other we just need to consider the regular part M_0 of M. In order to determine the correct orientations we transfer the problem from M_0 to $G/H \times (\mathbb{R} \setminus \mathbb{Z})$ by the map

$$\phi : G/H \times \mathbb{R} \to M, \quad (gH, t) \mapsto g \cdot \gamma(t).$$

The restriction of ϕ to each $G/H \times]\ell, \ell+1[$ with $\ell \in \mathbb{Z}$ is a diffeomorphism. We equip $\mathbb{R}$ with the standard orientation and G/H with an orientation such the restriction of $G/H \times]0,1[\to M_0$ of ϕ is orientation preserving. This defines a standard product orientation on $G/H \times \mathbb{R}$. For the rest of the paper, however, we will equip each $G/H \times]\ell, \ell+1[$ with the orientation inherited from M_0 by ϕ.

Lemma 3.2. *If* $\operatorname{codim} N_0$ *and* $\operatorname{codim} N_1$ *are both odd then all* $G/H \times]\ell, \ell+1[$ *inherit the standard product orientation from* M_0 *via* ϕ.

If $\operatorname{codim} N_0$ *and* $\operatorname{codim} N_1$ *are both even, only the pieces* $G/H \times]2\ell, 2\ell+1[$ *inherit the standard product orientation from* M_0 *via* ϕ.

If $\operatorname{codim} N_0$ *is odd and* $\operatorname{codim} N_1$ *is even then only the pieces* $G/H\times\,]4\ell, 4\ell+1[$ *and* $G/H\times\,]4\ell-1, 4\ell[$ *inherit the standard product orientation from* M_0 *via* ϕ.

If $\operatorname{codim} N_0$ *is even and* $\operatorname{codim} N_1$ *is odd then only the pieces* $G/H\times\,]4\ell, 4\ell+1[$ *and* $G/H\times\,]4\ell+1, 4\ell+2[$ *inherit the standard product orientation from* M_0 *via* ϕ.

Proof. Let σ_0 denote the geodesic reflection along N_0, i.e., σ_0 maps each point $g\cdot\gamma(t)$ in $M \setminus N_1$ to $g\cdot\gamma(-t)$. This equivariant diffeomorphism of $M\setminus N_1$ corresponds to $-\operatorname{id}$ in the normal bundle of the orbit N_0 and hence preserves the orientation of $M\setminus N_1$ if and only if $\operatorname{codim} N_1$ is even. We have the equivariant commutative diagram

$$\begin{array}{ccc} G/H\times\,]0,1[& \xrightarrow{(gH,t)\mapsto(gH,-t)} & G/H\times\,]-1,0[\\ \phi\big\downarrow & & \phi\big\downarrow \\ M_0 & \xrightarrow{\sigma_0} & M_0 \end{array}$$

Note that the map $]0,1[\,\to\,]-1,0[$, $t\mapsto -t$ reverses the orientation of $\mathbb{R}$. Hence, $G/H\times\,]-1,0[$ inherits the same orientation from M_0 as $G/H\times\,]0,1[$ if and only if σ_0 reverses orientation, i.e., if and only if $\operatorname{codim} N_0$ is odd. □ □

Corollary 3.3. *If* M *is orientable and* $\operatorname{codim} N_0$ *and* $\operatorname{codim} N_1$ *do not have the same parity, then the order* $|W|$ *of the Weyl group* W *of* M *is divisible by* 4.

Theorem 3.4. *The map* $\psi_k : M \to M$, $k = j|W|/2+1$ *has degree*

$$\deg\psi_k = \begin{cases} k & \textit{if } \operatorname{codim} N_0 \textit{ and } \operatorname{codim} N_1 \textit{ are both odd,} \\ +1 & \textit{otherwise,} \end{cases}$$

if j *is even, and degree*

$$\deg\psi_k = \begin{cases} k & \textit{if } \operatorname{codim} N_0 \textit{ and } \operatorname{codim} N_1 \textit{ are both odd,} \\ 0 & \textit{if } \operatorname{codim} N_0 \textit{ and } \operatorname{codim} N_1 \textit{ are both even, } |W|\notin 4\mathbb{Z}, \\ -1 & \textit{if } \operatorname{codim} N_0 \textit{ is even, } \operatorname{codim} N_1 \textit{ is odd, and } |W|\notin 8\mathbb{Z}, \\ +1 & \textit{otherwise,} \end{cases}$$

if j *is odd provided that* ψ_k *is well-defined in this case (see Lemma 2.1).*

Proof. The point $\gamma(\tau)$ with $\tau = 1/2$ is a regular value of the map ψ_k whose $|k|$ preimages are given by $\gamma(t_m)$ with

$$t_m = \frac{m|W|+\tau}{k},\ m\in\mathbb{Z}.$$

In order to compute whether the differential of ψ_k at $\gamma(t_m)$ preserves or reverses orientation we lift $\psi_k : M_0 \to M_0$ to the map

$$\tilde{\psi}_k : G/H\times\mathbb{R} \to G/H\times\mathbb{R}, \quad (gH,t)\to\big(gH,kt\big)$$

and answer the same question for the differential of $\tilde{\psi}_j$ at (eH, t_m) by applying Lemma 3.2 (note again that we use the orientation on $G/H\times(\mathbb{R}\setminus\mathbb{Z})$ induced from M_0 by ϕ). If $\operatorname{codim} N_0$ and $\operatorname{codim} N_1$ are both odd then all pieces $G/H\times]\ell,\ell+1[$ inherit the standard product orientation. Hence, $\deg\psi_k = k$ and we are done.

Thus assume that $\operatorname{codim} N_0$ or $\operatorname{codim} N_1$ are even. For each preimage p of $\gamma(\tau)$ there is precisely one $m\in\mathbb{Z}$ such that $p=\gamma(t_m)$ and $t_m\in]0,|W|[$. We now have to count how many t_m are contained in each of the intervals $]\ell,\ell+1[$

for $\ell \in \{0, 1, \ldots, |W| - 1\}$. A simple computation shows that $t_m \in]\ell, \ell + 1[$ is equivalent to

$$\frac{j}{2}\ell + \frac{\ell - \tau}{|W|} < m < \frac{j}{2}(\ell + 1) + \frac{\ell + 1 - \tau}{|W|}$$

if $j \geq 0$, and equivalent to

$$\frac{j}{2}(\ell + 1) + \frac{\ell + 1 - \tau}{|W|} < m < \frac{j}{2}\ell + \frac{\ell - \tau}{|W|}$$

