On the constants in some inverse inequalities for finite element functions

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Summary: We determine the constants in some inverse inequalities for finite element functions. These constants are crucial for the correct calibration of a posteriori error estimators.

Key words: Inverse inequalities; finite element functions; a posteriori error estimates

Résumé: Pour des éléments finis on calcule les constantes dans certaines inégalités inverses. Cettes constantes sont importantes pour l'étalonnage des estimateurs d'erreur a posteriori.

Mots clefs: Inégalités inverses; éléments finis; estimation d'erreur a posteriori

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1. Introduction and main results

Adaptive finite element methods based on a posteriori error estimates have become an undispensable tool in large scale scientific computing. Most known a posteriori error estimates yield two-sided bounds on the error which contain multiplicative constants. An explicit knowledge of these constants is mandatory for a correct calibration of the a posteriori error estimates. The constants usually depend in a multiplicative way on the norm of the differential operator and of its inverse, on the norm of suitable quasi-interpolation operators, and on constants in certain inverse inequalities for finite element functions. The norms of the quasi-interpolation operators have recently been estimated explicitely [4]. It is the aim of the present analysis to determine the constants in the inverse inequalties.

In order to describe our results, consider a d–dimensional simplex K and a $(d-1)$–dimensional face E thereof. Denote by h_K and h_E the diameters of K and of E, respectively. Number the vertices of K from 0 to d such that the vertices of E are numbered first. Denote by $\lambda_{K,0}, \ldots, \lambda_{K,d}$ the barycentric co-ordinates of K. I.e., $\lambda_{K,i}$ is the affine function that takes the value 1 at the i-th vertex and that vanishes at the other vertices. Set

$$\begin{aligned} \psi_K &:= (d+1)^{d+1} \prod_{i=0}^{d} \lambda_{K,i} \\ \psi_E &:= d^d \prod_{i=0}^{d-1} \lambda_{K,i}. \end{aligned} \tag{1.1}$$

The functions ψ_K and ψ_E attain their maximal value 1 at the barycentres of K and of E, respectively.

There are constants $\gamma_1, \ldots, \gamma_5$ such that the following inverse inequalities hold for all polynomials v and σ of degree at most k in d resp. $d-1$ variables defined on K resp. E [3; Lemma 3.3]:

$$\begin{aligned} \|v\|_{L^2(K)} &\leq \gamma_1 \; \|\psi_K^{1/2} v\|_{L^2(K)}, \\ \|\nabla(\psi_K v)\|_{L^2(K)} &\leq \gamma_2 \; h_K^{-1} \; \|v\|_{L^2(K)}, \\ \|\sigma\|_{L^2(E)} &\leq \gamma_3 \; \|\psi_E^{1/2} \sigma\|_{L^2(E)}, \\ \|\nabla(\psi_E \sigma)\|_{L^2(K)} &\leq \gamma_4 \; h_E^{-1/2} \; \|\sigma\|_{L^2(E)}, \\ \|\psi_E \sigma\|_{L^2(K)} &\leq \gamma_5 \; h_E^{1/2} \; \|\sigma\|_{L^2(E)}. \end{aligned} \tag{1.2}$$

From the proof of (1.2) it follows that $\gamma_1, \ldots, \gamma_5$ depend on the polynomial degree k and that γ_2, γ_4, and γ_5 in addition depend on the shape parameter h_K/ρ_K of K. Here, as usual, ρ_K denotes the diameter of the largest ball which may be inscribed into K.

It is our aim to derive sharp bounds on the constants $\gamma_1, \ldots, \gamma_5$ and to make explicit their dependence on the parameters K, E, k, and d. To this end denote by $\hat{K}$ the d–dimensional reference simplex, which has the origin and the end-points of the unit vectors as its vertices, and by $\hat{E}$ the $(d-1)$–dimensional face of $\hat{K}$ which is in the d-th co-ordinate plane $\{x_d = 0\}$. With these notations we can prove the following result:

1.1 Proposition. *Denote by $h_E^\perp$ the height of K above E. The constants $\gamma_1, \ldots, \gamma_5$ in inequalities (1.2) can be bounded by*

$$\begin{aligned} \gamma_1 &= \hat{\gamma}_1, \\ \gamma_2 &\leq \frac{h_K}{\rho_K} \hat{\gamma}_2, \\ \gamma_3 &= \hat{\gamma}_3, \\ \gamma_4 &\leq \begin{cases} \left\{ 2 \frac{h_E h_E^\perp}{\rho_K^2} \right\}^{1/2} \hat{\gamma}_4 & \text{if } d = 2, \\ \left\{ \sqrt{2} \frac{h_E h_E^\perp}{\rho_K^2} \right\}^{1/2} \hat{\gamma}_4 & \text{if } d \geq 3, \end{cases} \\ \gamma_5 &= \begin{cases} \left\{ \frac{h_E^\perp}{h_E} \right\}^{1/2} \hat{\gamma}_5 & \text{if } d = 2, \\ \left\{ \sqrt{2} \frac{h_E^\perp}{h_E} \right\}^{1/2} \hat{\gamma}_5 & \text{if } d \geq 3. \end{cases} \end{aligned} \tag{1.3}$$

