HOMOGENEITY RANK AND ATOMS OF ACTIONS

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Abstract. We introduce two related concepts for smooth actions of compact Lie groups: The homogeneity rank is a simple numerical invariant of the action. As one of our results we determine the precise range of this invariant for isometric actions on compact Riemannian manifolds with positive sectional curvature and exhibit special properties of the actions with maximal homogeneity rank. Atoms are special components of fixed point sets. They inherit actions with the same cohomogenity and homogeneity rank as the original action, but with trivial principal isotropy group. Other properties of the original action like polarity are reflected in the atoms. We determine the atoms in some interesting concrete cases. Not only for this purpose we give a detailed treatise on the structure of fixed point sets, in particular in cohomogeneity one manifolds.

Introduction

The most visuable measure for the amount of symmetry that is induced by a smooth action of a compact Lie group G on a manifold M is the cohomogeneity $\mathrm{cohom}(M, G)$ of the action, i.e., the codimension of a principal orbit or, equivalently, the dimension of the orbit space M/G. The cohomogeneity measures how much the action deviates from a transitive action.

In this paper we consider another numerical invariant of the action that we call the *homogeneity rank* of the action. It is defined by

$$\begin{aligned} \mathrm{rk}(M, G) &:= (\operatorname{rank} G - \operatorname{rank} H) - \mathrm{cohom}(M, G) \\ &= (\operatorname{rank} G - \operatorname{rank} H) + (\dim G - \dim H) - \dim M, \end{aligned}$$

where H denotes a principal isotropy group of the action. The homogeneity rank has the two basic properties that $\mathrm{rk}(M, G') \leq \mathrm{rk}(M, G)$ if G' is a closed subgroup of G and that

$$\mathrm{rk}(M_1 \times M_2, G_1 \times G_2) = \mathrm{rk}(M_1, G_1) + \mathrm{rk}(M_2, G_2)$$

for direct products of actions.

Theorem A. *For actions on manifolds of a fixed dimension n we have*

$$\mathrm{rk}(M^n, G) \in \{-n, -n+2, \ldots, n-2, n\}$$

and each of these values is attained for an effective action of a torus T^k on the torus T^n. If $G \times M^n \to M^n$ is effective and G is connected then:

- $\mathrm{rk}(M^n, G) = -n$ *if and only if* $G = \{\mathbb{1}\}$.

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- $\mathrm{rk}(M^n, G) = -n + 2$ *implies that* $G = \mathsf{T}^1$ *or* $G = \mathsf{SO}(3)$. *In the latter case the principal orbit is* $\mathbb{S}^2$ *or* $\mathbb{RP}^2$.
- $\mathrm{rk}(M^n, G) = n - 2$ *implies that the action is transitive or that* $G = \mathsf{T}^{n-1}$ *acts with cohomogeneity one. In the latter case the action is weakly equivalent to the direct product of the standard action of* T^{n-3} *on itself and a cohomogeneity one action of* T^2 *on a manifold* N^3.
- $\mathrm{rk}(M^n, G) = n$ *if and only if the torus* T^n *acts transitively on itself.*

Our motivation to introduce the homogeneity rank as an invariant of the action comes from an inequality of Bredon (see [Br], Theorem IV.5.3) where the homogeneity rank (without having any name) appears on the right hand side:

Theorem (Bredon)**.** *If the fixed point set* M^T *of a maximal torus* T *of* G *is nonempty then* $\dim M^T \leq -\mathrm{rk}(M, G)$.

Note that we can substitute the fixed point set of a maximal torus by the fixed point set of a generic element in G, i.e., an element that generates a maximal torus. Thus, if $\mathrm{rk}(M, G) > 0$ then generic transformations do not have fixed points. If $\mathrm{rk}(M, G) = 0$ generic transformations can only have isolated fixed points. Adding a few more simple arguments we can derive the following statement on the Euler characteristic of compact manifolds M:

Theorem B. *If* M *is compact and if* $\mathrm{rk}(M, G) \geq 0$ *then*

$$\chi(M) = \#M^T = \sum_{\text{all } G\text{-orbits } \mathcal{O}} \chi(\mathcal{O}),$$

where M^T *denotes the fixed point set of a maximal torus* T.

This theorem is a direct generalization of a classical result of Hopf and Samelson [HS] on the Euler characteristic of a homogeneous space. It also generalizes the formula for the Euler characteristic of an even-dimensional cohomogeneity one manifold (see [AP]).

Theorem B can also be read in the way that negative Euler characteristic obstructs actions with $\mathrm{rk}(M, G) \geq -1$ and positive Euler characteristic obstructs actions with $\mathrm{rk}(M, G) \geq 1$. The following result shows that the geometric property 'positive sectional curvature' (as opposed to nonnegative sectional curvature) restricts the range of the homogeneity rank for isometric actions as well:

Theorem C. *Let* M *be a compact Riemannian manifold with positive sectional curvature. Then for any isometric action* $G \times M \to M$ *of a compact Lie group* G *we have* $\mathrm{rk}(M, G) \leq 1$.

- *If* $\mathrm{rk}(M, G) = 1$ *then all orbits* $G \cdot p$ *with* $\mathrm{rank}\, G_p = \mathrm{rank}\, G - 1$ *are isolated and such orbits exist. All other orbits* $G \cdot p$ *have* $\mathrm{rank}\, G_p \leq \mathrm{rank}\, G - 2$.
- *If* $\mathrm{rk}(M, G) = 0$ *then all orbits* $G \cdot p$ *with* $\mathrm{rank}\, G_p = \mathrm{rank}\, G$ *are isolated and such orbits exist. Their number is bounded from above by the Euler characteristic of* M.

Note that in any fixed dimension n the homogeneity ranks of the standard actions $\mathsf{SO}(k) \times \mathbb{S}^n \to \mathbb{S}^n$, $k \leq n + 1$, attain all values in $\{-n, \ldots, -2, 0\}$ for even n and all values in $\{-n, \ldots, -1, 1\}$ for odd n. Hence, the inequality in Theorem C is optimal.

Theorem C is a generalization of the first part of a theorem of Grove and Searle on the symmetry rank (i.e., the rank of the isometry group) of compact manifolds with positive sectional curvature (see [GS1]): In case of an effective torus action $T\times M \to M$ we have $\mathrm{rk}(M,T) = 2\dim T - \dim M$ and hence $\dim T \leq [(\dim M + 1)/2]$. In the second part of their paper Grove and Searle classify the positively curved manifolds with maximal symmetry rank: These are diffeomorphic to spheres, lense spaces, or complex projective spaces.[1] The generalization of this rigidity result would be the classification of compact manifolds with positive sectional curvature and maximal homogeneity rank, i.e., $\mathrm{rk}(M,G) \in \{0,1\}$. A solution of this problem is beyond the scope of the present paper. The classification list would include all homogeneous spaces of positive sectional curvature, the Eschenburg space E^6, and the Eschenburg and Bazaikin spaces in dimension 7 and 13 that admit cohomogeneity one metrics with positive sectional curvature (cf. [GSZ]).

As a general tool for classification problems of this type we introduce the notion of an *atom* of an action. Atoms are connected components of fixed point sets for which the induced effective actions have trivial principal isotropy group and the same cohomogeneity and homogeneity rank as the action on the ambient space. The name 'atom' is motivated by the fact that if an action has trivial principal isotropy group then there do not exist any components of fixed point sets for which the induced actions have the same cohomogeneity as the action on the ambient space. We will show that atoms exist for all actions and that the induced action on an atom is polar if and only if the action on the ambient space is polar.

A similar and well-known concept is the reduction or the core of an action (see [St], [GS2]). The advantage of atoms is that the induced actions have by definition the same homogeneity rank as the action on the ambient space. In particular, the dimensions of the atoms and the dimension of the ambient space have the same parity. This property allows us, for example, to give a classification of the atoms of isometric cohomogeneity one actions on even-dimensional Riemannian manifolds with positive sectional curvature by Proposition 5.3. Since the induced actions on these atoms are polar, a consequence of this result is that all cohomogeneity one actions on even-dimensional manifolds with positive sectional curvature are polar.

Atoms are also useful to characterize the geometry or topology of concrete spaces, particularly if a space (like a homogeneous space) is constructed together with the action by means of a prospective orbit space, a Lie group, and closed subgroups along the orbit space. In this case the atoms may turn out to be well-known manifolds and may thus provide information about the original space. The main problem here is to determine the structure of the atoms from the construction data of the space. We will solve this problem in the case of cohomogeneity one manifolds.

As examples we will in particular consider a family of seven-dimensional cohomogeneity one manifolds that was recently constructed by Grove and Ziller (see [GZ1]). The members of this family admit invariant Riemannian metrics of non-negative sectional curvature. They can be indexed by two integers $n, l \in \mathbb{Z}$ with even sum. Each space $P_{n,l}$ is a principal $\mathbb{S}^3$-bundle over $\mathbb{S}^4$ whose Euler class is given by the integer $n(2l+1)$. Note that this integer classifies the principal bundle.