if $j < 0$. Suppose first that j is even. For $\ell > 0$ we have $\frac{\ell - \tau}{|W|} \in]0, 1[$ and $\frac{\ell+1-\tau}{|W|} \in]0, 1[$. Hence there are precisely $|j|$ of the t_m in each interval $]\ell, \ell + 1[$ and $|j + 1|/2$ in the interval $]0, 1[$. When counted with orientation using Lemma 3.2 the sum of preimages of $\gamma(\tau)$ is hence $+1$ since there are $|W|$ intervals $]\ell, \ell + 1[$ and $|W|$ is divisible by 2 if the codimensions of N_0 and N_1 are both even and $|W|$ is divisible by 4 if the codimensions of N_0 and N_1 have different parities. Note that in the case $j \geq 0$ there is one point more in the interval $]0, 1[$ than in the other intervals and this point has positive orientation, while in the case $j < 0$ there is one point less but the orientation of the real line has changed. Suppose now that j is odd. Then $\frac{\ell-\tau}{|W|} \in]0, \frac{1}{2}[$ if and only if $0 < \ell \leq |W|/2$ and $\frac{\ell+1-\tau}{|W|} \in]0, \frac{1}{2}[$ if and only if $0 \leq \ell < |W|/2$. Using these facts it is straightforward to show that there are $|j + 1|/2$ of the t_m in each of the intervals $]\ell, \ell + 1[$ with $\ell = 0$, $\ell = |W|/2$, odd $l < |W|/2$, or even $l > |W|/2$, while there are $|j - 1|/2$ of the t_m in each of the intervals $]\ell, \ell + 1[$ with even $0 < l < |W|/2$ or odd $l > |W|/2$. The claimed degrees follow now by applying Lemma 3.2. □ □

Proof of Corollary 2. Suppose that M has the rational cohomology ring of

$$\mathbb{S}^{l_1} \times \mathbb{S}^{l_2} \times \cdots \mathbb{S}^{l_m}, \quad l_1 < l_2 < \ldots < l_m,$$

and that $G \times M \to M$ is a cohomogeneity one action with $\operatorname{rank} G = \operatorname{rank} H$, i.e., $\chi(G/H) > 0$ where H is a principal isotropy group. By Lemma 2.1, M admits equivariant selfmaps ψ_k, $k = j|W|/2 + 1$, with degree k and Lefschetz number $L(\psi_k) = -j\chi(G/H)$ for every even integer j. By Dirichlet's theorem there are infinitely many prime numbers in the arithmetic progression $k = j|W|/2+1$. Hence, we can assume that k is a prime number. We now inspect the induced map ψ_k^* on the cohomology ring. Let $x_1, \ldots, x_m$ be generators of $H^*(M)$ with $x_i \in H^{l_i}(M)$. The equation $\deg \psi_k = k$ means $\psi_k^*(x_1 x_2 \cdots x_m) = k x_1 x_2 \cdots x_m$. Since k is a prime number and $l_1 < l_2 < \ldots < l_m$ it is clear that

$$\psi_k^*(x_j) = k\sigma_j x_j + \text{products of lower dimensional generators}$$

with $\sigma_j = \pm 1$ for precisely one $j \in \{1, \ldots, m\}$ and

$$\psi_k^*(x_i) = \sigma_i x_i + \text{products of lower dimensional generators}$$

with $\sigma_i = \pm 1$ for $i \in \{1, \ldots, m\}$, $i \neq j$. The total number of σ_i with $\sigma_i = -1$ for $i \in \{1, \ldots, m\}$ must be even. The Lefschetz number $L(\psi_k)$ is equal to $-j\chi(G/H)/2$ by Theorem 3.1. On the other hand, $L(\psi_k)$ is the alternating trace of ψ^* in cohomology. The simple cohomology ring structure implies

$$-j\chi(G/H)/2 = L(\psi_k) = \big(1 + (-1)^{l_j} k\sigma_j\big) \prod_{i \neq j} \big(1 + (-1)^{l_i} \sigma_i\big).$$

Since we can assume $j > 2$ and since $\chi(G/H) > 0$, we have $\sigma_i = (-1)^{l_i}$ for all $i \neq j$ and $\sigma_j = -(-1)^{l_j}$. Hence, $\chi(G/H) = 2^{m-1}|W|$ and Corollary 2 is proved. □ □

4. Basic examples

In this section we present basic examples for all the various cases in Theorem 3.1 and Theorem 3.4.

4.1. **Powers of spheres.** The standard example of a cohomogeneity one action on a compact manifold is the action of $\mathrm{SO}(n)$ on the sphere $\mathbb{S}^n \subset \mathbb{R} \times \mathbb{R}^n$. The regular orbits are great spheres and the two non-regular orbits are the two fixed points $(\pm 1, 0)$. The equivariant selfmaps are the k-powers of $\mathbb{S}^n \subset \mathbb{R} \times \mathbb{R}^n$:

$$(\cos t, v \sin t)^k = (\cos kt, v \sin kt).$$

Our construction in section 2 can be seen as a natural extension of the classical k-powers of spheres to general cohomogeneity one manifolds.

If n is odd then the rank of the principal isotropy group $\mathrm{SO}(n-1)$ of the $\mathrm{SO}(n)$-action on $\mathbb{S}^n$ is equal to the rank of $\mathrm{SO}(n)$. The Weyl group is isomorphic to $\mathbb{Z}_2$ and the non-principal isotropy groups are both equal to $\mathrm{SO}(n)$. From Theorem 3.4 we recover the known fact that the degree of the k-power of $\mathbb{S}^n$ is k if n is odd, and 0 or 1 if n is even, depending on whether k is even or odd. From Theorem 3.1 we recover the Lefschetz number $(1-k)$ if n is odd, and 1 or 2 if n is even, depending on whether k is even or odd.

It is easy to obtain explicit polynomial formulas for the k-powers of spheres: We define the functions c_k and s_k implicitly by

$$\cos kt = \begin{cases} c_k(\sin^2 t), & \text{for even } k, \\ c_k(\sin^2 t)\cos t, & \text{for odd } k \end{cases}$$

and

$$\sin kt = \begin{cases} s_k(\sin^2 t)\cos t \sin t, & \text{for even } k, \\ s_k(\sin^2 t)\sin t, & \text{for odd } k. \end{cases}$$

Applying the binomial formula to $\cos kt + \mathrm{i} \sin kt = (\cos t + \mathrm{i} \sin t)^k$ yields

$$c_k(r) = \sum_{i=0}^{[|k|/2]} (-1)^i \binom{|k|}{2i} r^i (1-r)^{[|k|/2]-i}$$

$$s_k(r) = \mathrm{sgn}(k) \sum_{i=0}^{[(|k|-1)/2]} (-1)^i \binom{|k|}{2i+1} r^i (1-r)^{[(|k|-1)/2]-i}$$

where $[a]$ denotes the largest integer less or equal to a. The k-powers of spheres $\mathbb{S}^n \subset \mathbb{R} \times \mathbb{R}^n$ are now given by

$$(x,y)^k = \big(c_k(1-x^2), s_k(1-x^2)xy\big) \text{ for even } k$$

and

$$(x,y)^k = \big(c_k(1-x^2)x, s_k(1-x^2)y\big) \text{ for odd } k.$$

The functions c_k and s_k will appear several times in other examples below. Additionally, there also appears the function $h_k(r) = \frac{1-c_k(r)}{r}$ if k is odd. Note that this function is also polynomial.

