Constant-Free A Posteriori Error Estimates

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Outline

The Basic Steps of A Priori Error Estimation

- Derive a variational formulation of the differential equation.
- Replace the infinite dimensional test and trial spaces of the variational problem by finite dimensional subspaces consisting of functions which are piece-wise polynomials on a partition into non-overlapping subdomains.
- Abstract results (e.g. Lemmas of Céa and Lax-Milgram) imply that the discrete problem admits a unique solution and that its error is proportional to the error of the best approximation with a constant depending on properties of the variational problem.
- Bound the error of the best approximation by the error of a suitable interpolation.

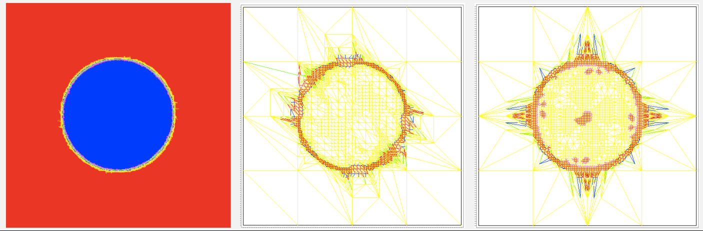
Drawbacks of A Priori Error Estimates

- They only yield information on the asymptotic behaviour of the error.
- They give no information on the actual size of the error and its spatial and temporal distribution.
- The error estimate is globally deteriorated by local singularities arising from e.g. re-entrant corners or interior or boundary layers.

Goals of A Posteriori Error Estimation and Adaptivity

- From the data of the differential equation and the computed solution of the discrete problem extract an easy-to-compute and precise information on the actual size of the error and its spatial and temporal distribution.
- Obtain an approximation for the solution of the differential equation with a given tolerance using a (nearly) minimal amount of unknowns.

Example: A Reaction-Diffusion Equation with an Interior Layer



	Triangles		Quadrilaterals	
	uniform	adaptive	uniform	adaptive
Unknowns	16129	2923	16129	4722
Triangles	32768	5860	0	3830
Quadrilaterals	0	0	16384	2814
Error	3.8%	3.5%	6.1%	4.4%

Standard Residual A Posteriori Error Estimates

- Prove the equivalence of error and residual.
- Derive an L^2-representation of the residual using integration by parts element-wise.
- Establish an upper bound for the dual norm of the residual using its Galerkin-orthogonality and error estimates for a suitable quasi-interpolation operator.
- Derive lower bounds for the dual norm of the residual using suitable local cut-off functions and inverse estimates.
- The upper and lower bounds involve multiplicative constants c^* and c_* which are not known explicitly and c^* may be larger than 1.

Constant-Free A Posteriori Error Estimates

- The theorem of Prager and Synge allows to express the error of any approximation in terms of vector-fields which are in equilibrium with the exterior force.
- Judiciously choose the vector-field.
- This yields an upper bound for the error with constant 1.
- The approach is completely different from the standard approach and superior.

Goals

- Constant-free a posteriori error estimates fit into the standard abstract framework.
- Contrary to standard residual estimates, constant-free estimates are not robust with respect to dominant reaction or convection terms.
- Using a suitable localization of the residual, the constant c^* of standard residual estimates can be expressed explicitly in terms of Poincaré constants which can be computed from geometric data of the partition.

Model Problem

$$\begin{aligned} -\Delta u + \kappa^2 u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned}$$

- First: $\kappa = 0$
- Later: $\kappa \gg 1$
- Energy norm:

$$|||u||| = \left\{ \|\nabla u\|^2 + \kappa^2 \|u\|^2 \right\}^{\frac{1}{2}}$$

Discretization

- $\mathcal{T}$: admissible, affine equivalent, shape regular partition
- $S_0^{k,0}(\mathcal{T})$: continuous, piece-wise polynomials of degree k vanishing on Γ
- $u_\mathcal{T} \in S_0^{k,0}(\mathcal{T})$: finite element approximation of u