Proposition E. *The atoms of each Grove–Ziller space $P_{n,l}$ are all diffeomorphic to the three-dimensional lense space $L(|n|,1)$ if $n \neq 0$ and to $\mathbb{S}^1 \times \mathbb{S}^2$ if $n = 0$.*

[1] Recent results on the Euler characteristics and fundamental groups of positively curved manifolds with almost maximal symmetry rank are due to Rong (see [R2]).

A consequence of this result is that the finitely many Grove–Ziller spaces $P_{n,l}$ with a fixed nonzero Euler class are distinguished by the fundamental groups of their atoms, i.e., even though the spaces are equivalent as principal bundles, the invariant metrics on them are essentially different. Another consequence is that the spaces $P_{0,l}$ (which are all equivalent to $\mathbb{S}^3 \times \mathbb{S}^4$ as principal bundles) do not admit any invariant metric with positive sectional curvature.

An essential ingredient in the proofs of Theorem C and Bredon's inequality besides the slice theorem is the

Isotropy rank lemma. *Let M be a sphere or, more generally, a Riemannian manifold with positive sectional curvature. Then for any isometric action $G \times M \to M$ there is a point $p \in M$ where $\operatorname{rank} G_p = \operatorname{rank} G$ if M is even-dimensional and a point $p \in M$ where $\operatorname{rank} G_p \geq \operatorname{rank} G - 1$ if M is odd-dimensional.*

Actually, this statement also holds for locally smooth actions on rational homology spheres and in this form it is stated by Bredon (see [Br], Theorem III.10.12). For manifolds with positive sectional curvature the isotropy rank lemma has according to our knowledge first been stated by Karsten Grove and Wolfgang Ziller (see also the survey of Grove [Gr] from which we have taken the name of the lemma).

A large part of the paper consists of a detailed treatise on the structure of fixed point sets in manifolds with actions of a certain cohomogeneity and/or homogeneity rank. When we prove Proposition E we demonstrate how this structure can be determined explicitely in cohomogeneity one manifolds.

The paper is organized as follows: In Section 1 we will combine the isotropy rank lemma with the slice theorem in order to prove an inequality for the dimension of the union $M_{(H)}$ of orbits of type (H) where H is any isotropy group of the action. This inequality is a fundamental technical tool in this paper and should be of value in other contexts as well. Section 2 is devoted to fixed point sets of isometries in complete Riemannian manifolds and the properties of the induced action on the components of the fixed point set. We will in particular prove the existence of atoms in this section. Section 3 discusses the structure of components of fixed point sets in cohomogeneity one manifolds. Among the examples given in this section are the Grove–Ziller actions of Proposition E. In Section 4 we will deduce Bredon's inequality from the inequality for unions of isotypic orbits in Section 1 and prove Theorem A and Theorem B. In Section 5 we will apply the preceding results to manifolds of positive sectional curvature. In particular, we will prove Theorem C and Proposition D in this section.

Many of the statements in this paper are formulated for isometric actions rather than for smooth actions. Of course, this is no restriction since for any smooth action of a compact Lie group on a (paracompact, Hausdorff) manifold there is an invariant Riemannian metric.

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1. An inequality for unions of isotypic orbits

In this section we will consider isometric actions of a compact Lie group G on a Riemannian manifold M. We will prove an inequality on the dimension of the union $M_{(H)}$ of orbits of type (H) where H is any isotropy group of the action.

We first give the basic inequality that will be improved in Theorem 1.3 below by self–application:

Lemma 1.1. *Let $G \times M \to M$ be an isometric action of a compact Lie group on a Riemannian manifold M with principal isotropy group* p.i.g. *Then*

$$\operatorname{rank} G_p - \operatorname{rank} \text{p.i.g.} \;\leq\; \operatorname{cohom}(M, G)$$

for all $p \in M$.

Proof. We will prove the theorem by induction on $k = \operatorname{cohom}(M, G)$. If $k = 0$ the statement holds obviously. Let $k > 0$ and $p \in M$ be any point. Set $H = G_p$ and let $V = \nu_p(G\cdot p)$ denote the normal space to the orbit $G\cdot p$ at p. It follows from the slice theorem that $\operatorname{cohom}(V, H) = k$ and that $\text{p.i.g.}(V, H) = \text{p.i.g.}(M, G)$. Now H acts on the unit sphere S in V with cohomogeneity $k-1$. By the induction assumption we see that $\operatorname{rank} H_v - \operatorname{rank} \text{p.i.g.} \leq k-1$ for all $v \in S$. By the isotropy rank lemma there exists a $v \in S$ with $\operatorname{rank} H_v \geq \operatorname{rank} H - 1$. Hence, $\operatorname{rank} H - \operatorname{rank} \text{p.i.g.} \leq k$ as desired. □

Corollary 1.2. *Let $G \times V \to V$ be a representation of the compact Lie group G on the real vector space V. Let H denote any isotropy group of this representation. Then*

$$\dim V \;\geq\; \dim G - \dim H + \operatorname{rank} G - \operatorname{rank} H.$$

Proof. It suffices to consider the case where H is a principal isotropy group. The inequality follows immediately from the previous one since the isotropy group of the zero vector is the full group G. □

We will now improve Lemma 1.1. Recall that the connected components of the union $M_{(H)}$ of orbits with type (H) (i.e., the union of orbits for which the isotropy groups are conjugate to the subgroup $H \subset G$) are submanifolds of M that are in general not closed. Their tangent spaces are given by

$$T_pM_{(H)} \;=\; T_p(G\cdot p) \,\oplus\, \big(\nu_p(G\cdot p)\big)^{G_p}, \tag{1}$$

where $\big(\nu_p(G\cdot p)\big)^{G_p}$ denotes the fixed point set of G_p in the normal space $\nu_p(G\cdot p)$ to the orbit $G\cdot p$. The dimension of the right summand in this decomposition is the dimension of the local projection of $M_{(H)}$ to the orbit space M/G. It is clear that if this projection has the full dimension of the orbit space M/G then $G\cdot p$ is a principal orbit. The following theorem makes this more quantitative:

Theorem 1.3. *Let $G \times M \to M$ be an isometric action of the compact group G on a Riemannian manifold M. Then for any $p \in M$ we have*

$$\dim\big(\nu_p(G\cdot p)\big)^{G_p} \;\leq\; \operatorname{cohom}(M, G) - (\operatorname{rank} G_p - \operatorname{rank} \text{p.i.g.}).$$

Hence, for the union $M_{(H)}$ of orbits of type (H) we have

$$\dim M_{(H)} \;\leq\; \dim M - (\dim H - \dim \text{p.i.g.}) - (\operatorname{rank} H - \operatorname{rank} \text{p.i.g.}),$$

where p.i.g. *denotes the principal isotropy group of the action.*

Proof. Set $H = G_p$ and $V = \nu_p(G \cdot p)$ and let W be the orthogonal complement of V^H in V. Then $\mathrm{p.i.g.}(W, H) = \mathrm{p.i.g.}(V, H) = \mathrm{p.i.g.}(M, G)$. We apply Corollary 1.2 to the representation $H \times W \to W$ and get

$$\dim W \;\geq\; \dim H - \dim \mathrm{p.i.g.} + \operatorname{rank} H - \operatorname{rank} \mathrm{p.i.g.}$$

and hence

$$\begin{aligned} \dim V^H &= \dim V - \dim W \\ &= \dim M - (\dim G - \dim H) - \dim W \\ &\leq \operatorname{cohom}(M, G) - (\operatorname{rank} H - \operatorname{rank} \mathrm{p.i.g.}). \end{aligned}$$

The second inequality follows from the first one and from (1). □

2. Fixed point sets

We consider an isometric action $G \times M \to M$ of a compact Lie group G on a complete Riemannian manifold M. We are interested in the structure of the components of fixed point sets

$$M^K = \{p \in M \mid g \cdot p = p \text{ for all } g \in K\}$$

where K is a subgroup of G. There are three basic facts that will be used frequently below: First, we have $M^K = M^{\bar{K}}$ where $\bar{K}$ denotes the closure of K in G. This fact will frequently be applied to subgroups of G that are generated by one element. Second, the normalizer $N(K) \subset G$ acts naturally on M^K and hence its identity component $N_0(K)$ acts naturally on each component of M^K. Third, the components of M^K are complete totally geodesic submanifolds of M. Thus each component is determined by a point and its tangent space at this point. For these and other standard facts about fixed point sets we refer to [Br] and [Ko].