4.2. **Selfmaps of degree -1 of $\mathbb{CP}^{2\ell+1}$.** An example of a cohomogeneity one action where the codimensions of the singular orbits have different parity and there are equivariant maps with degree -1 is given by the standard action of $\mathrm{SO}(1+m) \subset \mathrm{SU}(1+m)$ on $\mathbb{CP}^m$ for odd m. A normal geodesic is given by

$$\gamma(t) = [\cos t : \mathrm{i}\sin t : 0 : \ldots : 0].$$

The singular orbit at $t = 0$ is $\mathbb{RP}^m = \mathrm{SO}(1+m)/\mathrm{O}(m)$ and the singular isotropy groups at $t = \pi/4$ and $t = -\pi/4$ are both equal to $\mathrm{SO}(2)\mathrm{SO}(m-1)$. The normal geodesic γ is closed with period π. Hence, the Weyl group is the dihedral group of order 4 and $2j+1$-folding the distance to $N_0 = \mathbb{RP}^m$ yields a well-defined map with degree 1 and Lefschetz number $m+1$ if j is even and degree -1 and Lefschetz number 0 if j is odd. Note that reflecting the normal geodesics along the $\mathbb{RP}^m$ is just the involution induced by complex conjugation on $\mathbb{C}^{m+1}$. Note also that for even m selfmaps of $\mathbb{CP}^m$ with degree -1 cannot exist and this is perfectly matched by the fact that both singular orbits have even codimensions in this case.

4.3. **Two 7-manifolds with selfmaps of arbitrary degree.** We now discuss the spaces $M_1^7 = \mathrm{Sp}(2) \times_{\mathrm{Sp}(1)^2} \mathrm{Sp}(1)$ and $M_2^7 = \mathrm{SU}(3) \times_{\mathrm{U}(2)} \mathrm{SU}(2)$ where $\mathrm{U}(2)$ acts on $\mathrm{SU}(2)$ by conjugation and one of the factors of $\mathrm{Sp}(1)^2$ acts trivially on $\mathrm{Sp}(1)$ and the other by conjugation. The space M_1^7 is an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$. Such bundles are classified by their characteristic homomorphism $\mathbb{S}^3 \to \mathrm{SO}(4)$. Using two Cartan embeddings of $\mathbb{S}^4$ into $\mathrm{Sp}(2)$ it is not difficult to see that for M_1^7 this homomorphism is given by $q \mapsto C_q$ where q is a unit quaternion and $C_q : \mathbb{H} \to \mathbb{H}$ is conjugation by q. In particular, the bundle $M_1^7 \to \mathbb{S}^4$ has a section but is nontrivial. It is known that M_1^7 is not homotopy equivalent to $\mathbb{S}^3 \times \mathbb{S}^4$ but has the same cohomology and homotopy groups as $\mathbb{S}^3 \times \mathbb{S}^4$ [JW]. The group $\mathrm{Sp}(2)$ acts from the left on M_1^7 by cohomogeneity one. The principal orbits are diffeomorphic to $\mathbb{CP}^3 = \mathrm{Sp}(2)/\mathrm{Sp}(1)\mathrm{SO}(2)$ with Euler characteristic 4 and the singular orbits are both diffeomorphic to $\mathbb{S}^4$. We have $\operatorname{rank} G = \operatorname{rank} H$. The Weyl group is isomorphic to $\mathbb{Z}_2$. Hence, by Lemma 2.1 there exist equivariant selfmaps ψ_k with degree $\psi_k = k$ and Lefschetz number $L(\psi_k) = 2(1-k)$ for all integers k. This example thus illustrates Corollary 2, since $4 = \chi(G/H) = 2 \cdot |W|$.

The space M_2^7 is an $\mathbb{S}^3$-bundle over $\mathbb{CP}^2$. The group $\mathrm{SU}(3)$ acts fromt the left on M_2^7 by cohomogeneity one. The principal orbits are diffeomorphic to $\mathrm{SU}(3)/\mathrm{T}^2$ with Euler characteristic 6 and the singular orbits are both diffeomorphic to $\mathbb{CP}^2$. Since the bundle $M_2^7 \to \mathbb{CP}^2$ has a section, M_2^7 has the same cohomology and homotopy groups as $\mathbb{S}^3 \times \mathbb{CP}^2$. Moreover, the corresponding principal $\mathrm{SO}(3)$-bundle over $\mathbb{CP}^2$ is the Aloff-Wallach space $W_{1,1}^7$. The first Pontrjagin class of this principal bundle is $-3 \in H^4(\mathbb{CP}^2) \approx \mathbb{Z}$ [Zi]. Hence, the $\mathbb{S}^3$-bundle $M_2^7 \to \mathbb{CP}^2$ has $w_2 = 0$ and $p_1 = -3$. By the Dold-Whitney classification [DoWh] it is nontrivial. The Weyl group of the cohomogeneity one $\mathrm{SU}(3)$ action is isomorphic to $\mathbb{Z}_2$. Hence, there exist equivariant selfmaps ψ_k with degree $\psi_k = k$ and Lefschetz number $L(\psi_k) = 3(1-k)$ for all integers k in agreement with the homology of M_2^7.

4.4. **The general source of examples.** In the examples above we gave concrete cohomogeneity one actions on manifolds already constructed by different methods (homogeneous spaces and twisted products). A vast number of examples is not of this type. Cohomogeneity one manifolds M with $M/G \approx [0,1]$ can be (re)constructed from their group diagram: Given a compact Lie group G and closed subgroups $H \subset K_0, K_1 \subset G$ such that K_0/H and K_1/H are diffeomorphic to spheres there is a Riemannian cohomogeneity one manifold such that H is the generic isotropy group and K_0 and K_1 are adjacent non-principal isotropy groups along this geodesic. In general, a cohomogeneity one manifold cannot be described in a more natural way than by its group diagram. Using group diagrams it is easy to see that there are infinitely many families of cohomogeneity one manifolds that fulfill the different hypotheses of Theorem 3.1 and Theorem 3.4.

5. Selfmaps of irreducible symmetric spaces of compact type

We now apply our construction to the irreducible Riemannian symmetric spaces of compact type. Cohomogeneity one actions on these spaces were classified by Kollross [Ko]. Here we are particularly interested in the degree generating cohomogeneity one actions. In order to determine these actions from Kollross' list the main work is to compute the non-principal isotropy groups (or at least their dimensions). It turns out that there exist only a couple of degree generating actions on simply connected irreducible symmetric spaces of compact type, namely, only on spheres and on simple compact Lie groups. We will present some of these actions and the induced selfmaps in detail.