Here, $\hat{\gamma}_1, \ldots, \hat{\gamma}_5$ are the corresponding constants for the reference simplex $\hat{K}$ and its *face $\hat{E}$. They can be estimated by*

$$
\begin{aligned}
\hat{\gamma}_1 &\le [2(k+2)]^d \ \left[\left(\frac{d}{d+1} \right)^{d+1} \ d! \right]^{1/2}, \\
\hat{\gamma}_2 &\le d\sqrt{2d} \ \left(\frac{d+1}{d} \right)^{d+1} \ \left\{ 1 + \frac{1}{2}\sqrt{k(k+1)} \right\}, \\
\hat{\gamma}_3 &\le [2(k+2)]^{d-1} \ \left[\left(\frac{d-1}{d} \right)^{d} \ (d-1)! \right]^{1/2}, \\
\hat{\gamma}_4 &\le \begin{cases} \left\{ \frac{352}{27} + \frac{8}{3}k(k+1) \right\}^{1/2} & \text{if } d=2, \\ 2^{1/4} \left\{ \frac{9d-7}{8} \frac{(2d)^{2d}}{(2d-1)^{2d-1}} + \frac{1}{6} \frac{d^{2d}}{(d-1)^{2d-3}} k(k+1) \right\}^{1/2} & \text{if } d \ge 3, \end{cases} \\
\hat{\gamma}_5 &\le \begin{cases} \frac{24\sqrt{5}}{125} & \text{if } d=2, \\ 2^{-1/4} \ \frac{3}{2} \ \left(\frac{2d}{2d+1} \right)^{d} \ \frac{1}{\sqrt{2d+1}} & \text{if } d \ge 3. \end{cases}
\end{aligned}
\tag{1.4}
$$

We will prove the first part of Proposition 1.1 in Section 2. In Section 3 we establish a one-dimensional analogue of the first two inequalties in (1.2). Combining this result with a dimension-reduction argument, we will prove the second part of Proposition 1.1. in Section 4.

2. Transformation to the reference simplex

Given a d–dimensional simplex K and a $(d-1)$–dimensional face E, there is an orientation preserving affine transformation $F_K : \hat{x} \longrightarrow b_K + B_K\hat{x}$ which maps $\hat{K}$ onto K and its face $\hat{E}$ onto E. The transformations $v \longrightarrow \hat{v} := v \circ F_K$ and $\sigma \longrightarrow \hat{\sigma} := \sigma \circ F_K$ yield a one-to-one correspondence between polynomials v and σ of degree k in d resp. $d-1$ variables defined on K resp. E and polynomials $\hat{v}$ and $\hat{\sigma}$ of degree k in d resp. $d-1$ variables defined on $\hat{K}$ resp. $\hat{E}$. Denote by meas_d the d–dimensional Lebesgue measure. Since $\psi_{\hat{K}} = \psi_K \circ F_K$ and $\psi_{\hat{E}} = \psi_E \circ F_K$, the transformation rule for integrals yields

$$
\begin{aligned}
\|v\|_{L^2(K)} &= \left\{ \frac{\text{meas}_d(K)}{\text{meas}_d(\hat{K})} \right\}^{1/2} \ \|\hat{v}\|_{L^2(\hat{K})} \\
&\le \hat{\gamma}_1 \left\{ \frac{\text{meas}_d(K)}{\text{meas}_d(\hat{K})} \right\}^{1/2} \ \|\psi_{\hat{K}}^{1/2}\hat{v}\|_{L^2(\hat{K})} \\
&= \hat{\gamma}_1 \|\psi_K^{1/2} v\|_{L^2(K)}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\sigma\|_{L^2(E)} &= \left\{\frac{\operatorname{meas}_{d-1}(E)}{\operatorname{meas}_{d-1}(\hat{E})}\right\}^{1/2} \|\hat{\sigma}\|_{L^2(\hat{E})} \\
&\leq \hat{\gamma}_3 \left\{\frac{\operatorname{meas}_{d-1}(E)}{\operatorname{meas}_{d-1}(\hat{E})}\right\}^{1/2} \|\psi_{\hat{E}}^{1/2}\hat{\sigma}\|_{L^2(\hat{E})} \\
&= \hat{\gamma}_3 \|\psi_E^{1/2}\sigma\|_L^2(E).
\end{aligned}
$$

This establishes the results of Proposition 1.1 concerning γ_1 and γ_3.

Denote by $|||B_K^{-1}|||$ the spectral norm of B_K^{-1}. The transformation rule for integrals and the chain rule for differentiation yield

$$
\begin{aligned}
\|\nabla(\psi_K v)\|_{L^2(K)} &= \left\{\frac{\operatorname{meas}_d(K)}{\operatorname{meas}_d(\hat{K})}\right\}^{1/2} \|B_K^{-T}\nabla_{\hat{x}}(\psi_{\hat{K}}\hat{v})\|_{L^2(\hat{K})} \\
&\leq \left\{\frac{\operatorname{meas}_d(K)}{\operatorname{meas}_d(\hat{K})}\right\}^{1/2} |||B_K^{-1}|||\ \|\nabla_{\hat{x}}(\psi_{\hat{K}}\hat{v})\|_{L^2(\hat{K})} \\
&\leq \hat{\gamma}_2 \left\{\frac{\operatorname{meas}_d(K)}{\operatorname{meas}_d(\hat{K})}\right\}^{1/2} |||B_K^{-1}|||\ h_{\hat{K}}^{-1}\ \|\hat{v}\|_{L^2(\hat{K})} \\
&= \hat{\gamma}_2 |||B_K^{-1}|||\ h_{\hat{K}}^{-1}\ \|v\|_{L^2(K)}.
\end{aligned}
$$

Since [1; Theorem 3.1.3]

$$
|||B_K^{-1}||| \leq \frac{h_{\hat{K}}}{\rho_K}
$$

this etablishes the estimate for γ_2 given in Proposition 1.1.