Residual

- Define the residual as a continuous linear functional by
$$\begin{aligned} \langle R, v \rangle &= \int_\Omega f v - \int_\Omega \{ \nabla u_\mathcal{T} \cdot \nabla v + \kappa^2 u_\mathcal{T} v \} \\ &= \int_\Omega \{ \nabla (u - u_\mathcal{T}) \cdot \nabla v + \kappa^2 (u - u_\mathcal{T}) v \}. \end{aligned}$$
- Its dual norm is equivalent to the energy norm of the error $|||u - u_\mathcal{T}||| = |||R|||_*$.
- It admits the L^2-representation
$$\langle R, v \rangle = \int_\Omega r v + \int_\Sigma j v$$
with $r|_K = f + \Delta u_\mathcal{T} - \kappa^2 u_\mathcal{T}$ and $j|_E = -\mathbb{J}_E(\mathbf{n}_E \cdot \nabla u_\mathcal{T})$.
- It fulfils the Galerkin orthogonality $\langle R, v_\mathcal{T} \rangle = 0$ for all $v_\mathcal{T} \in S_0^{1,0}(\mathcal{T})$.

Theorem of Prager and Synge

- If
  - $\kappa = 0$,
  - u solves the model problem,
  - $U \in H^1(\Omega)$ is arbitrary,
  - $\rho \in H(\mathrm{div};\Omega)$ satisfies $-\operatorname{div}\rho = f$ in Ω
- then

  $\|\nabla u - \nabla U\|^2 = \|\rho - \nabla U\|^2 - \|\nabla u - \rho\|^2$

Proof

- For every $w \in H_0^1(\Omega)$:

  $$\int_\Omega \nabla u \cdot \nabla w = \int_\Omega f w = -\int_\Omega \operatorname{div} \rho w = \int_\Omega \rho \cdot \nabla w$$
- Insert $w = u$ and $w = U$:

  $$\begin{aligned} &\|\nabla u - \nabla U\|^2 + \|\nabla u - \rho\|^2 \\ &= 2\|\nabla u\|^2 - 2\int_\Omega \nabla u \cdot \nabla U + \|\nabla U\|^2 - 2\int_\Omega \rho \cdot \nabla u + \|\rho\|^2 \\ &= 2\int_\Omega \rho \cdot \nabla u - 2\int_\Omega \rho \cdot \nabla U + \|\nabla U\|^2 - 2\int_\Omega \rho \cdot \nabla u + \|\rho\|^2 \\ &= \|\nabla U\|^2 - 2\int_\Omega \rho \cdot \nabla U + \|\rho\|^2 \\ &= \|\nabla U - \rho\|^2 \end{aligned}$$

Standard Constant-Free Estimates

- Apply the theorem of Prager and Synge to $U = u_\mathcal{T}$.
- Then

  $$\|\nabla u - \nabla u_\mathcal{T}\| \le \|\nabla u_\mathcal{T} - \rho\|$$

  holds for every $\rho \in H(\mathrm{div};\Omega)$ with $-\operatorname{div}\rho = f$.
- Construct ρ judiciously.
- Myriads of constructions: Repin, Smolianski, Ern, Vohralik, Braess - Schöberl, ...

$H_\mathcal{T}(\mathrm{div};\Omega)$-Lifting of Residuals

- Assume that the continuous linear functional R satisfies
  - $\langle R, v\rangle = \int_\Omega r v + \int_\Sigma j v$ for all $v \in H_0^1(\Omega)$,
  - $\langle R, v_\mathcal{T}\rangle = 0$ for all $v_\mathcal{T} \in S_0^{1,0}(\mathcal{T})$.
- Then there is a vector field $\rho_\mathcal{T} \in H_\mathcal{T}(\mathrm{div};\Omega)$ with

  $\langle R, v\rangle = \int_\Omega \rho_\mathcal{T} \cdot \nabla v$ for all $v \in H_0^1(\Omega)$.
- $\rho_\mathcal{T}$ can be constructed by sweeping through the elements.
- If r and j are piece-wise polynomials $\rho_\mathcal{T}$ can be chosen from a broken RT- or BDM-space.
- $|\!|\!|R|\!|\!|_* \le \|\rho_\mathcal{T}\|$
- $\|\rho_\mathcal{T}\| \le c_s c_* \max_{K\in\mathcal{T}} \max\{1, h_K \kappa^2\} |\!|\!|R|\!|\!|_*$

Idea of the Proof

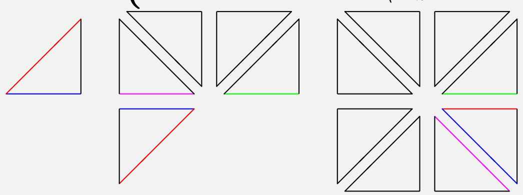
- Prove an auxiliary existence and stability result on elements.
- Construct $\rho_\mathcal{T}$ on patches of elements by marching through the elements.
- Put together the contributions of the patches.