5.1. **Some basic facts on the classification.** In this and the next subsection we describe some basic aspects of how to determine the degree generating cohomogeneity one actions on the irreducible symmetric spaces of compact type. Since the property "degree generating" just depends on the codimensions of the non-principal orbits, it suffices to determine the degree generating actions up to orbit equivalence. Two actions $G_1 \times M_1 \to M_1$ and $G_2 \times M_2 \to M_2$ are called *orbit equivalent* if there is an isometry $\Phi : M_1 \to M_2$ with $\Phi(G_1 \cdot p) = G_2 \cdot \Phi(p)$ for all $p \in M_1$.

Let $G \times M \to M$ be a cohomogeneity one action of a connected compact Lie group G on a compact Riemannian manifold M such that M/G is diffeomorphic to the interval $[0, 1]$. Let $\tilde{M} \to M$ be a covering with finitely many sheets. Then there exists a connected compact Lie group $\tilde{G}$ that covers G finitely and acts with cohomogeneity one on $\tilde{M}$ such that the diagram

$$\begin{array}{ccc} \tilde{G} \times \tilde{M} & \longrightarrow & \tilde{M} \\ \big\downarrow & & \big\downarrow \\ G \times M & \longrightarrow & M \end{array}$$

commutes (see [Bd1]). There are now two cases: In the first case the two non-principal orbits in $\tilde{M}$ project to distinct non-principal orbits in M. In the second case the two non-principal orbits in $\tilde{M}$ project to the same non-principal orbit in M and the other non-principal orbit in M is exceptional, i.e., its codimension is 1. In any of the two cases, the $\tilde{G}$-action on $\tilde{M}$ generates degree if and only if the G-action on M generates degree.

Clearly, the universal covering space $\tilde{M}$ of an irreducible Riemannian symmetric space of compact type M is again an irreducible Riemannian symmetric space of compact type and the covering $\tilde{M} \to M$ has finitely many sheets. Hence, it suffices to determine the degree generating cohomogeneity one actions on the irreducible Riemannian symmetric spaces of compact type up to coverings. Note, however, that the Weyl groups of $\tilde{M}$ and M are not necessarily isomorphic.

The irreducible Riemannian symmetric spaces of compact type split into two types: A space of type II is a simple compact Lie group G with its biinvariant metric (unique up to scaling). The identity component of the isometry group of G is finitely covered by $G \times G$ (the first factor acting by left translations, the second by right translations). Hence, in order to determine the cohomogeneity one actions on G it suffices to study actions of closed connected subgroups of $G \times G$.

The identity component of the isometry group of a space M of type I is a simple compact Lie group G (simple in the usual sense that the Lie algebra is simple, i.e., has no nontrivial normalizers) and the isotropy group K of a point $p \in M$ lies between the fixed point set of an involutive automorphism of G and its identity component. If a closed subgroup H of G acts on $M \approx G/K$ with cohomogeneity one then the action

$$(H \times K) \times G \to G, \quad (h, k) \bullet g = hgk^{-1}$$

is also of cohomogeneity one. More precisely, the isotropy group of the H-action on G/K at gK is

$$H_{gK} = H \cap gKg^{-1}$$

and the isotropy group of the $H \times K$-action on G at g is

$$(H \times K)_g = \{(h, k) \,|\, h \in H_{gK},\ k = g^{-1}hg\} \approx H_{gK}.$$

In particular, the codimension of the H-orbit through gK is equal to the codimension of the $H \times K$-orbit through g. Hence, the H-action on $M = G/K$ generates degree if and only if the $H \times K$-action on G generates degree. The Weyl group of these actions, however, are not necessarily isomorphic.

All in all we see that in order to classify the degree generating cohomogeneity one actions on the irreducible Riemannian symmetric spaces of compact type up to equivalence and coverings it suffices to determine for all simple compact Lie groups G up to coverings the subgroups of $G \times G$ that act on G with cohomogeneity one and two non-principal orbits of odd codimensions.

5.2. **Degree generating actions on simple compact Lie groups.** Kollross [Ko] determined for all simple compact Lie groups G (up to covering) all largest connected subgroups of $G \times G$ (up to conjugacy) that act on G with cohomogeneity one. We omit the lengthy but more or less straightforward computations and considerations that filter out the degree generating actions from this list and merely present the result.

We separate the degree generating cohomogeneity one actions on the compact simple Lie groups in two groups. The first group comprises the following actions:

$$\mathrm{SU}(3) \times \mathrm{SU}(3) \to \mathrm{SU}(3), \quad A \bullet B = ABA^{\mathrm{T}},$$
$$\mathrm{Sp}(2) \times \mathrm{Sp}(2) \times \mathrm{SU}(4) \to \mathrm{SU}(4), \quad (A, B) \bullet C = ACB^{-1},$$
$$\mathrm{SO}(2n-1) \times \mathrm{SO}(2n-1) \times \mathrm{SO}(2n) \to \mathrm{SO}(2n), \quad (A, B) \bullet C = ACB^{-1},$$
$$\mathrm{G}_2 \times \mathrm{G}_2 \times \mathrm{SO}(7) \to \mathrm{SO}(7), \quad (A, B) \bullet C = ACB^{-1}.$$

Note that second action is actually a covering action of the action $\mathrm{SO}(5) \times \mathrm{SO}(5) \times \mathrm{SO}(6) \to \mathrm{SO}(6)$. We list it separately because the formula for the selfmaps of the matrix group SU(4) turns out to be very simple (see subsection 5.6).

The remaining degree generating cohomogeneity one actions are given by the isotropy representations of the symmetric spaces

$$\mathrm{SU}(3)\times\mathrm{SU}(3)/\Delta\mathrm{SU}(3),\quad \mathrm{SU}(6)/\mathrm{Sp}(3),\quad \mathrm{E}_6/\mathrm{F}_4,$$
$$\mathrm{Sp}(2)\times\mathrm{Sp}(2)/\Delta\mathrm{Sp}(2),\quad \mathrm{G}_2\times\mathrm{G}_2/\Delta\mathrm{G}_2.$$

These representations provide homomorphisms

$$\mathrm{SU}(3)\to\mathrm{SO}(8),\quad \mathrm{Sp}(3)\to\mathrm{SO}(14),\quad \mathrm{F}_4\to\mathrm{SO}(26),$$
$$\mathrm{Sp}(2)\to\mathrm{SO}(10),\quad \mathrm{G}_2\to\mathrm{SO}(14)$$

that yield cohomogeneity one actions on the spheres

$$\mathbb{S}^7=\mathrm{SO}(8)/\mathrm{SO}(7),\quad \mathbb{S}^{13}=\mathrm{SO}(14)/\mathrm{SO}(13),\quad \mathbb{S}^{25}=\mathrm{SO}(26)/\mathrm{SO}(25),$$
$$\mathbb{S}^9=\mathrm{SO}(10)/\mathrm{SO}(9),\quad \mathbb{S}^{13}=\mathrm{SO}(14)/\mathrm{SO}(13).$$