With the same arguments we conclude that

$$
\begin{aligned}
\|\nabla(\psi_E\sigma)\|_{L^2(K)} &\leq \left\{\frac{\operatorname{meas}_d(K)}{\operatorname{meas}_d(\hat{K})}\right\}^{1/2} |||B_K^{-1}|||\ \|\nabla_{\hat{x}}(\psi_{\hat{E}}\hat{\sigma})\|_{L^2(\hat{E})} \\
&\leq \hat{\gamma}_4 \left\{\frac{\operatorname{meas}_d(K)}{\operatorname{meas}_d(\hat{K})}\right\}^{1/2} |||B_K^{-1}|||\ h_{\hat{E}}^{-1/2}\ \|\hat{\sigma}\|_{L^2(\hat{E})} \\
&= \hat{\gamma}_4 \left\{\frac{\operatorname{meas}_d(K)}{\operatorname{meas}_d(\hat{K})}\frac{\operatorname{meas}_{d-1}(\hat{E})}{\operatorname{meas}_{d-1}(E)}\right\}^{1/2} |||B_K^{-1}|||\ h_{\hat{E}}^{-1/2}\ \|\sigma\|_{L^2(E)} \\
&\leq \hat{\gamma}_4 \left\{\frac{\operatorname{meas}_d(K)}{\operatorname{meas}_d(\hat{K})}\frac{\operatorname{meas}_{d-1}(\hat{E})}{\operatorname{meas}_{d-1}(E)}\frac{h_{\hat{K}}^2}{\rho_K^2}\frac{h_E}{h_{\hat{E}}}\right\}^{1/2} h_E^{-1/2}\ \|\sigma\|_{L^2(E)}.
\end{aligned}
$$

Since

$$\begin{aligned} d\,\mathrm{meas}_d(K) &= h_E^{\perp}\mathrm{meas}_{d-1}(E), \\ d\,\mathrm{meas}_d(\hat{K}) &= \mathrm{meas}_{d-1}(\hat{E}), \\ h_{\hat{K}} &= \sqrt{2}, \\ h_{\hat{E}} &= \begin{cases} 1 & \text{if } d=2, \\ \sqrt{2} & \text{if } d \geq 3, \end{cases} \end{aligned} \tag{2.1}$$

this proves the estimate for γ_4 of Proposition 1.1.
The transformation rule for integrals finally yields

$$\begin{aligned} \|\psi_E\sigma\|_{L^2(K)} &= \left\{ \frac{\mathrm{meas}_d(K)}{\mathrm{meas}_d(\hat{K})} \right\}^{1/2} \|\psi_{\hat{E}}\hat{\sigma}\|_{L^2(\hat{E})} \\ &\leq \hat{\gamma}_5 \left\{ \frac{\mathrm{meas}_d(K)}{\mathrm{meas}_d(\hat{K})} \right\}^{1/2} h_{\hat{E}}^{1/2}\ \|\hat{\sigma}\|_{L^2(\hat{E})} \\ &\leq \hat{\gamma}_5 \left\{ \frac{\mathrm{meas}_d(K)}{\mathrm{meas}_d(\hat{K})} \frac{\mathrm{meas}_{d-1}(\hat{E})}{\mathrm{meas}_{d-1}(E)} \frac{h_{\hat{E}}}{h_E} \right\}^{1/2} h_E^{1/2}\ \|\sigma\|_{L^2(E)}. \end{aligned}$$

Together with (2.1) this establishes the estimate of γ_5 given in Proposition 1.1.

3. Some inverse inequalities for univariate polynomials

Denote by L_k the k-th Legendre polynomial with leading coefficient 1. Consider two integers $0 < \ell \leq k$. Since $(1-x^2)L_\ell'(x)$ vanishes at $x = \pm 1$, integration by parts yields

$$\int_{-1}^{1} (1-x^2)L_k'(x)L_\ell'(x)dx = -\int_{-1}^{1} L_k(x)\Big[(1-x^2)L_\ell'(x)\Big]'dx.$$

Since $\big[(1-x^2)L_\ell'(x)\big]'$ is a polynomial of degree ℓ with leading coefficient $-\ell(\ell+1)$, the orthogonality of the Legendre polynomials implies that

$$\int_{-1}^{1} (1-x^2)L_k'(x)L_\ell'(x)dx = \begin{cases} k(k+1)\|L_k\|^2_{L^2((-1,1))} & \text{if } \ell = k, \\ 0 & \text{if } \ell < k. \end{cases} \tag{3.1}$$

Now consider a polynomial p of degree k. It may be written in the form

$$p = \sum_{\ell=0}^{k} \alpha_\ell L_\ell.$$

The orthogonality of the Legendre polynomials and equation (3.1) imply that

$$\|p\|^2_{L^2((-1,1))} = \sum_{\ell=0}^{k} \alpha_\ell^2\ \|L_\ell\|^2_{L^2((-1,1))}$$

and

$$
\begin{aligned}
\|(1-x^2)^{1/2}p'\|^2_{L^2((-1,1))} &= \int_{-1}^{1}(1-x^2)p'(x)^2dx \\
&= \sum_{\ell=0}^{k}\alpha_\ell^2\ \ell(\ell+1)\ \|L_\ell\|^2_{L^2((-1,1))} \\
&\le k(k+1)\ \|p\|^2_{L^2((-1,1))}.
\end{aligned}
$$

This establishes:

3.1 Proposition. *The following inverse inequality holds for all univariate polynomials p of degree k and all integers k*

$$
\|(1-x^2)^{1/2}p'\|_{L^2((-1,1))}\ \le\ \sqrt{k(k+1)}\ \|p\|_{L^2((-1,1))}.
$$

Since any open, non-void interval (a,b) may be transformed affinely to $(-1,1)$ via $x \longrightarrow -1+2\frac{x-a}{b-a}$, we obtain from Proposition 3.1:

3.2 Corollary. *The following inverse inequality holds for all intervals (a,b), all univariate polynomials p of degree k, and all integers k*

$$
\|(x-a)^{1/2}(b-x)^{1/2}p'\|_{L^2((a,b))}\ \le\ \sqrt{k(k+1)}\ \|p\|_{L^2((a,b))}.
$$