A Single Element

- Assume that $f \in L^2(K)$ and $g \in L^2(\partial K)$ satisfy
  $$\int_K f + \int_{\partial K} g = 0.$$
- Then there is $\rho_K \in H(\operatorname{div}; K)$ with
  - $-\operatorname{div} \rho_K = f$ on K,
  - $\rho_K \cdot \mathbf{n}_K = g$ on ∂K.
- ρ_K satisfies the stability estimate
  $$\|\rho_K\| \le \frac{1}{\pi} h_K \|f\| + \frac{\sqrt{2\pi + 1}}{\pi} \left(\frac{h_K |\partial K|}{|K|} \right)^{\frac{1}{2}} h_K^{\frac{1}{2}} \|g\|.$$
- Proof:
  - $\rho_K = \nabla v_K$ with $-\Delta v_K = f$ on K and $\mathbf{n}_K \cdot \nabla v_K = g$ on ∂K.
  - Transform to the reference element and back.
  - Use the H^1-stability of the Neumann problem on the reference element.

A Patch of Elements

- Sweep through the elements K sharing a given vertex z.
- Apply the previous result to
  $$f = \lambda_z r \text{ and } g = \begin{cases} \lambda_z j & \text{on } (\partial K \cap \sigma_z) \setminus (E \cup E'), \\ \alpha_{E'} & \text{on } E', \\ \lambda_z j - \alpha_E & \text{on } E, \\ 0 & \text{on } \partial K \setminus \sigma_z \end{cases}$$



- The construction is feasible since $\langle R, \lambda_z \rangle = 0$.

Global Assembly

- The previous step yields vector-fields ρ_z with
  $$\int_{\omega_z} \rho_z \cdot \nabla v = \int_{\omega_z} \lambda_z r v + \int_{\sigma_z} \lambda_z j v.$$
- Set $\rho_\mathcal{T} = \sum_z \rho_z$.
- Then
  $$\begin{aligned} \int_\Omega \rho_\mathcal{T} \cdot \nabla v &= \sum_z \int_{\omega_z} \rho_z \cdot \nabla v \\ &= \sum_z \left\{ \int_{\omega_z} \lambda_z r v + \int_{\sigma_z} \lambda_z j v \right\} \\ &= \sum_z \langle R, \lambda_z v \rangle \\ &= \langle R, v \rangle. \end{aligned}$$

Localization of Residuals

- Since $\sum_z \lambda_z = 1$, we have for every $v \in H_0^1(\Omega)$
  $\langle R, v \rangle = \sum_z \langle R, \lambda_z v \rangle$,
  $$\sum_z \|\lambda_z^{\frac{1}{2}} \nabla v\|^2 = \sum_z \int_\Omega \lambda_z |\nabla v|^2 = \|\nabla v\|^2.$$
- The Galerkin orthogonality yields for every $v_z \in \mathbb{R}$ with $v_z \lambda_z \in S_0^{1,0}(\mathcal{T})$
  $\langle R, \lambda_z v \rangle = \langle R, \lambda_z (v - v_z) \rangle$.
- v_z can be chosen such that
  $\|\lambda_z^{\frac{1}{2}}(v - v_z)\| \le c(\omega_z) h_z \|\lambda_z^{\frac{1}{2}} \nabla v\|$
  $$\Big\{ \sum_{E \subset \sigma_z} h_E^{\perp} \|\lambda_z^{\frac{1}{2}}(v - v_z)\|_E^2 \Big\}^{\frac{1}{2}} \le c(\sigma_z) h_z \|\lambda_z^{\frac{1}{2}} \nabla v\|$$
  with $h_z = \operatorname{diam}(\omega_z) = \operatorname{diam}(\sigma_z)$ and $h_E^{\perp} = \frac{\int_{\omega_E} \lambda_z}{\int_E \lambda_z}$.