2.1. **Transitive actions.**

If the action of the group is transitive then the following result is known:

Theorem 2.1 (see [Br], Corollary II.5.7)**.** *Let $K \subset H \subset G$ be compact Lie groups. Then the orbit space of the $N(K)$-action on $(G/H)^K$ is finite.*

The proof of this theorem is not constructive. Only in special cases the complete structure of the fixed point set is given explicitely. For example, the fixed point set of H in G/H is $N(H)/H$ and the fixed point set of a maximal torus T of H in G/H is $\big(N(T) \cdot H\big)/H \approx N(T)/(N(T) \cap H)$. In general, the components of $(G/H)^K$ can differ even by dimension.

2.2. **The infinitesimal structure of fixed point sets.**

Since each component of a fixed point set M^K is determined by a point and its tangent space at this point it is sufficient to discuss the infinitesimal structure of the component. In order to do this, we first give a technical definition: For any subset $A \subset M$ we set

$$A_0 := \{p \in A \mid A \subset M_{(G_p)} \text{ in some neighborhood of } p\}.$$

Note that M_0 just denotes the union of principal G-orbits and that $A \cap M_0 \subset A_0$.

Lemma 2.2. *The subset A_0 is open and dense in A.*

Proof. The set A_0 is open by construction. Since there are only finitely many orbit types in a sufficiently small distance tube around any orbit, we can find a point $p \in A$ where the isotropy group is locally minimal (i.e., if there is a point $q \in A$ close to p such that G_q can be conjugated into G_p then G_q and G_p are already conjugate) in an arbitrary small neighborhood of any point of A. By the slice theorem we have $A \subset M_{(G_p)}$ in a neighborhood around p and therefore $p \in A_0$. Hence, A_0 is dense. □

In any point $p \in M^K$ the action of $K \subset G_p$ on T_pM clearly leaves $T_p(G \cdot p)$ invariant and therefore also the normal space $\nu_p(G \cdot p)$. Hence,

$$T_pM^K = \big(T_p(G \cdot p)\big)^K \oplus \big(\nu_p(G \cdot p)\big)^K.$$

We identify the two summands in this decomposition:

Lemma 2.3. *We have*

$$\big(T_p(G \cdot p)\big)^K = T_p\big(N(\bar{K}) \cdot p\big) \text{ for all } p \in M^K,$$
$$\big(\nu_p(G \cdot p)\big)^K = \big(\nu_p(G \cdot p)\big)^{G_p} \text{ for all } p \in (M^K)_0,$$

where $\nu_p(G \cdot p)$ denotes the normal space to the orbit $G \cdot p$.

Proof. The first identity follows immediately from Theorem 2.1. By definition we have $M^K \subset M_{(G_p)}$ in a neighborhood of any point $p \in (M^K)_0$, hence $T_pM^K \subset T_pM_{(G_p)}$. This gives us the inclusion '$\subset$' in the second identity. The converse inclusion follows from $K \subset G_p$. □

2.3. **Properties of the induced action.** We will now derive some immediate consequences of Lemma 2.3 and the inequality for unions of isotypic orbits in Section 1.

Corollary 2.4. *Suppose $M^K \neq \emptyset$. Then for the action of $N_0(\bar{K})$ on each connected component V of M^K the following holds:*

1. *The set V_0 is a subset of the union of all principal $N_0(\bar{K})$-orbits in V.*
2. *For all $p \in V_0$ we have*
$$\operatorname{cohom}\big(V, N_0(\bar{K})\big) \leq \operatorname{cohom}(M, G) - \big(\operatorname{rank} G_p - \operatorname{rank} \text{p.i.g.}(M, G)\big).$$
3. *We have $\operatorname{cohom}\big(V, N_0(\bar{K})\big) = \operatorname{cohom}(M, G)$ if and only if $V \cap M_0 \neq \emptyset$.*
4. *If $\operatorname{rank} N(\bar{K}) = \operatorname{rank} G$ (e.g., if $\bar{K}$ is contained in a maximal torus of G) then*
$$\operatorname{rk}\big(V, N_0(\bar{K})\big) \geq \operatorname{rk}(M, G).$$
5. *If $V \cap M_0 \neq \emptyset$ then the natural map $V/N_0(\bar{K}) \to M/G$ is surjective.*

Proof. It follows from the decomposition of T_pM^K in Lemma 2.3 that for any $p \in (M^K)_0$ the isotropy group acts trivially on the normal space. Hence, the orbit $N_0(\bar{K}) \cdot p$ is a principal orbit in V.

The second and third statements are straightforward consequences of Theorem 1.3 and the discussion preceding it. The fourth statement follows from the first two since $\operatorname{rank} N(\bar{K}) = \operatorname{rank} G$ and $\operatorname{rank} \text{p.i.g.}\big(V, N_0(\bar{K})\big) \leq \operatorname{rank} G_p$ for $p \in V_0$. The fifth statement can be deduced from Lemma 2.3 and Lemma 2.12. □

Proposition 2.5. *Suppose $V \cap M_0 \neq \emptyset$ for some component V of M^K and let W denote any component of $V \cap M_0$. Then the natural map $\phi : W/N_0(\bar{K}) \to M_0/G$ is a Riemannian covering map with finitely many sheets.*

Proof. First note that the fiber of the map ϕ is finite over each point in M/G because of Theorem 2.1. Note also that $W/N_0(\bar{K})$ inherits a canonical Riemannian manifold structure from W because W consists of principal orbits of the $N_0(\bar{K})$-action on V. Since the horizontal spaces of the two Riemannian submersions $W \to W/N_0(\bar{K})$ and $M \to M/G$ are identical, the map ϕ is a local isometry.

We now need the following lemma that can be found in Kleiner's thesis [Kl]: If $c : [0,1] \to M$ is a shortest curve from the orbit $G \cdot p$ to the orbit $G \cdot q$, then the isotropy groups $G_{c(t)}$ are constant for $0 < t < 1$ and equal to the subgroup of G that fixes all points of c. This means in particular, that M_0 is G-convex in the sense that a shortest curve between the orbits $G \cdot p$ and $G \cdot q$ with $p, q \in M_0$ is entirely contained in M_0.

Using this lemma it follows as in Lemma 2.12 that the composed map $W \hookrightarrow M_0 \to M_0/G$ is surjective, and the proof that ϕ is a Riemannian covering map proceeds as in the case of a local isometry between complete Riemannian manifolds. □

If the map ϕ in the preceding proposition is an isometry (e.g., if M_0/G is simply connected) then its extension $\bar{W}/N_0(\bar{K}) \to M/G$ is an isometry of metric spaces, i.e., M/G is isometric to the closure of an open subset of $V/N_0(\bar{K})$.

Remark 2.6. Note that $N_0(\bar{K})$ is not necessarily the full subgroup of G that leaves a component V of the fixed point set M^K invariant. The full invariance group $I(V) = \{\psi \in G \,|\, \psi(V) = V\}$ is the normalizer of the subgroup $F(V)$ of all elements of G that fix V pointwise. Since V is totally geodesic, $F(V)$ is easy to determine in the tangent space of any point $p \in V$ as the subgroup of the isotropy group G_p that fixes all $v \in T_pV$. Corollary 2.4 and Proposition 2.5 remain valid if $N_0(\bar{K})$ is substituted by $I(V)$.

Remark 2.7. There is the important special case where K is generated by one element $\psi \in G$, i.e., $K = \langle\psi\rangle = \{\psi^k \,|\, k \in \mathbb{Z}\}$. If ψ is contained in the identity component G_0 of G then $\bar{K}$ is a finite cyclic group or a torus in G and the centralizer $C(\psi)$ of ψ in G is the union of all maximal tori of G that contain ψ. In particular, $C(\psi)$ is connected, $N_0(\bar{K}) = C(\psi)$, and $\operatorname{rank} C(\psi) = \operatorname{rank} G$.

2.4. **Atoms of actions.** A useful method in the theory of transformation groups is the reduction to trivial principal isotropy group. This reduction is done by passing from the manifold M to its so called core or reduction (see [GS2], [St]). The core is the closure of the fixed point set $(M_0)^H$ in M where H denotes a principal isotropy group for the isometric action $G \times M \to M$. The group $N(H)/H$ acts on the core with trivial principal isotropy group and the orbit space of this action is isometric to M/G.

The disadvantage of this method is that it usually does not preserve the homogeneity rank and even not the parity of the dimension. For manifolds with positive sectional curvature for example the parity of the dimension is very significant. For this reason we suggest a different reduction to trivial principal isotropy group that preserves the homogeneity rank and hence by Theorem A in particular the parity of the dimension.

We first give the natural definition for the object that we are seeking:

Definition 2.8. An *atom* of the G-action on M is a component V of a fixed point set M^K with $V \cap M_0 \neq \emptyset$ and such that the principal isotropy group of the action $I_0(V) \times V \to V$ acts trivially on V and $\mathrm{rk}(V, I_0(V)) = \mathrm{rk}(M, G)$. Here, $I_0(V)$ denotes the identity component of the subgroup $I(V)$ of G that leaves V invariant.