Denote by $1 > x_{1,\ell} > \ldots > x_{\ell,\ell} > -1$ the zeros of L_ℓ and by $\omega_{1,\ell},\ldots,\omega_{\ell,\ell}$ the weights of the corresponding Gaussian quadrature formula. Consider a non-negative polynomial q of degree k. Denote by

$$
\ell(k) := \left\lceil \frac{k+3}{2} \right\rceil
$$

the smallest integer greater than or equal to $\frac{k+3}{2}$. Since $2\ell(k)-1 \ge k+2$, we have

$$
\begin{aligned}
\int_{-1}^{1} q(x)dx &= \sum_{i=1}^{\ell(k)} \omega_{i,\ell(k)}\ q(x_{i,\ell(k)}), \\
\int_{-1}^{1}(1-x^2)q(x)dx &= \sum_{i=1}^{\ell(k)} \omega_{i,\ell(k)}\ (1-x_{i,\ell(k)}^2)\ q(x_{i,\ell(k)}).
\end{aligned}
$$

Since the weights $\omega_{1,\ell},\ldots,\omega_{\ell,\ell}$ and the polynomial q are non-negative, we conclude that

$$
\begin{aligned}
\int_{-1}^{1}(1-x^2)q(x)dx &\ge (1-x_{1,\ell(k)}^2)\ \sum_{i=1}^{\ell(k)} \omega_{i,\ell(k)}\ q(x_{i,\ell(k)}) \\
&= (1-x_{1,\ell(k)}^2)\ \int_{-1}^{1} q(x)dx
\end{aligned}
$$

or – equivalently –

$$\int_{-1}^{1} q(x)dx \leq \frac{1}{1-x_{1,\ell(k)}^2} \int_{-1}^{1} (1-x^2)q(x)dx.$$

Since [2; Theorem VI.6.21.3]

$$x_{1,\ell(k)} \leq \cos\left(\frac{\pi}{2\ell(k)}\right)$$

and since

$$\sin z \geq \frac{2}{\pi}\, z \quad \text{on } [0, \frac{\pi}{2}],$$

this establishes:

3.3 Proposition. *The following inverse inequality holds for all univariate non-negative polynomials q of degree k and all integers k*

$$\int_{-1}^{1} q(x)dx \leq \left\lceil \frac{k+3}{2} \right\rceil^2 \int_{-1}^{1} (1-x^2)q(x)dx.$$

Invoking the affine transformation of a given interval (a, b) to $(-1, 1)$, Proposition 3.3 leads to:

3.4 Corollary. *The following inverse inequality holds for all intervals (a, b), all univariate non-negative polynomials q of degree k, and all integers k*

$$\int_{a}^{b} q(x)dx \leq \left\lceil \frac{k+3}{2} \right\rceil^2 \left(\frac{2}{b-a}\right)^2 \int_{a}^{b} (x-a)(b-x)q(x)dx.$$

Since the square of a polynomial of degree k is a non-negative polynomial of degree $2k$ and since

$$\left\lceil \frac{2k+3}{2} \right\rceil = k+2,$$

Corollary 3.4 finally implies:

3.5 Corollary. *The following inverse inequality holds for all intervals (a, b), all univariate polynomials p of degree k, and all integers k*

$$\|p\|_{L^2((a,b))} \leq \frac{2}{b-a}\,(k+2)\,\|(x-a)^{1/2}(b-x)^{1/2}p\|_{L^2((a,b))}.$$

4. Inverse inequalities on the reference simplex

In this section we want to establish the second part of Proposition 1.1. Since our main tool is a dimension-reduction argument, we will sometimes label quantities with an index d in order to stress their dependence on the space dimension. Throughout this section v and σ denote generic polynomials of degree k in d resp. $d-1$ variables defined on $\hat{K}$ resp. $\hat{E}$. We decompose vectors $x \in \mathbb{R}^d$ in the form $x = (x', x_d)$ with $x' \in \mathbb{R}^{d-1}$.

In order to estimate $\hat{\gamma}_1$, we first observe that the interval $[0, 1]$ is the 1–dimensional reference simplex $\hat{K}_1$ and that the function $4x(1-x)$ is the corresponding function $\psi_{\hat{K}_1}$ of (1.1). Corollary 3.5 therefore yields

$$\hat{\gamma}_{1,1} \leq k + 2. \tag{4.1}$$

Now, fix a $d \geq 2$. For any point $x \in \hat{K}_d$ we have

$$1 \geq \sum_{i=1}^{d} x_i \geq d \min_{1 \leq i \leq d} x_i.$$

This implies that

$$\hat{K}_d \subset \bigcup_{i=1}^{d} \hat{K}_{d,i} \tag{4.2}$$

where

$$\hat{K}_{d,i} := \hat{K}_d \cap \left\{ x \in \mathbb{R}^d : x_i \leq \frac{1}{d} \right\}.$$

Assume that we dispose of a constant δ_d such that

$$\|v\|_{L^2(\hat{K}_{d,d})} \leq \delta_d \|\psi_{\hat{K}_d}^{1/2} v\|_{L^2(\hat{K}_d)} \tag{4.3}$$

holds for all polynomials v. Since the right-hand side of (4.3) is invariant under permutations of the co-ordinates, Equations (4.2) and (4.3) imply that

$$\begin{aligned} \|v\|_{L^2(\hat{K}_d)} &\leq \left\{ \sum_{i=1}^{d} \|v\|^2_{L^2(\hat{K}_{d,i})} \right\}^{1/2} \\ &\leq \left\{ \sum_{i=1}^{d} \delta_d^2 \|\psi_{\hat{K}_d}^{1/2} v\|^2_{L^2(\hat{K}_d)} \right\}^{1/2} \\ &= \sqrt{d}\, \delta_d \|\psi_{\hat{K}_d}^{1/2} v\|_{L^2(\hat{K}_d)} \end{aligned}$$

holds for all polynomials v. This yields

$$\hat{\gamma}_{1,d} \leq \sqrt{d}\, \delta_d. \tag{4.4}$$

In order to determine δ_d we invoke Fubini's theorem:

$$\|v\|^2_{L^2(\hat{K}_{d,d})} = \int_0^{1/d} \left\{ \int_{\hat{K}_d \cap \{x_d = t\}} v(x)^2 dx \right\} dt.$$