As explained in 5.1, these actions in turn yield the following cohomogeneity one actions on simple compact Lie groups:

$$\mathrm{SU}(3)\times\mathrm{SO}(7)\times\mathrm{SO}(8)\to\mathrm{SO}(8),$$
$$\mathrm{Sp}(3)\times\mathrm{SO}(13)\times\mathrm{SO}(14)\to\mathrm{SO}(14),$$
$$\mathrm{F}_4\times\mathrm{SO}(25)\times\mathrm{SO}(26)\to\mathrm{SO}(26),$$
$$\mathrm{Sp}(2)\times\mathrm{SO}(9)\times\mathrm{SO}(10)\to\mathrm{SO}(10),$$
$$\mathrm{G}_2\times\mathrm{SO}(13)\times\mathrm{SO}(14)\to\mathrm{SO}(14).$$

Theorem 5.1. *The nine actions above are up to coverings and orbit equivalence the only degree generating cohomogeneity one actions on simple compact Lie groups.*

Proof. Determination of all degree generating actions from Kollross' list [Ko]. □
□

A list of all degree generating cohomogeneity one actions on irreducible Riemannian symmetric spaces of compact type can now easily be obtained using the constructions in subsection 5.1. In particular, degree generating cohomogeneity one actions exist on no other irreducible Riemannian symmetric spaces of compact type than on simple compact Lie groups and spheres up to coverings.

5.3. **Selfmaps of spheres.** A complete list of degree generating cohomogeneity one actions on spheres up to orbit equivalence is given in Table 1. The information about the Weyl groups and the non-principal orbits can be retrieved from the detailed tables of cohomogeneity one actions on compact rank one symmetric spaces in [GWZ]. They are also partly computed here in the later subsections.

group acting	sphere acted on	Weyl group	selfmap degrees
$\mathrm{SO}(2n-1)$	$\mathbb{S}^{2n-1}$	D_1	$\mathbb{Z}$
$\mathrm{SU}(3)$	$\mathbb{S}^7$	D_3	$3\mathbb{Z}+1$
$\mathrm{Sp}(3)$	$\mathbb{S}^{13}$	D_3	$3\mathbb{Z}+1$
F_4	$\mathbb{S}^{25}$	D_3	$3\mathbb{Z}+1$
$\mathrm{Sp}(2)$	$\mathbb{S}^9$	D_4	$4\mathbb{Z}+1$
G_2	$\mathbb{S}^{13}$	D_6	$6\mathbb{Z}+1$

TABLE 1. Degree generating cohomogeneity one actions on spheres.

Although topologically all spheres admit selfmaps of arbitrary degree only odd dimensional spheres admit cohomogeneity one selfmaps of degree $\neq -1, 0, 1$.

5.4. **Selfmaps of compact Lie groups.** The most natural selfmaps of compact Lie groups G are the power maps $\rho_k : A \mapsto A^k$. These maps are equivariant with respect to conjugation. The cohomogeneity of the adjoint action of G is equal to the rank of G. The power maps ρ_k are determined by their restriction to a maximal torus $T^r \subset G$. Using this fact it is easy to see that the degree of ρ_k is k^r and thus rather few integers can be realized as degrees of power maps.

Our construction provides highly symmetric selfmaps of some compact Lie groups whose degrees supplement the sparse list of integers provided by the power maps. For SU(3) we will in particular get a complete list of all possible degrees of selfmaps.

We will work out the geometry of the cohomogeneity one actions on SU(3), SU(4), SO($2n$), and SO(7) and the related selfmaps in detail in the following subsections. For the selfmaps of SU(3), SU(4), SO($2n$) we obtain explicit formulas that are *coordinate polynomial* in the sense that the entries of the target matrix are polynomials in the entries of the argument matrix. For the cohomogeneity one actions on SO(8), SO(10), SO(14), and SO(26) we merely list which integers can be realized by cohomogeneity one selfmaps. The results are summarized in Table 2.

group acting	group acted on	Weyl group	selfmap degrees
SU(3)	SU(3)	D_2	$2\mathbb{Z}+1$
$\mathrm{Sp}(2) \times \mathrm{Sp}(2)$	SU(4)	D_2	$2\mathbb{Z}+1$
$\mathrm{SO}(2n-1) \times \mathrm{SO}(2n-1)$	SO($2n$)	D_1	$2\mathbb{Z}+1$
$\mathrm{G}_2 \times \mathrm{G}_2$	SO(7)	D_3	$3\mathbb{Z}+1$
$\mathrm{SU}(3) \times \mathrm{SO}(7)$	SO(8)	D_3	$6\mathbb{Z}+1$
$\mathrm{Sp}(3) \times \mathrm{SO}(13)$	SO(14)	D_3	$6\mathbb{Z}+1$
$\mathrm{F}_4 \times \mathrm{SO}(25)$	SO(26)	D_3	$6\mathbb{Z}+1$
$\mathrm{Sp}(2) \times \mathrm{SO}(9)$	SO(10)	D_4	$8\mathbb{Z}+1$
$\mathrm{G}_2 \times \mathrm{SO}(13)$	SO(14)	D_6	$12\mathbb{Z}+1$

Table 2. Degree generating cohomogeneity one actions on simple compact Lie groups.

Note that the final column in Table 2 does not change when we lift the actions from the special orthogonal groups to the spin groups: For the $\mathrm{G}_2 \times \mathrm{G}_2$-action on Spin(7) this is in detail explained in subsection 5.8. For all other actions the order of the Weyl groups double but opposite isotropy groups along a fixed normal geodesic become equal.

Note also that a cohomogeneity one action on a sphere has the same Weyl group as the induced action on the special orthogonal group. Nevertheless, only half of the selfmaps lift from $\mathbb{S}^{2n-1}$ to SO($2n$) or Spin($2n$).

5.5. **Selfmaps of** SU(3)**.** We consider the cohomogeneity one action

$$\mathrm{SU}(3) \times \mathrm{SU}(3) \to \mathrm{SU}(3), \quad (A, B) \mapsto ABA^{\mathrm{T}}.$$

A straightforward computation shows that

$$\gamma(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a normal geodesic for this action, i.e. passes through all orbits perpendicularly. This geodesic is closed with period 2π. At times $t = 0$ and $t = \pi$ it passes through the singular orbit SU(3)/SO(3) that consists of the symmetric matrices in SU(3). At $t = \frac{\pi}{2} + \pi\mathbb{Z}$ it passes through the other singular orbit SU(3)/SU(2) where SU(2) is embedded in the upper left corner of SU(3). The isotropy group of $\gamma(t)$ for all other times t is SO(2) embedded in the upper left corner of SU(3) in the standard way. The Weyl group is isomorphic to $D_2 \approx \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that the singular isotropy groups at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ are not just conjugate but equal.