Thus, if V is an atom and we pass to the effective action corresponding to $I_0(V) \times V \to V$ then the principal isotropy group of this effective action is trivial. Note that by Corollary 2.4 the action $I_0(V) \times V \to V$ has the same cohomogeneity as the action $G \times M \to M$. The orbit spaces of these two actions need not be isometric. However, by Proposition 2.5 they are closely related.

Examples 2.9. The atoms of a compact homogeneous space G/H are compact Lie groups whose rank is $\mathrm{rank}\, G - \mathrm{rank}\, H$. For any isometric torus action $T \times M \to M$ the only atom is M itself.

We will now show that atoms always exist. In order to do this we fix a point $p \in M_0$ and construct a finite sequence of subactions $G_j \times V_j \to V_j$ of the action $G \times M \to M$ with $p \in V_j$. The isotropy groups of p for these subactions will be denoted by H_j.

Algorithm 2.10. Set $V_1 = M$ and $G_1 = G_0$, the identity component of G. Until H_j acts trivially on V_j proceed inductively in the following way:

- Choose an element $\psi_j \in H_j$ that acts nontrivially on V_j, if possible from the identity component of H_j.
- Let V_{j+1} be the component of $V_j^{\psi_j}$ that contains p and let $G_{j+1} = I_0(V_{j+1})$ be the identity component of the subgroup of G that consists of all elements which leave V_{j+1} invariant.

Note that V_{j+1} is the component of $M^{\langle \psi_1, \dots, \psi_j \rangle}$ that contains p.

Lemma 2.11. *Each sequence of subactions $G_j \times V_j \to V_j$ obtained by Algorithm 2.10 has the following elementary properties:*

1. *The length l of the sequence does not exceed $\dim M$ since $dim V_{j+1} < \dim V_j$.*
2. *The inclusion maps $V_j \hookrightarrow M$ are equivariant totally geodesic embeddings.*
3. $\mathrm{cohom}(V_j, G_j) = \mathrm{cohom}(M, G)$ *for all j.*
4. *The orbit $G_j \cdot p$ is a principal orbit of each action $G_j \times V_j \to V_j$.*
5. $\mathrm{rank}\, G_j = \mathrm{rank}\, G$ *and* $\mathrm{rank}\, H_j = \mathrm{rank}\, H$ *for all j.*
6. $\mathrm{rk}(V_j, G_j) = \mathrm{rk}(M, G)$ *for all j.*
7. *The final submanifold V_l is an atom of the action $G \times M \to M$.*

Proof. We will only prove the fifth statement, the other statements are obvious or direct consequences of Corollary 2.4. Since G_j is connected, the centralizer $C_{G_j}(\psi_j)$ of ψ_j in G_j is the union of all the maximal tori of G_j that contain ψ_j. Hence, $\mathrm{rank}\, G_{j+1} = \mathrm{rank}\, G_j$ since $C_{G_j}(\psi_j)$ leaves V_{j+1} invariant. If ψ_j is in the identity component of H_j then by the analog reasoning we have $\mathrm{rank}\, H_{j+1} = \mathrm{rank}\, H_j$. If the identity component $(H_j)_0$ of H_j acts trivially on V_j, then clearly $(H_j)_0 \subset H_{j+1}$ and hence we have $\mathrm{rank}\, H_{j+1} = \mathrm{rank}\, H_j$ as well. □

In concrete cases atoms may be easier to find than with Algorithm 2.10. In particular, it is often not necessary to calculate the full invariance groups.

2.5. **Polar actions.** An isometric action $G \times M \to M$ of a compact Lie group G on a complete Riemannian manifold M is called polar, if there is a properly embedded submanifold (called section) that meets all orbits and intersects all of them perpendicularly. A section is automatically totally geodesic.

The following lemma is a well-known partial converse:

Lemma 2.12. *Let V be a complete totally geodesic submanifold of M with $T_pV = \nu_p(G \cdot p)$ for some $p \in V$. Then V meets all G-orbits.*

Proof. Consider any other orbit $G \cdot q$ and suppose that q is chosen such that a minimal geodesic c between $G \cdot p$ and $G \cdot q$ that starts in p ends in q. Then $\dot{c}(0) \in T_pV$ by the first variation formula. Since V is totally geodesic it follows that $q \in V$. □

Using our discussion of the infinitesimal structure of fixed point sets it is easy to reprove the following simple criterion for polarity (see [PT], Corollary 5.5). We will need this criterion in Section 5.

Proposition 2.13 (Palais-Terng)**.** *Let $G \times M \to M$ be an isometric action for which* $\operatorname{rank} G = \operatorname{rank} \text{p.i.g.}(M, G)$*, i.e.,* $\operatorname{rk}(M, G) = -\operatorname{cohom}(M, G)$*. Then the action is polar.*

Proof. Let T be a maximal torus of G. Then $N(T) \cdot p$ is discrete for all $p \in M^T$. It follows from Lemma 2.3 that $T_pM^T \subset \nu_p(G \cdot p)$ for all $p \in M^T$. Now let V be a component of M^T that has a nonempty intersection with the union of the principal orbits M_0 and let $p \in V \cap M_0$. Since the orbit through p is principal, the isotropy group G_p acts trivially on the normal space to the orbit. Hence, $T_pV = \nu_p(G \cdot p)$. Therefore, V meets all orbits by the preceding lemma. □

The component V of M^T is a section for the action $G \times M \to M$ and it clearly is an atom of the action $G \times M \to M$ as well.

Examples of polar actions where the ranks of all isotropy groups are the same are numerous. Among them are the actions of connected compact Lie groups on themselves by conjugation.

We will now show that whether any given action is polar or not can be decided by inspecting an atom of the action (or a more general fixed point set).

Proposition 2.14. *Let $G \times M \to M$ be an isometric action and K be a closed subgroup of G such that $V \cap M_0 \neq \emptyset$ for some component V of M^K. Then the action $G \times M \to M$ is polar if and only if the action $N_0(\bar{K}) \times V \to V$ is polar.*

Proof. Suppose that Σ is a section of the action $N_0(\bar{K}) \times V \to V$ that passes through $p_0 \in V \cap M_0$. Then $T_{p_0}\Sigma = \nu_{p_0}(G \cdot p_0)$ by Lemma 2.3 and hence Σ meets all G-orbits in M by Lemma 2.12. For arbitrary $p \in \Sigma$ we have

$$T_p\Sigma \subset \nu_p(N_0(\bar{K}) \cdot p) \cap T_pV = \nu_p(G \cdot p)^K \subset \nu_p(G \cdot p)$$

by Lemma 2.3 and hence Σ intersects all G-orbits perpendicularly.

Conversely, let Σ be a section of the action $G \times M \to M$ that passes through $p_0 \in V \cap M_0$. Then

$$T_{p_0}\Sigma = \nu_{p_0}(G \cdot p_0) = \nu_{p_0}(G \cdot p_0)^K \subset T_{p_0}V$$

and hence $\Sigma \subset V$ since V and Σ are totally geodesic. The equation above also says that the normal space to the orbit $N_0(\bar{K}) \cdot p_0$ in V is equal to $T_{p_0}\Sigma$. Hence, Σ meets all $N_0(\bar{K})$-orbits in V and it is clear that it intersects all of them perpendicularly, since it even intersects the G-orbits perpendicularly. □

Remark 2.15. Note that recent progress (cf. [HLO], [GZ2]) suggests to call an action polar if there exists an immersed submanifold that meets all orbits perpendicularly. Proposition 2.14 clearly remains true for this weakened notion of polarity.

3. Fixed point sets in cohomogeneity one manifolds

In this section we will show how the structure of the components of fixed point sets in cohomogeneity one manifolds can be determined by the results of the previous section. We will only consider the most complicated typ of actions here, namely those, where the orbit space is a closed interval. At the beginning, we will summarize the facts that are relevant for this section and Section 5. As references for these facts we refer to [Mo], [Nm], [Br], [AA].