Fix a $t \in \left[0, \frac{1}{d}\right]$. Since $\hat{K}_d \cap \{x_d = t\}$ is the image of the $(d-1)$–dimensional reference simplex $\hat{K}_{d-1}$ under the transformation $\mathbb{R}^{d-1} \ni x' \longrightarrow ((1-t)x', t) \in \mathbb{R}^d$, we have

$$\int_{\hat{K}_d \cap \{x_d = t\}} v(x)^2 dx = (1-t)^{d-1} \int_{\hat{K}_{d-1}} v((1-t)x', t)^2 dx'.$$

Since $w(x') := v((1-t)x', t)$ is a polynomial of degree k in $d-1$ variables on $\hat{K}_{d-1}$, we may apply Proposition 1.1 in dimension $d-1$ and obtain

$$\int_{\hat{K}_{d-1}} v((1-t)x', t)^2 dx' \leq \hat{\gamma}^2_{1,d-1} \int_{\hat{K}_{d-1}} \psi_{\hat{K}_{d-1}}(x') \, v((1-t)x', t)^2 dx'.$$

Since

$$\begin{aligned} \psi_{\hat{K}_{d-1}}(x') &= d^d \prod_{i=0}^{d-1} \lambda_{\hat{K}_{d-1}, i}(x') \\ &= d^d \prod_{i=0}^{d-1} \left\{ \lambda_{\hat{K}_d, i}((1-t)x', t) \, \frac{1}{1-t} \right\}, \end{aligned}$$

we arrive at

$$\|v\|^2_{L^2(\hat{K}_{d,d})} \leq \hat{\gamma}^2_{1,d-1} \, d^d \int_0^{1/d} \left\{ \int_{\hat{K}_d \cap \{x_d = t\}} (1-t)^{-d} \, v(x)^2 \prod_{i=0}^{d-1} \lambda_{\hat{K}_d, i}(x) dx \right\} dt.$$

Since

$$p(t) := \int_{\hat{K}_d \cap \{x_d = t\}} (1-t)^{-d} \, v(x)^2 \prod_{i=0}^{d-1} \lambda_{\hat{K}_d, i}(x) dx$$

is a positive univariate polynomial of degree $2k$, we obtain from Corollary 3.4

$$\begin{aligned} \int_0^{1/d} p(t) dt &\leq \left\lceil \frac{2k+3}{2} \right\rceil^2 \int_0^{1/d} (2d)^2 \, t \, (\frac{1}{d} - t) \, p(t) dt \\ &\leq (k+2)^2 \, (2d)^2 \, \frac{1}{d} \int_0^{1/d} t \, p(t) dt. \end{aligned}$$

Since $t = \lambda_{\hat{K}_d | \{x_d = t\}}$, this leads to

$$\begin{aligned} &\|v\|^2_{L^2(\hat{K}_{d,d})} \\ &\leq \hat{\gamma}^2_{1,d-1} \, d^d \, (k+2)^2 \, 4d \int_0^{1/d} \left\{ \int_{\hat{K}_d \cap \{x_d = t\}} (1-t)^{-d} \, v(x)^2 \, t \prod_{i=0}^{d-1} \lambda_{\hat{K}_d, i}(x) dx \right\} dt \\ &\leq \hat{\gamma}^2_{1,d-1} \, d^d \, (k+2)^2 \, 4d \left(\frac{d}{d-1} \right)^d \int_{\hat{K}_d} v(x)^2 \prod_{i=0}^{d} \lambda_{\hat{K}_d, i}(x) dx \\ &= \hat{\gamma}^2_{1,d-1} \, d^d \, (k+2)^2 \, 4d \left(\frac{d}{d-1} \right)^d \frac{1}{(d+1)^{d+1}} \int_{\hat{K}_d} v(x)^2 \, \psi_{\hat{K}_d}(x) dx. \end{aligned}$$

Hence, we have shown that

$$\delta_d \leq 2(k+2)\ \hat{\gamma}_{1,d-1}\ \left[\frac{d^{d+1}\ d^d}{(d-1)^d\ (d+1)^{d+1}}\right]^{1/2}.$$

Together with (4.4) this yields the recursion

$$\hat{\gamma}_{1,d} \leq 2(k+2)\ \hat{\gamma}_{1,d-1}\left[\frac{d^{2d+2}}{(d-1)^d\ (d+1)^{d+1}}\right]^{1/2}. \tag{4.5}$$

From estimates (4.1) and (4.5) we conclude by induction that

$$\hat{\gamma}_{1,d} \leq \Big[2(k+2)\Big]^d\ \left[\left(\frac{d}{d+1}\right)^{d+1}\ d!\right]^{1/2}.$$

This establishes the first inequality in (1.4)

Since $\hat{E}_d \simeq \hat{K}_{d-1}$ and since $\psi_{\hat{E}_d|\hat{E}_d} = \psi_{\hat{K}_{d-1}}$, we have

$$\hat{\gamma}_{3,d} = \hat{\gamma}_{1,d-1}.$$

This establishes the third inequality in (1.4).