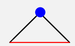
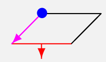
A Vertex-Oriented Residual Error Estimate

- The previous results yield the upper bound
  $$|||R|||_* \le \Big\{ \sum_z \eta_z^2 \Big\}^{\frac{1}{2}}$$
  with
  $$\eta_z = c(\omega_z) \alpha_z \|\lambda_z^{\frac{1}{2}} r\| + c(\sigma_z) \Big\{ \sum_{E \subset \sigma_z} \alpha_z h_z (h_E^{\perp})^{-1} \|\lambda_z^{\frac{1}{2}} j\|_E^2 \Big\}^{\frac{1}{2}},$$
  $\alpha_z = \min\{h_z, \kappa^{-1}\}$.
- η_z can be bounded from above by the standard element-oriented residual error estimator.
- Inverse estimates for local cut-off functions prove the standard lower bounds.

Poincaré and Friedrichs Inequalities

- Set $v_z = \dfrac{\int_{\omega_z} \lambda_z v}{\int_{\omega_z} \lambda_z}$ if $z \in \Omega$ and $v_z = 0$ if $z \in \Gamma$.
- Then $c(\omega_z)$ is the Poincaré or Friedrichs constant of ω_z with weight function λ_z.
- The Friedrichs constant can be expressed in terms of the corresponding Poincaré constant.
- $c(\omega_z) = \frac{1}{\pi}$ if ω_z is convex.
- If ω_z is not convex, $c(\omega_z)$ can be arbitrarily large and can be bounded explicitly and sharply in terms of the number of elements in ω_z and the ratio of the maximal over the minimal distance to z of all vertices on $\partial\omega_z \setminus \{z\}$.

Trace Equalities and Inequalities

- For every element K and every face E of K set
  - $\gamma_{K,E}(x) = x - a_{K,E}$ if K is a simplex,
  
  - $\gamma_{K,E}(x) = \frac{(x - a_{K,E}) \cdot \mathbf{n}_{K,E}}{\mathbf{m}_{K,E} \cdot \mathbf{n}_{K,E}} \mathbf{m}_{K,E}$ if K is a parallelepiped.
  
- Then the trace equality
  $$\frac{1}{|E|} \int_E w - \frac{1}{|K|} \int_K w = \frac{1}{\nu_K |K|} \int_K \gamma_{K,E} \cdot \nabla w$$
  holds with $\nu_K = d$ for simplices and $\nu_K = 1$ for parallelepipeds.
- The trace equality implies the trace inequality
  $h_E^{\perp} \|\lambda_z^{\frac{1}{2}} v\|_E^2 \le \|\lambda_z^{\frac{1}{2}} v\|_K^2 + \frac{2h_K}{\nu_K + 1} \|\lambda_z^{\frac{1}{2}} v\|_K \|\lambda_z^{\frac{1}{2}} \nabla v\|_K$.
- This allows to express $c(\sigma_z)$ in terms of $c(\omega_z)$.

The Role of the L^2-Representation

- An L^2-representation holds for all systems in divergence form.
- The contribution j of the skeleton Σ is the difficult part to handle.
- Constant-free estimates take care of this term by lifting it to $H_{\mathcal{T}}(\mathrm{div};\Omega)$.
- Residual estimates control this term with the help of trace inequalities.

The Role of the Galerkin Orthogonality

- The assumption of Galerkin orthogonality can be dropped.
- This gives rise to an additional consistency error.
- If the consistency error is due some Petrov-Galerkin stabilization or to an inexact solution of the discrete problem, it can be controlled by the error estimator.

The Role of Robustness

- Robustness is mandatory for singularly perturbed problems with dominant low order terms.
- The lack of robustness of the constant-free estimates is a structural drawback.
- It is due to the fact that the vector-field $\rho_{\mathcal{T}}$ only controls the principal part of the differential operator.
- Full robustness can be recovered by combining the constant-free estimates with standard robust residual estimates with explicit constants (cf. Ern et al).

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