3.1. **Preliminaries.** Let M be a compact Riemannian manifold and G be a compact Lie group that acts isometrically on M such that M/G is a closed interval. The orbits that project to the boundary points of the interval are singular or exceptional and the others are principal. Any shortest curve from one non-principal orbit to the other extends to a geodesic $\mathbb{R} \to M$ that intersects each orbit perpendicularly. We will call such a geodesic a Cartan geodesic. The group G clearly acts transitively on the set of Cartan geodesics. We fix a Cartan geodesic and denote the principal isotropy group along this geodesic by H and the isotropy groups of two adjacent intersection points p_-, p_+ with non-principal orbits by H_- and H_+. The subgroups H_- and H_+ act transitively on the unit spheres S_-, S_+ in the normal slices at p_-, p_+, respectively, and the groups $N_{H_-}(H)/H$ and $N_{H_+}(H)/H$ act simply transitively on the fixed point sets S_-^H and S_+^H. Consequently, $N_{H_-}(H)/H$ and $N_{H_+}(H)/H$ are isomorphic to $\mathbb{S}^0 = \mathbb{Z}_2$, $\mathbb{S}^1$, or $\mathbb{S}^3$, and there exist unique involutions σ_-, σ_+ in these groups. The two involutions $\sigma_-, \sigma_+ \in N(H)/H$ generate the Weyl group, i.e., the group of elements of G that leave the Cartan geodesic invariant modulo the group of elements of G that fix the Cartan geodesic pointwise. If the Weyl group is the dihedral group D_k then the Cartan geodesic is an embedding of a circle with length $2k \operatorname{diam}(M/G)$ and k is the number of intersection points with each non-principal orbit. If the Weyl group is the dihedral group D_∞ then the Cartan geodesic is an injective immersion of $\mathbb{R}$. All singular isotropy groups along the Cartan geodesic are given by $H_-^j = \rho^j H_- \rho^{-j}$ and $H_+^j = \rho^j H_+ \rho^{-j}$ where $j \in \mathbb{Z}$ and $\rho := \sigma_+ \circ \sigma_-$ is a primitive rotation/translation in the Weyl group.

The manifold M is the union of the two closed distance tubes M_- and M_+ around the two singular orbits with radius $r = \operatorname{diam}(M/G)/2$. The two parts M_- and M_+ intersect in their common boundary, i.e., in the principal orbit that projects to the midpoint of the interval. By the slice theorem M_- and M_+ are equivariantly diffeomorphic to $G \times_{H_-} D_-$ and $G \times_{H_+} D_+$ where D_- and D_+ denote the disks with radius r in the normal slices at p_- and p_+. The identification between the two boundaries $G \times_{H_-} S_-$ and $G \times_{H_+} S_+$ coming from M is given by $[g, v_-] \sim [g, v_+]$ where v_- denotes the tangent vector of the fixed Cartan geodesic at p_- and v_+ the negative tangent vector at p_+.

Now let G be a compact Lie group G and $H \subset H_-, H_+ \subset G$ three closed subgroups such that H_-/H and H_+/H are spheres of dimension l_-, l_+, respectively.

From the classification of transitive actions on spheres it follows that there are unique orthogonal representations $H_- \times \mathbb{R}^{l_-+1} \to \mathbb{R}^{l_-+1}$, $H_+ \times \mathbb{R}^{l_++1} \to \mathbb{R}^{l_++1}$ such that the isotropy groups of some vectors $v_- \in \mathbb{S}^{l_-}$, $v_+ \in \mathbb{S}^{l_+}$ are H. If we identify the boundaries of the two unit disk bundles $G \times_{H_-} D_-^{l_-+1}$ and $G \times_{H_+} D_+^{l_++1}$ by an arbitrary equivariant diffeomorphism, i.e., by the map $[g, v_-] \mapsto [g, nv_+]$ for some $n \in N(H)$, then the equivalence classes (with respect to equivariant diffeomorphism) of the resulting cohomogeneity one manifolds correspond to the connected components of the double coset space (see [Nm])

$$N(H) \cap N(H_-) \setminus N(H) \,/\, N(H) \cap N(H_+).$$

In particular, the cohomogeneity one manifolds obtained by the glueing process do not depend on the choices of $v_- \in (\mathbb{S}^{l_-})^H$ and $v_+ \in (\mathbb{S}^{l_+})^H$.

In the following we will always associate to a tupel (G, H, H_-, H_+) as above the manifold M that is obtained by the identification $[g, v_-] \mapsto [g, v_+]$. It is easy to see that there exist G-invariant metrics on M for which the curve given by $t \mapsto [\mathbb{1}, tv_-] \in G \times_{H_-} D_-$ for $0 \le t \le 1$ and $t \mapsto [\mathbb{1}, (2-t)v_+] \in G \times_{H_+} D_+$ extends to a Cartan geodesic.

If G is connected and $H = \{\mathbb{1}\}$ (i.e., if the manifold is an atom) then the cohomogeneity one manifolds associated to $(G, \mathbb{1}, H_-, H_+)$ and $(G, \mathbb{1}, H'_-, H'_+)$ are equivariantly diffeomorphic if and only if H'_- is conjugate to H_- in G and H'_+ is conjugate to H_+ (or the same holds if H'_- and H'_+ are interchanged). This will be used in the proof of Proposition 5.3. Note that the application of an external automorphism of G to the tupel $(G, \mathbb{1}, H_-, H_+)$ induces a weak equivalence of the associated manifolds.

3.2. **The structure of components of fixed point sets.** Let M be a compact Riemannian manifold and $G \times M \to M$ be an isometric action such that M/G is a closed interval. Let $K \subset G$ be a subgroup with $M^K \neq \emptyset$. By Corollary 2.4 the group $\tilde{G} := N_0(\bar{K})$ acts on any component V of M^K either transitively or with cohomogeneity one. In the first case, V is located in one of the two non-principal orbits and the structure of V is clearly determined by $\tilde{G}$ and any of its isotropy groups in V. In the second case V contains regular points, i.e., points in the union M_0 of principal G-orbits. We fix a point $p \in V \cap M_0$. By Lemma 2.3 the totally geodesic submanifold V contains the Cartan geodesic through p. Of course, this geodesic is also a Cartan geodesic for the action $\tilde{G} \times V \to V$. Using Corollary 2.4, we see that the points of the Cartan geodesic that are regular with respect to the G-action are regular with respect to the $\tilde{G}$-action as well. However, singular or exceptional points with respect to the G-action may become regular points for the $\tilde{G}$-action, i.e., the orbit space $V/\tilde{G}$ can be larger than M/G. There are now two different cases:

1. (1) There are non-principal isotropy groups along the Cartan geodesic with respect to the $\tilde{G}$-action. In this case V can be reconstructed as described above from $\tilde{G}$, the principal isotropy group $\tilde{H} = \tilde{G} \cap G_p$, and two adjacent non-principal isotropy groups $\tilde{H}_-$ and $\tilde{H}_+$.
2. (2) All points on the Cartan geodesic have the same isotropy group with respect to the $\tilde{G}$-action. In this case the orbit space of the action $\tilde{G} \times V \to V$ is a circle, and $V \to V/\tilde{G} = \mathbb{S}^1$ is a fiber bundle with fiber $\tilde{G}/\tilde{H}$. These bundles are classified by the components of $N_{\tilde{G}}(\tilde{H})/\tilde{H}$. The relevant component

for V is determined by an element $\tilde{\tau} \in N_{\tilde{G}}(\tilde{H})/\tilde{H}$ which is given as follows: Set $k_0 = \inf\{k \in \mathbb{N} \mid \rho^k \in N_{\tilde{G}}(\tilde{H})/\tilde{H}\}$. If $k_0 < \infty$ then $\tilde{\tau} = \rho^{k_0}$. Otherwise, $\tilde{\tau}$ is the identity in $\tilde{G}$.

In the following examples we will only encounter the first case. However, it is easy to see that the fixed point set of the whole principal isotropy group in each of these examples is just a Cartan geodesic. The identity component of the normalizer is the trivial group and the orbit space clearly is a circle.

3.3. **Principal $\mathbb{S}^3$-bundles over $\mathbb{S}^4$.** Grove and Ziller defined in [GZ1] a family of seven-dimensional cohomogeneity one manifolds $P_{n,l}$ ($n, l \in \mathbb{Z}$ with $n + l$ even) by the following construction data:

$$\begin{aligned} G &= \mathbb{S}^3 \times \mathbb{S}^3, & H_- &= \{(e^{ip_-\theta}, e^{i\theta}) \mid \theta \in \mathbb{R}\} \cup \{(je^{ip_-\theta}, je^{i\theta}) \mid \theta \in \mathbb{R}\}, \\ H &= \Delta\{\pm 1, \pm i, \pm j, \pm k\}, & H_+ &= \{(e^{jp_+\theta}, e^{j\theta}) \mid \theta \in \mathbb{R}\} \cup \{(ke^{jp_+\theta}, ke^{j\theta}) \mid \theta \in \mathbb{R}\}. \end{aligned}$$

Here $p_- = 2n + 2l + 1$, $p_+ = -2n + 2l + 1$ are both in $1 + 4\,\mathbb{Z}$ and Δ denotes the diagonal embedding. Grove and Ziller showed that each of these spaces is a principal $\mathbb{S}^3$-bundle over $\mathbb{S}^4$ whose Euler class is given by the integer $n(2l + 1)$ and that this integer classifies the type of the principal bundle.