Theorem 5.2. SU(3) *admits selfmaps* ψ_k *with degree* k *and Lefschetz number* 0 *for all odd integers* k. *These selfmaps are equivariant with respect to the* SU(3) *action above. An explicit formula is given by*

$$\psi_k(B) = \tfrac{1}{2}s_k(r)(B + B^{\mathrm{T}}) + \tfrac{1}{2}c_k(r)(B - B^{\mathrm{T}}) + \tfrac{1}{4}\tfrac{1-s_k(r)}{r}\begin{pmatrix}\bar b_{23}-\bar b_{32}\\ \bar b_{31}-\bar b_{13}\\ \bar b_{12}-\bar b_{21}\end{pmatrix}\begin{pmatrix}\bar b_{23}-\bar b_{32}\\ \bar b_{31}-\bar b_{13}\\ \bar b_{12}-\bar b_{21}\end{pmatrix}^{\mathrm{T}}$$

where $r = 1 - \frac{1}{4}(\operatorname{trace} B - 1)^2$ *and* s_k *and* c_k *are the functions defined in subsection 4.1. Note that* $(1 - s_k(r))/r$ *is polynomial in* r.

Proof. The existence of the selfmaps and their properties follow immediately from Lemma 2.1, Theorem 3.1, and Theorem 3.4. In order to determine an explicit formula, let $B = (b_{i\ell})$ be a fixed matrix in SU(3). We want first to solve the equation $B = A\gamma(t)A^{\mathrm{T}}$ for $t \in [0, \frac{\pi}{2}]$ and $A \in \mathrm{SU}(3)$, then to multiply t by an odd integer k and, finally, to determine $B' = A\gamma(kt)A^{\mathrm{T}}$. Symbolically, we illustrate this as follows

$$B \rightsquigarrow (A, t) \rightsquigarrow (A, kt) \rightsquigarrow B'.$$

We will see that it is not necessary to determine the entries of A explicitly (with all their ambiguity). We can stop at an intermediate implicit level and then return.

Let $A = (z, w, v)$ with $z, w, v \in \mathbb{C}^3$. Then $B = A\gamma(t)A^{\mathrm{T}}$ becomes

$$\begin{aligned} b_{ii} &= (z_i^2 + w_i^2)\cos t + v_i^2,\\ b_{i,i+1} &= (z_i z_{i+1} + w_i w_{i+1})\cos t + v_i v_{i+1} - \bar v_{i-1}\sin t,\\ b_{i,i-1} &= (z_i z_{i-1} + w_i w_{i-1})\cos t + v_i v_{i-1} + \bar v_{i+1}\sin t \end{aligned}$$

with indices cyclic modulo 3. It is straightforward to see that

$$\begin{aligned} \cos t &= \tfrac{1}{2}(\operatorname{trace} B - 1),\\ 2\bar v_1 \sin t &= b_{32} - b_{23},\\ 2\bar v_2 \sin t &= b_{13} - b_{31},\\ 2\bar v_3 \sin t &= b_{21} - b_{12},\\ (z_i^2 + w_i^2)\cos t + v_i^2 &= b_{ii},\\ (z_2 z_3 + w_2 w_3)\cos t + v_2 v_3 &= \tfrac{1}{2}(b_{23} + b_{32}),\\ (z_3 z_1 + w_3 w_1)\cos t + v_3 v_1 &= \tfrac{1}{2}(b_{31} + b_{13}),\\ (z_1 z_2 + w_1 w_2)\cos t + v_1 v_2 &= \tfrac{1}{2}(b_{12} + b_{21}). \end{aligned}$$

We obtain

$$\begin{aligned} b'_{11} &= (z_1^2 + w_1^2)\cos(2j+1)t + v_1^2 \\ &= (z_1^2 + w_1^2)s_{2j+1}(\sin^2 t)\cos t + v_1^2 s_{2j+1}(\sin^2 t) + v_1^2(1 - s_{2j+1}(\sin^2 t)) \\ &= b_{11}s_{2j+1}(\sin^2 t) + \tfrac{1}{4}(\bar b_{23} - \bar b_{32})^2 \cdot \tfrac{1-s_{2j+1}(\sin^2 t)}{\sin^2 t} \end{aligned}$$

and analogous expressions for b_{22} and b_{33}. Note that these are polynomials in the matrix entries of B since $\sin^2 t = 1 - \frac{1}{4}(\operatorname{trace} B - 1)^2$ is a polynomial in $\operatorname{trace} B$ and, e. g., $\bar b_{23} = b_{31}b_{12} - b_{11}b_{32}$, since $B \in \mathrm{SU}(3)$. Similarly, we obtain polynomial expressions for the off-diagonal terms of $B' = \psi_{2j+1}(B)$. All in all this leads to the claimed formula. □ □

Corollary 5.3. *For every $m \in \mathbb{N}$ and every odd integer $(2\ell+1)$ there is a coordinate polynomial selfmap $\mathrm{SU}(3) \to \mathrm{SU}(3)$ of degree $4^m(2\ell+1)$.*

Proof. The maps ψ_k are defined for odd k and have degree k. The power maps $\rho_k : A \mapsto A^k$ have degree k^2. □ □

Lemma 5.4. *Let $\phi : \mathrm{SU}(3) \to \mathrm{SU}(3)$ be any map. Then $\deg\phi = 4^m \cdot (\text{odd number})$ for some $m \in \mathbb{N}$.*

Proof. The cohomology ring $H^*(\mathrm{SU}(3);\mathbb{Z})$ is isomorphic to the cohomology ring $H^*(\mathbb{S}^3\times\mathbb{S}^5;\mathbb{Z})$, i.e., to $\Lambda(x,y)$ with generators x in dimension 3 and y in dimension 5. If $\phi : \mathrm{SU}(3) \to \mathrm{SU}(3)$ is any map, we have the commutative diagram

$$\begin{array}{ccc} H^3(\mathrm{SU}(3);\mathbb{Z}_2) & \xrightarrow{\phi^*} & H^3(\mathrm{SU}(3);\mathbb{Z}_2) \\ \downarrow{\scriptstyle \mathrm{Sq}^2} & & \downarrow{\scriptstyle \mathrm{Sq}^2} \\ H^5(\mathrm{SU}(3);\mathbb{Z}_2) & \xrightarrow{\phi^*} & H^5(\mathrm{SU}(3);\mathbb{Z}_2) \end{array}$$

Vertically, the second Steenrod square Sq^2 yields an isomorphism (see [Bd2], p. 469). The diagram thus says that $\phi^*(y)$ is an even multiple of y if and only if $\phi^*(x)$ is an even multiple of x. Hence, $\phi^*(xy)$ is in $4^m(2\ell+1)\cdot xy$ for some $m \in \mathbb{N}$ and $\ell \in \mathbb{Z}$. □ □

Lemma 5.5. *We have $\rho_k^*(x) = kx$ and $\rho_k^*(y) = ky$ while $\psi_k^*(x) = kx$ and $\psi_k^*(y) = y$. Hence, none of the ρ_k is homotopic to any of the ψ_k.*