We now turn to the constant $\hat{\gamma}_2$. From the triangle inequality we have

$$\|\partial_d(\psi_{\hat{K}_d} v)\|_{L^2(\hat{K}_d)} \leq \|\psi_{\hat{K}_d}\ \partial_d v\|_{L^2(\hat{K}_d)} + \|v\ \partial_d \psi_{\hat{K}_d}\|_{L^2(\hat{K}_d)}. \tag{4.6}$$

Here, ∂_i denotes the partial derivative w.r.t. the i-th variable.
Consider first the first term on the right-hand side of (4.6). The function

$$\varphi(x) = \left(1 - \sum_{i=1}^{d} x_i\right)\ x_d\ \prod_{i=1}^{d-1} x_i^2$$

is non-negative on $\hat{K}_d$ and vanishes on the boundary $\partial\hat{K}_d$. Hence it attains its maximum at an interior point of $\hat{K}_d$. The partial derivatives of φ are

$$\begin{aligned}
\partial_i\varphi &= \left(2 - 3x_i - \sum_{\substack{j=1\\ j\neq i}}^{d} 2x_j\right)\ x_i\ x_d\ \prod_{\substack{j=1\\ j\neq i}}^{d-1} x_j^2 \quad, 1 \leq i \leq d-1,\\
\partial_d\varphi &= \left(1 - \sum_{j=1}^{d-1} x_j - 2x_d\right)\ \prod_{j=1}^{d-1} x_j^2.
\end{aligned}$$

By symmetry all critical point of φ are therefore of the form $(a, \ldots, a, b)$ and must satisfy

$$\begin{aligned}
0 &= 2 - 2b - (2d-1)a\\
0 &= 1 - 2b - (d-1)a.
\end{aligned}$$

This yields

$$a = \frac{1}{d} \; , \; b = \frac{1}{2d}$$

and therefore

$$\max_{x \in \hat{K}_d} |\varphi(x)| = \frac{1}{4d^{2d}}.$$

Since

$$\begin{aligned} \psi_{\hat{K}_d}^2 &= (d+1)^{2(d+1)} \left(1 - \sum_{i=1}^d x_i\right)^2 \prod_{i=1}^d x_i^2 \\ &= (d+1)^{2(d+1)} \varphi(x) \; x_d \; \left(1 - \sum_{i=1}^d x_i\right), \end{aligned}$$

this estimate implies that

$$\begin{aligned} \|\psi_{\hat{K}_d} \, \partial_d v\|_{L^2(\hat{K}_d)}^2 &= (d+1)^{2(d+1)} \int_{\hat{K}_d} \varphi(x) \; x_d \; \left(1 - \sum_{i=1}^d x_i\right) \; |\partial_d v|^2 dx \\ &\le \frac{(d+1)^{2(d+1)}}{4d^{2d}} \int_{\hat{K}_d} x_d \; \left(1 - \sum_{i=1}^d x_i\right) \; |\partial_d v|^2 dx. \end{aligned}$$

Denote by $|.|_1$ the ℓ^1-norm on $\mathbb{R}^d$. From Fubini's theorem and Corollary 3.2 we conclude that

$$\begin{aligned} &\int_{\hat{K}_d} x_d \; \left(1 - \sum_{i=1}^d x_i\right) \; |\partial_d v|^2 dx \\ &= \int_{\hat{K}_{d-1}} \left\{ \int_0^{1-|x'|_1} x_d \; (1 - |x'|_1 - x_d) \; |\partial_d v(x', x_d)|^2 dx_d \right\} dx' \\ &\le \int_{\hat{K}_{d-1}} \left\{ k(k+1) \; \int_0^{1-|x'|_1} v(x', x_d)^2 dx_d \right\} dx' \\ &\le k(k+1) \int_{\hat{K}_d} v(x)^2 dx. \end{aligned}$$

Combining the last two estimates, we obtain

$$\|\psi_{\hat{K}_d} \, \partial_d v\|_{L^2(\hat{K}_d)} \le \frac{(d+1)^{d+1}}{2d^d} \; \sqrt{k(k+1)} \; \|v\|_{L^2(\hat{K}_d)}. \tag{4.7}$$

Now we turn to the second term on the right-hand side of (4.6). Consider the function

$$\varphi(x) = \left(1 - 2x_d - \sum_{i=1}^{d-1} x_i\right) \prod_{i=1}^{d-1} x_i.$$

Since

$$\partial_d \varphi = -2 \prod_{i=1}^{d-1} x_i,$$

the function φ attains its extrema on $\partial \hat{K}_d$. Obviously it vanishes on the faces $\hat{K}_d \cap \{x_i = 0\}$ with $1 \le i \le d-1$. On the face $\hat{E}_d = \hat{K}_d \cap \{x_d = 0\}$ it obviously coincides with $d^{-d} \psi_{\hat{E}_d}$ and is therefore bounded in absolute value by d^{-d}. On the face $\hat{K}_d \cap \{|x|_1 = 1\}$ we finally have $\varphi = -d^{-d} \psi_{\hat{E}_d}$. Therefore, $|\varphi|$ does not exceed d^{-d} on this face, too. In conclusion we have

$$\max_{x \in \hat{K}_d} |\varphi(x)| = d^{-d}.$$

Since

$$\partial_d \psi_{\hat{K}_d} = (d+1)^{d+1} \varphi,$$

this proves that

$$\|v \, \partial_d \psi_{\hat{K}_d}\|_{L^2(\hat{K}_d)} \le \frac{(d+1)^{d+1}}{d^d} \, \|v\|_{L^2(\hat{K}_d)}. \tag{4.8}$$

From (4.6) – (4.8) we obtain

$$\|\partial_d(\psi_{\hat{K}_d} v)\|_{L^2(\hat{K}_d)} \le \frac{(d+1)^{d+1}}{d^d} \, \left\{ 1 + \frac{1}{2} \sqrt{k(k+1)} \right\} \, \|v\|_{L^2(\hat{K}_d)}.$$

Since the ratio $\|\nabla(\psi_{\hat{K}_d} v)\|_{L^2(\hat{K}_d)} / \|v\|_{L^2(\hat{K}_d)}$ is invariant under permutations of the co-ordinates and since $h_{\hat{K}_d} = \sqrt{2}$, this proves that

$$\hat{\gamma}_2 \le \sqrt{2d} \, \frac{(d+1)^{d+1}}{d^d} \, \left\{ 1 + \frac{1}{2} \sqrt{k(k+1)} \right\}$$

and thus establishes the second inequality of (1.4).