Proposition 3.1. *The atoms of each space $P_{n,l}$ are all diffeomorphic to the three-dimensional lense space $L(|n|, 1)$ if $n \neq 0$ and to $\mathbb{S}^1 \times \mathbb{S}^2$ if $n = 0$.*

Proof. We consider the fixed point set of the transformation given by the element $(i, i) \in \mathbb{S}^3 \times \mathbb{S}^3$; the transformations given by (j, j) and (k, k) are treated similarly. The identity component of the normalizer of the subgroup of $\mathbb{S}^3 \times \mathbb{S}^3$ generated by (i, i) is the group $\tilde{G} = \{(e^{i\theta_1}, e^{i\theta_2}) \mid \theta_1, \theta_2 \in \mathbb{R}\}$. The principal isotropy group of the action of $\tilde{G}$ along the Cartan geodesic is $\tilde{H} = \Delta\{\pm 1, \pm i\}$. In order to determine the singular isotropy groups we first have to calculate the singular isotropy groups $H_-^0 = H_-$, $H_+^0 = H_+$, H_-^1, H_+^1, $H_-^2, \ldots$ of the $\mathbb{S}^3 \times \mathbb{S}^3$-action along the Cartan geodesic. The unique elements of order 2 in $N_{H_-}(H)/H$ and $N_{H_+}(H)/H$ are given by $\sigma_- = (e^{ip_-\pi/4}, e^{i\pi/4})$ and $\sigma_+ = (e^{jp_+\pi/4}, e^{j\pi/4})$, respectively. The composition $\rho := \sigma_+ \circ \sigma_-$ corresponds to a rotation by $2\pi/3$ if n is even and $\pi/3$ if n is odd. Using this rotation we obtain:

$$\begin{aligned} H_-^1 &= \rho H_- \rho^{-1} &= \{(e^{kp_-\theta}, e^{k\theta}) \mid \theta \in \mathbb{R}\} \cup \{(ie^{kp_-\theta}, ie^{k\theta}) \mid \theta \in \mathbb{R}\}, \\ H_+^1 &= \rho H_+ \rho^{-1} &= \{(e^{ip_+\theta}, e^{i\theta}) \mid \theta \in \mathbb{R}\} \cup \{(je^{ip_+\theta}, je^{i\theta}) \mid \theta \in \mathbb{R}\}, \ldots \end{aligned}$$

Since $\tilde{G} \cap H_+ = \tilde{G} \cap H_-^1 = \tilde{H}$ are principal isotropy groups for the action of $\tilde{G}$ the data of the component of the fixed point set is given by

$$\begin{aligned} \tilde{G} &= \{(e^{i\theta_1}, e^{i\theta_2}) \mid \theta_1, \theta_2 \in \mathbb{R}\}, & \tilde{H}_- &= \tilde{G} \cap H_- = \{(e^{ip_-\theta}, e^{i\theta}) \mid \theta \in \mathbb{R}\}, \\ \tilde{H} &= \Delta\{\pm 1, \pm i\}, & \tilde{H}_+ &= \tilde{G} \cap H_+^1 = \{(e^{ip_+\theta}, e^{i\theta}) \mid \theta \in \mathbb{R}\}. \end{aligned}$$

In order to determine the data of the corresponding effective action we identify $\tilde{G}$ with $\mathbb{R}^2/(4\mathbb{Z})^2$. The torus that acts effectively is now given by the lattice $\mathbb{Z}\,(4, 0) + \mathbb{Z}\,(0, 4) + \mathbb{Z}\,(1, 1)$. This lattice is generated by the two vectors $\binom{p_-}{1}$ and $\binom{-4}{0}$. The singular directions $\binom{p_-}{1}$ and $\binom{p_+}{1}$ are given by the coordinates $(1, 0)$ and $(1, n)$ with respect to these two vectors. The result follows now from the classification of three-dimensional cohomogeneity one manifolds by Neumann (see [Nm]). $\square$

If $n+l \neq 0$ (i.e., $p_- \neq 1$) then the transformation given by $(1, e^{2\pi i/p_-}) \in H_-$ fixes $\mathbb{S}^3 \times \mathbb{S}^1/\{(e^{ip_-\theta}, e^{i\theta})\} \approx \mathbb{S}^3$ in the singular orbit G/H_-. Analogously, if $n - l \neq 0$ (i.e., $p_+ \neq 1$) then the transformation $(1, e^{2\pi i/p_+}) \in H_+$ fixes an $\mathbb{S}^3$ in G/H_+.

It is an interesting question for which of the spaces $P_{n,l}$ there exists a transformation that fixes an $\mathbb{S}^3$ in each of the singular orbits. With a few elementary computations one can show that this is the case if and only if there exist two numbers $m_- \in \mathbb{Z} \smallsetminus (\mathbb{Z} \cdot p_-)$ and $m_+ \in \mathbb{Z} \smallsetminus (\mathbb{Z} \cdot p_+)$ such that $\frac{m_-}{p_-} + \frac{m_+}{p_+} \in \mathbb{Z}$. If this condition holds then the transformation given by $(1, e^{2\pi i m_-/p_-})$ has the desired two components. An example is the space $P_{5,-3}$ with $p_- = 5$ and $p_+ = -15$ where the transformation given by $(1, e^{2\pi i/5})$ is contained in H_- and in H_+. Note that for any invariant metric on these spaces the $\mathbb{S}^3$ in the one singular orbit is contained in the cut locus of any point of the $\mathbb{S}^3$ in the other singular orbit (see [Ko]).

3.4. **Principal $\mathbb{S}^3 \times \mathbb{S}^3$-bundles over $\mathbb{S}^4$.** Grove and Ziller defined in [GZ1] also a family of ten-dimensional cohomogeneity one manifolds. The members of this family can be indexed by four integers n_1, l_1, n_2, l_2 where both $n_1 + l_1$ and $n_2 + l_2$ are even. Each space $P_{n_1,l_1;n_2,l_2}$ is a principal $\mathbb{S}^3 \times \mathbb{S}^3$-bundle over $\mathbb{S}^4$. The type of this principal bundle is classified by the two integers $n_1(2l_1 + 1)$ and $n_2(2l_2 + 1)$. The construction data for the space $P_{n_1,l_1;n_2,l_2}$ is given by

$$\begin{aligned}
G &= \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3, \\
H &= \Delta\{\pm 1, \pm i, \pm j, \pm k\}, \\
H_- &= \{(e^{ip_-\theta}, e^{iq_-\theta}, e^{i\theta}) \mid \theta \in \mathbb{R}\} \cup \{(je^{ip_-\theta}, je^{iq_-\theta}, je^{i\theta}) \mid \theta \in \mathbb{R}\}, \\
H_+ &= \{(e^{jp_+\theta}, e^{jq_+\theta}, e^{j\theta}) \mid \theta \in \mathbb{R}\} \cup \{(ke^{jp_+\theta}, ke^{jq_+\theta}, ke^{j\theta}) \mid \theta \in \mathbb{R}\}
\end{aligned}$$

where $p_- = 2n_1+2l_1+1$, $p_+ = -2n_1+2l_1+1$, $q_- = 2n_2+2l_2+1$, $q_+ = -2n_2+2_2+1$ are all in $1 + 4\,\mathbb{Z}$.

Proposition 3.2. *The atoms of each space $P_{n_1,l_1;n_2,l_2}$ are all diffeomorphic to the product $\mathsf{T}^2 \times \mathbb{S}^2$ if $n_1 = n_2 = 0$ and to the product $\mathbb{S}^1 \times L(\gcd(|n_1|, |n_2|), 1)$ otherwise.*

Proof. We consider the fixed point set of (i, i, i) only. Analogously to the previous proof the data of the component of the fixed point set is given by

$$\begin{aligned}
\tilde{G} &= \{(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \mid \theta_1, \theta_2, \theta_3 \in \mathbb{R}\}, & \tilde{H}_- &= \{(e^{ip_-\theta}, e^{iq_-\theta}, e^{i\theta}) \mid \theta \in \mathbb{R}\}, \\
\tilde{H} &= \Delta\{\pm 1, \pm i\}, & \tilde{H}_+ &= \{(e^{ip_+\theta}, e^{iq_+\theta}, e^{i\theta}) \mid \theta \in \mathbb{R}\}.
\end{aligned}$$

We identify $\tilde{G}$ with $\mathbb{R}^3/(4\mathbb{Z})^3$. The torus that acts effectively is now given by the lattice

$$\mathbb{Z}\,(4,0,0) + \mathbb{Z}\,(0,4,0) + \mathbb{Z}\,(0,0,4) + \mathbb{Z}\,(1,1,1).$$

This lattice is generated by the three vectors $\begin{pmatrix} p_- \\ q_- \\ 1 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}$. The singular directions $\begin{pmatrix} p_- \\ q_- \\ 1\!\!1 \end{pmatrix}$ and $\begin{pmatrix} p_+ \\ q_+ \\ 1\!\!1 \end{pmatrix}$ are given by the coordinates $(1, 0, 0)$ and $(1, n_1, n_2)$ with respect to this basis. We now use Parker's classification of compact four-dimensional cohomogeneity one manifolds (see [Pa]). If $n_1 = n_2 = 0$ then we can use the classification list directly to see that the atom is $\mathsf{T}^2 \times \mathbb{S}^2$. Otherwise, we have to apply an automorphism of T^3: Set $n_1' := n_1/\gcd(|n_1|, |n_2|)$ and $n_2' :=$