Proof. For ρ_k we refer to [Ho]. It is easy to see that the subgroup SU(2) embedded in the upper left corner of SU(3) is invariant under ψ_k and ψ_k restricts to a selfmap of SU(2) with degree k. The commutative diagram

$$\begin{array}{ccc} H^3(\mathrm{SU}(3);\mathbb{Z}) & \xrightarrow{\psi_k} & H^3(\mathrm{SU}(3);\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^3(\mathrm{SU}(2);\mathbb{Z}) & \xrightarrow{\times k} & H^3(\mathrm{SU}(2);\mathbb{Z}) \end{array}$$

shows $\psi_k^*(x) = kx$. Since $\deg\psi_k = k$, the generator xy of $H^8(\mathrm{SU}(3;\mathbb{Z})$ is mapped to kxy. Hence, $\psi_k^*(y) = y$. □ □

5.6. **Selfmaps of** SU(4)**.** Let Sp(2) be the subgroup of SU(4) defined by the equation $JC = \bar{C}J$ or, equivalently, by $C + J\bar{C}J = 0$ where

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We consider the action

$$\mathrm{Sp}(2) \times \mathrm{Sp}(2) \times \mathrm{SU}(4) \to \mathrm{SU}(4), \quad (A, B) \bullet C = ACB^{-1}.$$

A normal geodesic $\gamma : \mathbb{R} \to \mathrm{SU}(4)$ is given by

$$\gamma(t) = \exp \begin{pmatrix} it & 0 & 0 & 0 \\ 0 & it & 0 & 0 \\ 0 & 0 & -it & 0 \\ 0 & 0 & 0 & -it \end{pmatrix}.$$

The isotropy groups at $\gamma(t)$ with $t \neq \frac{\pi}{2}\mathbb{Z}$ are given by

$$\Delta(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) = \left\{ \left(\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right), \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \right) \,\middle|\, A,\, B \ 2 \times 2\text{-matrices} \right\},$$

at $\gamma(0)$ by

$$\Delta\mathrm{Sp}(2) = \left\{ \left(\left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right), \left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right) \right) \,\middle|\, A,\, B,\, C,\, D \ 2 \times 2\text{-matrices} \right\},$$

and at $\gamma(\frac{\pi}{2})$ by

$$\tilde{\Delta}\mathrm{Sp}(2) = \left\{ \left(\left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right), \left(\begin{smallmatrix} A & -C \\ -B & D \end{smallmatrix}\right) \right) \,\middle|\, A,\, B,\, C,\, D \ 2 \times 2\text{-matrices} \right\}.$$

Theorem 5.6. SU(4) *admits selfmaps* ψ_k *with degree* k *and Lefschetz number* 0 *for all odd integers* k*. These maps are equivariant with respect to the cohomogeneity one action by* $\mathrm{Sp}(2) \times \mathrm{Sp}(2)$*. An explicit formula is given by*

$$\psi_k(C) = \tfrac{1}{2} s_k(r)(C - J\bar{C}J) + \tfrac{1}{2} c_k(r)(C + J\bar{C}J)$$

where $r = \frac{1}{16}|C + J\bar{C}J|^2$, $|C|^2 = \mathrm{trace}(\bar{C}^{\mathrm{T}} C)$ *and* s_k *and* c_k *are the functions defined in subsection 4.1.*

Proof. The existence of the maps follows immediately from Lemma 2.1, Theorem 3.1, and Theorem 3.4. The formula can be derived as in the case of SU(3) or verified by computing $\psi_k(A\gamma(t)B^{-1}) = A\gamma(kt)B^{-1}$ for $A, B \in \mathrm{Sp}(2)$. □ □

Corollary 5.7. *For every* $m \in \mathbb{N}$ *and every integer* ℓ *there is a coordinate polynomial selfmap* $\mathrm{SU}(4) \to \mathrm{SU}(4)$ *of degree* $8^m(2\ell + 1)$*.*

5.7. **Selfmaps of** SO(n)**.** We consider the action

$$\mathrm{SO}(n-1) \times \mathrm{SO}(n-1) \times \mathrm{SO}(n), \quad (A, B) \bullet C = ACB^{-1}$$

where $\mathrm{SO}(n-1)$ is embedded in the lower right corner of $\mathrm{SO}(n)$. A normal geodesic $\gamma : \mathbb{R} \to \mathrm{SO}(n)$ is given by

$$\gamma(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix}$$

where $\mathbb{1}$ denotes the $(n-2) \times (n-2)$ identity matrix. The isotropy groups at $\gamma(t)$ with $t \neq \pi\mathbb{Z}$ are given by

$$\Delta(\mathrm{SO}(n-2)) = \{(A, A) \mid A \in \mathrm{SO}(n-2)\},$$

at $\gamma(0)$ by

$$\Delta(\mathrm{SO}(n-1)) = \{(A, A) \mid A \in \mathrm{SO}(n-1)\},$$

and at $\gamma(\pi)$ by

$$\tilde{\Delta}(\mathrm{SO}(n-1)) = \{(A, \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_{n-2} \end{pmatrix} A \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_{n-2} \end{pmatrix}) \,\big|\, A \in \mathrm{SO}(n-1)\},.$$

Theorem 5.8. $\mathrm{SO}(n)$ *admits an infinite family of* $\mathrm{SO}(n-1)\times\mathrm{SO}(n-1)$*-equivariant selfmaps* ψ_k *indexed by odd integers* k*. If* n *is even, then* ψ_k *has degree* k*. If* n *is odd, then* ψ_k *has degree* 1*. An explicit formula is given by*

$$\psi_k \begin{pmatrix} a & w^{\mathrm{T}} \\ v & B \end{pmatrix} = s_k(1-a^2) \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} + c_k(1-a^2) \begin{pmatrix} 0 & w^{\mathrm{T}} \\ v & 0 \end{pmatrix} + \frac{1-s_k(1-a^2)}{1-a^2} \begin{pmatrix} 0 & 0 \\ 0 & (1-a^2)B + avw^{\mathrm{T}} \end{pmatrix}$$

where s_k *and* c_k *are the functions defined in subsection 4.1. Note that* $(1-s_k(r))/r$ *is polynomial in* r*.*

Proof. The existence of the maps and their properties follow immediately from Lemma 2.1, Theorem 3.1, and Theorem 3.4. The formulas can be derived as in the case of SU(3) or verified by computing $\psi_k(A\gamma(t)B^{-1}) = A\gamma(kt)B^{-1}$ for $A, B \in \mathrm{SO}(n-1)$. □ □

Corollary 5.9. *For every* $m \in \mathbb{N}$ *and every integer* ℓ *there is a coordinate polynomial selfmap* $\mathrm{SO}(2n) \to \mathrm{SO}(2n)$ *of degree* $2^{nm}(2\ell+1)$*.*

This result is certainly not optimal: SO(4) is diffeomorphic to $\mathrm{SO}(3) \times \mathbb{S}^3$ and hence admits selfmaps of arbitrary degree. The universal covering space of SO(6) is SU(4). This implies that SO(6) does not admit selfmaps of degree 2. We do not know whether SO(6) admits selfmaps of degree 4.