Next we estimate the constant $\hat{\gamma}_4$. Here, we must treat the derivative ∂_d and the remaining derivatives seperately.
Since σ and the barycentric co-ordinates $\lambda_{\hat{K}_d,1}, \ldots, \lambda_{\hat{K}_d,d-1}$ do not depend on x_d, we obtain

$$\begin{aligned} \partial_d(\psi_{\hat{E}_d} \sigma) &= d^d \, (\partial_d \lambda_{\hat{K}_d,0}) \, \prod_{i=1}^{d-1} \lambda_{\hat{K}_d,i} \, \sigma \\ &= - \, d^d \, \prod_{i=1}^{d-1} \lambda_{\hat{K}_d,i} \, \sigma. \end{aligned}$$

Together with Fubini's theorem this yields

$$\begin{aligned} \|\partial_d(\psi_{\hat{E}_d} \sigma)\|^2_{L^2(\hat{K}_d)} &= d^{2d} \, \int_{\hat{E}_d} \left\{ \int_0^{1-|x'|_1} \prod_{i=1}^{d-1} \lambda^2_{\hat{K}_d,i} \, \sigma^2 dx_d \right\} dx' \\ &= d^{2d} \, \int_{\hat{E}_d} (1 - |x'|_1) \, \prod_{i=1}^{d-1} \lambda^2_{\hat{K}_d,i} \, \sigma^2 dx'. \end{aligned}$$

Consider the function

$$\varphi(x') = (1 - |x'|_1) \prod_{i=1}^{d-1} \lambda^2_{\hat{K}_d,i} = \left(1 - \sum_{i=1}^{d-1} x_i\right) \prod_{i=1}^{d-1} x_i^2$$

on $\hat{E}_d \simeq \hat{K}_{d-1}$. It is non-negative and vanishes on the boundary $\partial\hat{K}_{d-1}$. Hence it attains its maximum at an interior point of $\hat{K}_{d-1}$. The derivatives of φ are

$$\partial_i\varphi = \left(2 - 2\sum_{\substack{j=1\\j\neq i}}^{d-1} x_j - 3x_i\right) x_i \prod_{\substack{j=1\\j\neq i}}^{d-1} x_j^2 \quad, 1 \le i \le d-1.$$

By symmetry, any critical point of φ therefore is of the form $(a, \ldots, a)$ and satisfies

$$2 - (2d-1)a = 0.$$

This yields

$$a = \frac{2}{2d-1}$$

and therefore

$$\max_{x' \in \hat{K}_{d-1}} |\varphi(x')| = \frac{2d-1}{4} \left(\frac{2}{2d-1}\right)^{2d}.$$

Hence, we obtain

$$\|\partial_d(\psi_{\hat{E}_d}\sigma)\|^2_{L^2(\hat{K}_d)} \le \frac{2d-1}{4} \left(\frac{2d}{2d-1}\right)^{2d} \|\sigma\|^2_{L^2(\hat{E}_d)}. \tag{4.9}$$

For the estimation of the remaining derivatives it suffices to consider the derivative w.r.t. x_1 since the ratio $\|\nabla(\psi_{\hat{E}_d}\sigma)\|_{L^2(\hat{K}_d)}/\|\sigma\|_{L^2(\hat{E}_d)}$ is invariant under permutations of the first $d-1$ co-ordinates.
From the triangle inequality we have

$$\|\partial_1(\psi_{\hat{E}_d}\sigma)\|_{L^2(\hat{K}_d)} \le \|\psi_{\hat{E}_d}\partial_1\sigma\|_{L^2(\hat{K}_d)} + \|\sigma\partial_1\psi_{\hat{E}_d}\|_{L^2(\hat{K}_d)}. \tag{4.10}$$

For the first term on the right-hand side of (4.10) we obtain from Fubini's theorem

$$\begin{aligned}\|\psi_{\hat{E}_d}\partial_1\sigma\|^2_{L^2(\hat{K}_d)} &= d^{2d} \int_{\hat{E}_d} \left\{ \int_0^{1-|x'|_1} (1 - |x'|_1 - x_d)^2 \prod_{i=1}^{d-1} x_i^2 \, |\partial_1\sigma(x')|^2 dx_d \right\} dx' \\ &= \frac{1}{3} d^{2d} \int_{\hat{E}_d} (1 - |x'|_1)^3 \prod_{i=1}^{d-1} x_i^2 \, |\partial_1\sigma(x')|^2 dx'.\end{aligned}$$

Since $\hat{E}_d \simeq \hat{K}_{d-1}$ and since

$$d^{2d} (1 - |x'|_1)^3 \prod_{i=1}^{d-1} x_i^2 \le \psi_{\hat{K}_{d-1}}(x')^2 \quad \text{on } \hat{E}_d$$

we may apply estimate (4.7) in dimension $d-1$ and get

$$\|\psi_{\hat{E}_d}\partial_1\sigma\|_{L^2(\hat{K}_d)} \le \frac{\sqrt{3}}{6}\ \frac{d^d}{(d-1)^{d-1}}\ \sqrt{k(k+1)}\ \|\sigma\|_{L^2(\hat{E}_d)}. \tag{4.11}$$

Since

$$\partial_1\psi_{\hat{E}_d} = d^d\ (1-|x'|_1 - x_1 - x_d)\ \prod_{i=2}^{d-1} x_i$$

we obtain by Fubini's theorem for the second term on the right-hand side of (4.10)

$$\begin{aligned}\|\sigma\partial_1\psi_{\hat{E}_d}\|^2_{L^2(\hat{K}_d)} &= d^{2d}\ \int_{\hat{E}_d}\left\{\int_0^{1-|x'|_1}(1-|x'|_1-x_1-x_d)^2\ \prod_{i=2}^{d-1}x_i^2\ \sigma(x')^2dx_d\right\}dx' \\ &= \frac{1}{3}d^{2d}\ \int_{\hat{E}_d}\left[(1-|x'|_1-x_1)^3+x_1^3\right]\ \prod_{i=2}^{d-1}x_i^2\ \sigma(x')^2dx'.\end{aligned}$$