$n_2/\gcd(|n_1|,|n_2|)$. Let a_1, a_2 be integers with $a_1 n_1' + a_2 n_2' = 1$. The automorphism given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & -n_2' & n_1' \end{pmatrix}$$

transforms the coordinates of the singular isotropy groups into the coordinates $(1,0,0)$ and $(1,\gcd(|n_1|,|n_2|),0)$. □

3.5. **The Berger space $B^7 = \mathsf{SO}(5)/\mathsf{SO}(3)$.** The Berger space B^7 is one of the three exceptional homogeneous spaces where the normal homogeneous metric has positive sectional curvature. The embedding of $\mathsf{SO}(3)$ into $\mathsf{SO}(5)$ is given by the action of $\mathsf{SO}(3)$ on the traceless symmetric 3×3-matrices by conjugation. The action of a standard $\mathsf{SO}(4)$ from the left has cohomogeneity one and if one passes to the corresponding ineffective $\mathbb{S}^3 \times \mathbb{S}^3$-action then the cohomogeneity one data of B^7 is given by

$$\begin{aligned} G &= \mathbb{S}^3 \times \mathbb{S}^3, & H_- &= \{(e^{-3i\theta}, e^{i\theta}) \mid \theta \in \mathbb{R}\} \cup \{(je^{-3i\theta}, je^{i\theta}) \mid \theta \in \mathbb{R}\}, \\ H &= \Delta\{\pm 1, \pm i, \pm j, \pm k\}, & H_+ &= \{(e^{j\theta}, e^{-3j\theta}) \mid \theta \in \mathbb{R}\} \cup \{(ke^{j\theta}, ke^{-3j\theta}) \mid \theta \in \mathbb{R}\}. \end{aligned}$$

The Weyl group of this cohomogeneity one space is D_3.

Proposition 3.3. *The atoms of B^7 are all diffeomorphic to $\mathbb{RP}^3$.*

The proof of this fact is similar to the proofs above. If B^7 is equipped with the (up to scaling) unique normal homogeneous metric then the sectional curvature pinching of B^7 is $1/37$ (see [El]) and the minimum of the sectional curvature is attained on planes tangential to the atoms. The atoms inherit Berger metrics with pinching $1/17$. This can be seen in the following way: The Berger space can equivalently be described as the quotient of $\mathsf{Sp}(2)$ by a maximal $\mathsf{SU}(2)$ (for example the one of [El]). In this description, the cohomogeneity one atoms above are the fixed point sets of the conjugation by i, j, or k. The centralizer of each of these elements is a $\mathsf{U}(2) \subset \mathsf{Sp}(2)$ which intersects with the maximal $\mathsf{SU}(2)$ in a common circle. Hence, the isotropy groups of the transitive $\mathsf{U}(2)$-action on the atoms are cricles and the induced metrics are Berger metrics.

There are two more three-dimensional fixed point sets in B^7: The transformation given by the element $(1, e^{2\pi i/3})$ fixes $\mathbb{S}^3 \times \mathbb{S}^1/\{(e^{-3i\theta}, e^{i\theta})\} \approx \mathbb{S}^3$ in the singular orbit G/H_-. Note that this transformation does not have a fixed point in the other singular orbit G/H_+. If B^7 is equipped with a normal homogeneous metric then the minimum and the maximum of the sectional curvature of B^7 are attained on planes tangential to this fixed point set. Analogously, the fixed point set of $(e^{2\pi i/3}, 1)$ is an $\mathbb{S}^3$ that is located in G/H_+.

4. Implications by the size of the homogeneity rank

We will now derive Bredon's inequality as a consequence of the inequality for unions of isotypic orbits in Section 1 and the infinitesimal structure of fixed point sets. We will then use Bredon's inequaltity to derive properties of the action if the range of the homogeneity rank attains certain values.

Theorem 4.1 (see [Br], Theorem IV.5.3)**.** *If the fixed point set M^T of a maximal torus T of G is nonempty then* $\dim M^T \leq -\operatorname{rk}(M,G)$.

Proof. The inequality follows immediately from Theorem 1.3 and Lemma 2.3, since $N(T)\cdot p$ is discrete for $p \in M^T$. □

Lemma 4.2. *For manifolds of fixed dimension n we have*

$$\operatorname{rk}(M^n, G) \in \{-n, -n+2, \dots, n-2, n\}.$$

Proof. From the definition of the homogeneity rank it follows immediatly that $-n \leq \operatorname{rk}(M,G) \leq n$. It is a well-known fact that for any compact Lie group $\dim G$ has the same parity as $\operatorname{rank} G$. This implies that $\operatorname{rk}(M,G)$ has the same parity as n. □

Lemma 4.3. *If $G \times M^n \to M^n$ is effective with $\operatorname{rk}(M,G) = n-2k$ then*

$$\operatorname{rank} G \leq n-k, \quad \operatorname{rank} \text{p.i.g.}(M,G) \leq k, \quad \textit{and} \quad \operatorname{cohom}(M,G) \leq k.$$

Proof. The effectivity of the action implies that G acts effectively on each principal orbit. A consequence of Bredon's inequality is that

$$\dim G/H \;\geq\; \operatorname{rank} G + \operatorname{rank} H \tag{2}$$

if the action of G on G/H is effective (see [Br], Corollary IV.5.4). Hence,

$$\begin{aligned} 2\operatorname{rank} G - n \;\leq\; \operatorname{rk}(M,G) \;&\leq 2(\dim G - \dim H - \operatorname{rank} H) - n \\ &\leq\; n - 2\operatorname{rank} H \end{aligned} \tag{3}$$

where H denotes the principal isotropy group of the action. □

It is now easy to prove Theorem A from the introduction. We will only explain the most complicated case $\operatorname{rk}(M,G) = n-2$. In this case $\operatorname{rank} H \leq 1$. We see from (3) that G acts transitively or with cohomogeneity one and that the latter can only happen if $\operatorname{rank} H = 0$. In the cohomogeneity one case it follows from the definition of the homogeneity rank that $\dim G = \operatorname{rank} G = n-1$ and hence that $G = \mathsf{T}^{n-1}$ and $H = \{\mathbb{1}\}$. These actions are easy to classify: If there are only principal orbits then the action is equivalent to the direct product of the standard action of T^{n-1} on itself and a one dimensional manifold. If there are non-principal isotropy groups then these determine the action up to equivalence. We can apply an external automorphism of T^{n-1} to move them into a standard T^2-factor of T^{n-1}. It follows from elementary considerations in the non-compact case and Neumann's classification list [Nm] in the compact case that all possible isotropy groups are already attained for the direct products of the standard action of T^{n-3} on itself and a cohomogeneity one action of T^2 on a manifold N^3.

Theorem 4.4. *If M is compact and if $\operatorname{rk}(M,G) \geq 0$ then*

$$\chi(M) = \#M^T = \sum_{\textit{all } G\textit{-orbits } \mathcal{O}} \chi(\mathcal{O}),$$

where T denotes a maximal torus of G.

Proof. We can assume that G is connected. If M is not orientable, we can lift the action of G to the two-fold orientable covering space of M and the action of the maximal torus will have twice as many fixed points as on M (see [Br], Chapter I). Hence, we can assume that M is orientable.

Since $M^T = M^\psi$, where $\psi \in T$ generates the torus, we see from Bredon's inequality that the isometry ψ is a Lefschetz map, i.e., it has only finitely many fixed points. The Lefschetz number $L(\psi)$ is equal to the number of fixed points (see [Ko]). Since ψ is homotopic to the identity in G we have $L(\psi) = L(\mathrm{id}) = \chi(M)$

and hence $\chi(M) = \#M^T$. The same argument applies to the fixed point set of the maximal torus T in each orbit, which proves the second part of the equation. □

The Euler characteristic of a homogeneous space G/H has first been determined by Hopf and Samelson (see [HS]). They proved that the Euler characteristic is equal to the number of fixed points of the left multiplication with a generic element in G. If $\operatorname{rank} H < \operatorname{rank} G$ then there are no fixed points and the Euler characteristic is zero. If $\operatorname{rank} H = \operatorname{rank} G$ then the fixed points are given explicitely by $(N(T)/T) \cdot H$ and thus the Euler characteristic of G/H divides the order of the Weyl group, i.e., the Euler characteristic of G/T where T is a maximal torus.

Since $\operatorname{rk}(M, G) = \operatorname{rank} G - \operatorname{rank} H \geq 0$ for transitive actions, Theorem 4.4 is a direct generalization of this classical result of Hopf and Samelson.