5.8. **Selfmaps of** Spin(7) **and** SO(7)**.** Let $\mathbb{H}$ denote the quaternions and $\mathbb{O}$ the octonions. As usual, we represent the octonions by pairs of quaternions with the multiplication $(u_1, v_1)\cdot(u_2, v_2) = (u_1u_2 - \bar{v}_2v_1, v_2u_1 + v_1\bar{u}_2)$ and the conjugation $\overline{(u,v)} = (\bar{u}, -v)$. Moreover, we identify the imaginary octonions $\mathrm{Im}\,\mathbb{O}$ with $\mathbb{R}^7$ by mapping the orthonormal basis $(0,1)$, $(\mathbf{i},0)$, $(0,\mathbf{i})$, $(\mathbf{j},0)$, $(0,\mathbf{j})$, $(\mathbf{k},0)$, $(0,\mathbf{k})$ of $\mathrm{Im}\,\mathbb{O}$ to the standard basis of $\mathbb{R}^7$. The exceptional Lie group G_2 is the group of automorphisms of the octonions. It is a subgroup of SO(7). The action of G_2 on $\mathbb{R}^7$ is transitive on the unit sphere $\mathbb{S}^6$. With our conventions the isotropy group of the first standard basis vector of $\mathbb{R}^7$ is the $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ in the lower right corner of SO(7), i.e., the set of matrices in SO(6) that commute with

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We now consider the action

$$\mathrm{G}_2 \times \mathrm{G}_2 \times \mathrm{SO}(7) \to \mathrm{SO}(7), \quad (A,B)\bullet C = ACB^{-1}.$$

The orbit through the unit element $\mathbb{1} \in \mathrm{SO}(7)$ clearly is the subgroup $\mathrm{G}_2 \subset \mathrm{SO}(7)$. A normal geodesic $\gamma : \mathbb{R} \to \mathrm{SO}(7)$ with $\gamma(0) = \mathbb{1}$ is given by

$$\gamma(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & D(2t) & 0 & 0 \\ 0 & 0 & D(2t) & 0 \\ 0 & 0 & 0 & D(2t) \end{pmatrix}$$

where $D(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ denotes the standard 2×2 rotation matrix. Note that $\gamma(t)$ is the matrix that corresponds to conjugation with $(\cos t, -\sin t) \in \mathbb{S}^7 \subset \mathbb{O}$ on the imaginary octonions and that $\gamma(t) \in \mathrm{G}_2$ if and only if $t \in \mathbb{Z} \cdot \frac{\pi}{3}$. Hence the Weyl group of this cohomogeneity one action is D_3. The isotropy groups at $\gamma(t)$ are given by

$$\begin{aligned} \Delta(\mathrm{SU}(3)) &= \{(A, A) \mid A \in \mathrm{SU}(3)\}, && \text{for } t \neq \mathbb{Z} \cdot \tfrac{\pi}{6}, \\ \Delta_\ell \mathrm{G}_2 &= \{(A, \gamma(-\ell\tfrac{\pi}{3}) A \gamma(j\ell\tfrac{\pi}{3})) \mid A \in \mathrm{G}_2\}, && \text{for } t = \ell\tfrac{\pi}{3}, \\ \Delta_\ell \widehat{\mathrm{SU}(3)} &= \{(A, \gamma(-\ell\tfrac{\pi}{3}) A \gamma(\ell\tfrac{\pi}{3})) \mid A \in \widehat{\mathrm{SU}(3)}\}, && \text{for } t = \tfrac{\pi}{2} + \ell\tfrac{\pi}{3} \end{aligned}$$

where $\widehat{\mathrm{SU}(3)} = \mathrm{G}_2 \cap \mathrm{O}(6)$ is the $\mathbb{Z}_2$-extension of $\mathrm{SU}(3)$ in G_2 by the matrix

$$\mathrm{diag}(-1, 1, -1, 1, -1, 1, -1).$$

We see that the isotropy group at $t = \frac{\pi}{2}$ is a subgroup of the isotropy group at $t = 0$.

The action of $\mathrm{G}_2 \times \mathrm{G}_2$ on $\mathrm{SO}(7)$ lifts to the action

$$\mathrm{G}_2 \times \mathrm{G}_2 \times \mathrm{Spin}(7) \to \mathrm{Spin}(7), \quad (A, B) \bullet C = ACB^{-1}.$$

The subgroup $\mathrm{G}_2 \subset \mathrm{SO}(7)$ lifts to two copies of G_2 in $\mathrm{Spin}(7)$ (a subgroup and translated copy). Let $\tilde{\gamma}$ be the lift of γ through the unit element. Then the isotropy groups at $\tilde{\gamma}(t)$ are given by

$$\begin{aligned} \Delta(\mathrm{SU}(3)) &= \{(A, A) \mid A \in \mathrm{SU}(3)\}, && \text{for } t \neq \mathbb{Z} \cdot \tfrac{\pi}{3}, \\ \Delta_\ell \mathrm{G}_2 &= \{(A, \gamma(-\ell\tfrac{\pi}{3}) A \gamma(j\ell\tfrac{\pi}{3})) \mid A \in \mathrm{G}_2\}, && \text{for } t = \ell\tfrac{\pi}{3}. \end{aligned}$$

Note that $\tilde{\gamma}$ has period 2π. Hence, the Weyl group of the action on $\mathrm{Spin}(7)$ is also D_3. The isotropy group at $\tilde{\gamma}(t)$ equals the isotropy group at $\tilde{\gamma}(t + \pi)$.

Theorem 5.10. $\mathrm{SO}(7)$ *and* $\mathrm{Spin}(7)$ *admit infinite familes of* $\mathrm{G}_2 \times \mathrm{G}_2$*-equivariant selfmaps* ϕ_k *indexed by integers of the form* $k = 3j + 1$*. The maps* ϕ_k *have degree* k *and Lefschetz number* 0.

Of course, we can compose the maps ϕ_k with the orientation reversing diffeomorphism $C \mapsto C^{-1}$ of $\mathrm{SO}(7)$ and obtain maps of degree $3j - 1$. Since the rank of $\mathrm{SO}(7)$ is 3 the power maps $C \mapsto C^m$ have degree m^3.

Corollary 5.11. *All integers except possibly* $\pm 3^{3\ell+1}$ *and* $\pm 3^{3\ell+2}$ *for* $\ell \geq 0$ *appear as degrees of selfmaps of* $\mathrm{SO}(7)$ *and* $\mathrm{Spin}(7)$.

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