Define the function φ on $[0,1]$ by

$$\varphi(t) = (1-2t)^3 + t^3.$$

An elementary calculation yields

$$0 < \varphi(t) \le 1 \quad \forall t \in [0,1].$$

If $d=2$, we therefore have

$$(1-|x'|_1-x_1)^3 + x_1^3 = \varphi(x_1) \le 1 \quad \text{on } \hat{E}_2.$$

If $d \ge 3$, we set for abreviation

$$z := \sum_{i=2}^{d-1} x_i \quad \text{and} \quad t := \frac{x_1}{1-z}.$$

For any interior (w.r.t. $\mathbb{R}^{d-1}$) point of $\hat{E}_d$, we then conclude that

$$(1-|x'|_1-x_1)^3 + x_1^3 = (1-z)^3\ \varphi(t) \le (1-z)^3.$$

By continuity this also holds on the boundary of $\hat{E}_d$. Hence, we arrive at

$$\|\sigma\partial_1\psi_{\hat{E}_d}\|^2_{L^2(\hat{K}_d)} = \frac{1}{3}d^{2d}\ \int_{\hat{E}_d}\left(1-\sum_{i=2}^{d-1}x_i\right)^3\ \prod_{i=2}^{d-1}x_i^2\ \sigma(x')^2dx'. \tag{4.12}$$

If $d = 2$, we obviously have

$$\left(1 - \sum_{i=2}^{d-1} x_i\right)^3 \prod_{i=2}^{d-1} x_i^2 = 1.$$

If $d \geq 3$, we must consider the function

$$\varphi(y) = \left(1 - \sum_{i=1}^{d-2} y_i\right)^3 \prod_{i=1}^{d-2} y_i^2$$

on $\hat{K}_{d-2}$. Since φ vanishes on the boundary $\partial\hat{K}_{d-2}$, it attains its maximum at an interior point. Since its derivatives are

$$\partial_j \varphi = \left(2 - 2\sum_{i=1}^{d-2} y_i - 3y_j\right) \left(1 - \sum_{i=1}^{d-2} y_i\right)^2 \; y_j \prod_{\substack{i=1 \\ i \neq j}}^{d-2} y_i^2 \quad , 1 \leq j \leq d-2,$$

all critical points are of the form $(a, \ldots, a)$ and satisfy

$$0 = 2 - (2d-1)a.$$

Hence, we obtain

$$\max_{y \in \hat{K}_{d-2}} |\varphi(y)| = \frac{27}{16} \left(\frac{2}{2d-1}\right)^{2d} (2d-1). \tag{4.13}$$

Obviously, this estimate also holds for $d = 2$.
Combining this with inequality (4.12), we obtain

$$\|\sigma \partial_1 \psi_{\hat{E}_d}\|_{L^2(\hat{K}_d)} \leq \frac{3}{4} \left(\frac{2d}{2d-1}\right)^d \sqrt{2d-1}\, \|\sigma\|_{L^2(\hat{E}_d)}. \tag{4.14}$$

From estimates (4.9) – (4.11) and (4.14) and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ we finally conclude that

$$\begin{aligned} \|\nabla(\psi_{\hat{E}_d}\sigma)\|_{L^2(\hat{K}_d)} \leq & \left\{ \frac{2d-1}{4} \left(\frac{2d}{2d-1}\right)^{2d} \right. \\ & + (d-1)\Big[\frac{\sqrt{3}}{6} \frac{d^d}{(d-1)^{d-1}} \sqrt{k(k+1)} \\ & \left. + \frac{3}{4}\left(\frac{2d}{2d-1}\right)^d \sqrt{2d-1}\Big]^2 \right\}^{1/2} \|\sigma\|_{L^2(\hat{E}_d)} \\ \leq & \left\{ \frac{9d-7}{8} (2d-1) \left(\frac{2d}{2d-1}\right)^{2d} \right. \\ & \left. + \frac{(d-1)^3}{6} \left(\frac{d}{d-1}\right)^{2d} k(k+1) \right\}^{1/2} \|\sigma\|_{L^2(\hat{E}_d)}. \end{aligned}$$

Since

$$h_{\hat{E}_d} = \begin{cases} 1 & \text{if } d = 2, \\ \sqrt{2} & \text{if } d \geq 3. \end{cases}$$

This proves the estimate of $\hat{\gamma}_4$ of Proposition 1.1.

Finally, we turn to the constant $\hat{\gamma}_5$. From Funbini's theorem we have

$$\begin{aligned} \|\psi_{\hat{E}_d}\sigma\|^2_{L^2(\hat{K}_d)} &= d^{2d} \int_{\hat{E}_d} \left\{ \int_0^{1-|x'|_1} (1-|x'|_1 - x_d)^2 \prod_{i=1}^{d-1} x_i^2 \; \sigma(x')^2 dx_d \right\} dx' \\ &= \frac{1}{3} d^{2d} \int_{\hat{E}_d} (1-|x'|_1)^3 \prod_{i=1}^{d-1} x_i^2 \; \sigma(x')^2 dx'. \end{aligned}$$

From estimate (4.13) we conclude that

$$\max_{x' \in \hat{E}_d} (1-|x'|_1)^3 \prod_{i=1}^{d-1} x_i^2 = \frac{27}{16} \left(\frac{2}{2d+1} \right)^{2d+2} (2d+1).$$

This implies that

$$\|\psi_{\hat{E}_d}\sigma\|_{L^2(\hat{K}_d)} \leq \frac{3}{2} \left(\frac{2d}{2d+1} \right)^d \frac{1}{\sqrt{2d+1}} \|\sigma\|_{L^2(\hat{E}_d)}.$$

Recalling the size of $h_{\hat{E}_d}$ this proves the last estimate of Proposition 1.1.

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