5. Manifolds with positive sectional curvature

In this section we will apply the preceding results to Riemannian manifolds with positive sectional curvature.

5.1. **Maximal homogeneity rank.**

We combine the inequality for unions of isotypic orbits with the isotropy rank lemma. Note that an orbit is called isolated if there aren't any orbits of the same type in a neighborhood of this orbit.

Theorem 5.1. *Let M be a compact Riemannian manifold with positive sectional curvature. Then for any isometric action $G \times M \to M$ of a compact Lie group G we have $\operatorname{rk}(M, G) \leq 1$.*

- *If $\operatorname{rk}(M, G) = 1$ then all orbits $G \cdot p$ with $\operatorname{rank} G_p = \operatorname{rank} G - 1$ are isolated and such orbits exist. All other orbits $G \cdot p$ have $\operatorname{rank} G_p \leq \operatorname{rank} G - 2$.*
- *If $\operatorname{rk}(M, G) = 0$ then all orbits $G \cdot p$ with $\operatorname{rank} G_p = \operatorname{rank} G$ are isolated and such orbits exist. Their number is bounded from above by the Euler characteristic of M.*

Proof. The isotropy rank lemma and Lemma 1.1 imply that $\operatorname{rk}(M, G) \leq 1$. If $\operatorname{rk}(M, G) = 1$ and $\operatorname{rank} G_p \geq \operatorname{rank} G - 1$ for some $p \in M$ then Theorem 1.3 implies

$$\begin{aligned} \dim\big(\nu_p(G \cdot p)\big)^{G_p} &\leq \operatorname{cohom}(M, G) - (\operatorname{rank} G_p - \operatorname{rank} \text{p.i.g.}) \\ &\leq \operatorname{cohom}(M, G) - (\operatorname{rank} G - \operatorname{rank} \text{p.i.g.}) - 1 \;=\; 0. \end{aligned} \tag{4}$$

This shows that $G \cdot p$ is isolated and that $\operatorname{rank} G_p = \operatorname{rank} G - 1$ since equality must hold throughout in (4). The case $\operatorname{rk}(M, G) = 0$ is considered analogously. □

5.2. **Positivity of the Euler characteristic.**

For the sake of completeness we quote the following result from [PS]:

Theorem 5.2. *Let M be a compact even-dimensional Riemannian manifold with positive (nonnegative) sectional curvature and $G \times M \to M$ be an isometric action of a compact Lie group G with $\operatorname{rk}(M, G) \geq -5$. Then M has positive (nonnegative) Euler characteristic.*

5.3. **Cohomogeneity one manifolds.** Our aim in this subsection is to show how homogeneity rank and the notion of atoms give some structure to the classification of compact cohomogeneity one manifolds with positive sectional curvature. This classification is still open. Partial results exist (see [Se], [PV1], [PV2], [Ve]). So far, a new manifold of positive sectional curvature has not been discovered by cohomogeneity one methods. However, Grove, Shankar, and Ziller found many interesting cohomogeneity one actions on the known examples (see [GSZ]).

Let M be a compact Riemannian manifold with positive sectional curvature and $G \times M \to M$ be an isometric action with cohomogeneity one. It follows from Theorem 5.1 and from the definition of the homogeneity rank that $-1 \le \mathrm{rk}(M, G) \le 1$. We will discuss each of three cases separately. We refer to Section 3 for the required facts about cohomogeneity one manifolds.

The case $\mathrm{rk}(M, G) = -1$. In this case, $\dim M$ is odd, all isotropy groups have the same rank, and the action is polar by Proposition 2.13, i.e., the Cartan geodesic is closed. The only atom for such an action is a Cartan geodesic. An example for a nontrivial action with $\mathrm{rk}(M, G) = -1$ is the action of $\mathsf{SU}(3)$ on the unit sphere in the traceless Hermitian 3×3-matrices by conjugation. In this case the principal isotropy group is a maximal torus of $\mathsf{SU}(3)$ and the singular orbits are complex projective planes.

The case $\mathrm{rk}(M, G) = 0$. In this case, $\dim M$ is even, the rank of the principal isotropy group is one less than the rank of G and at least one of the singular isotropy groups has the same rank as G. The structure of the atoms is determined by the following result:

Proposition 5.3. *Let M be a compact even-dimensional Riemannian manifold with positive sectional curvature, and let G be a compact connected Lie group that acts isometrically with cohomogeneity one on M such that the isotropy group of the principal orbits is trivial. Then the universal covering space of M is equivariantly diffeomorphic to $\mathbb{S}^2$, $\mathbb{S}^4$, or $\mathbb{CP}^2$ with one of the following three actions:*

(1) *The standard action of* $\mathsf{SO}(2)$ *on* $\mathbb{S}^1 \subset \mathbb{R}^2$ *suspended to* $\mathbb{S}^2$.
(2) *The standard action of* $\mathsf{SU}(2)$ *on* $\mathbb{S}^3 \subset \mathbb{C}^2$ *suspended to* $\mathbb{S}^4$.
(3) *The standard isotropy action of* $\mathsf{SU}(2) \subset \mathsf{U}(2)$ *on* $\mathbb{CP}^2$.

In particular, the isometric action $G \times M \to M$ is polar.

Proof. Note that if M is not orientable, we can lift the action of G to the two-fold orientable covering space of M (see [Br], Chapter I), which is in this case the universal covering space. Thus, it suffices to consider the case where M is simply connected. Since the principal isotropy group is trivial the singular isotropy groups H_- and H_+ have to be spheres, i.e., $\mathbb{S}^0$, $\mathbb{S}^1$, or $\mathbb{S}^3$. Because M is simply connected, exceptional orbits cannot appear (see [Br] again). Hence, $\mathbb{S}^0 = \mathbb{Z}_2$ is excluded. Since $\mathrm{rk}(M, G) = 0$ we have $\mathrm{rank}\, G = 1$ and hence G is $\mathsf{SO}(2) \approx \mathbb{S}^1$, $\mathsf{SU}(2) \approx \mathbb{S}^3$, or $\mathsf{SO}(3)$. The possible combinations are now:

(1) $G = \mathsf{SO}(2)$, $H_- = \mathsf{SO}(2)$, $H_+ = \mathsf{SO}(2)$: This is the $\mathsf{SO}(2)$-action on $\mathbb{S}^2$.
(2) $G = \mathsf{SU}(2)$, $H_- = \mathsf{SU}(2)$, $H_+ = \mathsf{SU}(2)$: This is the $\mathsf{SU}(2)$-action on $\mathbb{S}^4$.
(3) $G = \mathsf{SU}(2)$, $H_- = \mathsf{SU}(2)$, $H_+ = \mathsf{SO}(2)$: This is the isotropy action on $\mathbb{CP}^2$.
(4) $G = \mathsf{SU}(2)$, $H_- = \mathbb{S}^1_1$, $H_+ = \mathbb{S}^1_2$: Theorem 4.4 implies that $\chi(M) = 4$.
(5) $G = \mathsf{SO}(3)$, $H_- = \mathbb{S}^1_1$, $H_+ = \mathbb{S}^1_2$: Theorem 4.4 implies that $\chi(M) = 4$.

By a result of Hsiang and Kleiner [HK] any 4-manifold M of positive sectional curvature with continuous symmetry has $\chi(M) \leq 3$. Hence, the forth and the fifth case can be excluded. Note for the third case that all $\mathbb{S}^1$ in $\mathsf{SU}(2)$ are conjugate. $\square$

It is easy to see that the manifolds in the fourth and the fifth case are $\mathbb{CP}^2 \# - \mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$, respectively. Note that the isometric $\mathsf{SU}(2)$-actions on all four dimensional manifolds with nonnegative curvature are classified in the unpublished parts of Kleiner's thesis [Kl].

Since the action on all possible atoms is polar, all cohomogeneity one actions on positively curved manifolds with $\mathrm{rk}(M, G) = 0$ are polar, i.e., the Cartan geodesics are closed.

The case $\mathrm{rk}(M, G) = 1$. This is the most complicated but also most interesting case. The dimension of M is odd, the rank of the principal isotropy group is two less than the rank of G, the rank of one of the singular isotropy groups is one less than the rank of G, and the rank of the other singular isotropy groups is at least one less than the rank of G. The known examples are several actions on spheres, the action on the Berger space B^7 discussed in Section 3, and the cohomogeneity one actions that exist on some of the Eschenburg and Bazaikin spaces (see [GSZ]). Note that the Eschenburg spaces are atoms for the actions on the Bazaikin spaces. A general classification of the atoms in the case $\mathrm{rk}(M, G) = 1$ does not exist so far. If the principal isotropy group is trivial, then $\mathrm{rank}\, G = 2$. The manifolds on which $G = \mathsf{T}^2$ acts with cohomogeneity one and trivial principal isotropy group are classified (see [Nm]). For the other rank 2 groups we do not have an analogous classification.

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