Preamble. This is a reprint of the article:

M. Jost, M. Schulze Darup and M. Mönnigmann. Optimal and suboptimal event-triggering in linear model predictive control. In *Proc. of the 2015 European Control Conference*, pp. 1153–1158, 2015.

The digital object identifier (DOI) of the original article is:

10.1109/ECC.2015.7330695

Optimal and suboptimal event-triggering in linear model predictive control

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Abstract

We present two event-triggered MPC laws that do not require to solve a quadratic program (QP) in every time step but only upon certain events. We prove one of the control laws results in exactly the same closed-loop behavior as classical MPC. The second control law requires even fewer QPs per time. It is suboptimal w.r.t. the MPC cost function, but still results in asymptotically stable closed-loop behavior. We illustrate the event-triggered MPC laws with two examples.

1 Introduction

Model predictive control (MPC) is an established method for the control of multivariable constrained systems. Because MPC is based on perpetually solving optimal control problems, the computational cost of MPC is often a bottleneck.

As one possible remedy, MPC may be combined with ideas from event-triggered control, where feedback is not applied periodically but only when the system requires attention (see [1] for an introduction). Several authors investigated event-triggered MPC schemes before. One idea builds on triggering an event if the difference between the predicted and the real state trajectory becomes too large [2–4]. In a second family of approaches the optimal input sequence is recalculated when the rate of change of the MPC cost function, which serves as a Lyapunov function, is not sufficient anymore [5–8].

The present paper also combines event-triggered control and MPC, but the proposed method is new to the knowledge of the authors. The central ideas can be summarized as follows: (i) MPC is based on solving a QP for the current state in every time step. The optimal solution is usually understood to provide the optimal input for the current state (a point in the state space). The optimal solution does, however, not only provide the optimal solution for a *point* in the state space, but an affine control law that is optimal

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for an entire *polytope* in the state space. (ii) As long as future states do not leave this polytope, the affine control law can be reused, and therefore no QP needs to be solved. If the system leaves the polytope, this is considered to be the event that triggers solving the next QP. This simple idea gives rise to the first event-triggered MPC control law proposed here. (iii) Roughly speaking, the affine control law can be reused even if the system has left its domain polytope as long as the MPC cost function, which is a Lyapunov function under typical assumptions, still decreases. This idea can be used to construct another condition that triggers solving the next QP. The resulting control law, while suboptimal by construction, can be shown to result in an asymptotically stable closed-loop system.

The paper is organized as follows. First, we detail the problem formulation and state preliminaries in Section 2. The main result of the paper, i.e., the two novel event-triggered MPC schemes for constrained linear systems sketched above, are presented in Section 3. Two examples are given in Section 4 and an outlook is stated in Section 5.

1.1 Notation

For an arbitrary matrix $M \in \mathbb{R}^{a\times b}$, $M^{\mathcal{I}}$ with $\mathcal{I} \subseteq \{1, ..., a\}$ denotes the submatrix with the rows indicated by $\mathcal{I}$. Let M^i be short for $M^{\{i\}}$. Define $R_+ = \{r \in \mathbb{R}\,|r > 0\}$. A *polytope* is understood to be a set that results from the intersection of a finite number of halfspaces. Note that this definition implies polytopes are convex. Let $\mathcal{Q} = \{1, \ldots, q\}$, where q is the number of constraints of the MPC problem introduced in (4) below.

2 Problem Statement and Preliminaries

Consider a linear discrete-time state space system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k), \quad x(0) = x_0 \\ y(k) &= C\,x(k) \end{aligned} \tag{1}$$

with states $x(k) \in \mathbb{R}^n$, inputs $u(k) \in \mathbb{R}^m$, outputs $y(k) \in \mathbb{R}^p$ and system matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. Assume the states and inputs of (1) are subject to constraints

$$u(k) \in \mathcal{U} \subset \mathbb{R}^m \quad \text{and} \quad x(k) \in \mathcal{X} \subset \mathbb{R}^n \tag{2}$$

for all $k \in \mathbb{N}$, where the properties of $\mathcal{U}$, $\mathcal{X}$ are summarized in Assumption 1 below. MPC regulates the system (1) to the origin while respecting the constraints (2) by solving the optimal control problem

$$\begin{array}{ll} \min\limits_{X,U} & x(N)'P\,x(N) + \sum\limits_{i=0}^{N-1} x(i)'Q\,x(i) + u(i)'R\,u(i) \\ \text{s.t.} & x(0) = x, \\ & x(i+1) = Ax(i) + Bu(i),\ i = 0, \ldots, N-1, \\ & x(i) \in \mathcal{X}, \quad i = 0, \ldots, N-1, \\ & u(i) \in \mathcal{U}, \quad i = 0, \ldots, N-1, \\ & x(N) \in \mathcal{T}, \end{array} \tag{3}$$

on a receding horizon[1], where $X = (x'(1), \ldots, x'(N))'$, $U = (u'(0), \ldots, u'(N-1))'$ are introduced for convenience and where the weighting matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, and

[1] Note that $x(0) = x$ in (3) is an abuse of notation that is supposed to mean that initial condition for the prediction internal to (3) is set to the current actual state x. Since the predicted states are eliminated (see (4)), we do not introduce a more elaborate notation here.

$P \in \mathbb{R}^{n\times n}$ define the cost function. $\mathcal{T}$ is a terminal set and N is the horizon. We make the following assumptions throughout the paper.

Assumption 1: *Assume the matrices Q, R, and P are symmetric and positive definite. Assume the pair (A, B) is stabilizable. Finally, assume $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{T} \subseteq \mathcal{X}$ are convex and compact polytopes that contain the origin as an interior point.*

By eliminating the state variables, the optimal control problem (3) can be stated in the compact form

$$\min_{U} \; V(x, U) \quad \text{s.t.} \quad G\,U - E\,x \leq w, \tag{4}$$

with the cost function

$$V(x, U) = \frac{1}{2}U'H\,U + x'F\,U + \frac{1}{2}x'Yx, \tag{5}$$

where $Y \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times mN}$, $H \in \mathbb{R}^{mN\times mN}$, $G \in \mathbb{R}^{q\times mN}$, $w \in \mathbb{R}^{q}$, $E \in \mathbb{R}^{q\times n}$, and where q denotes the number of constraints [9]. It can be shown that H is positive definite under the assumptions stated above.

We call (4) feasible for a given $x \in \mathbb{R}^n$ if there exists a U such that the constraints are fulfilled. Let $\mathcal{X}_f \subset \mathcal{X}$ be the set of states for which (4) is feasible. Since H is positive definite, the solution to (4) is unique for all $x \in \mathcal{X}_f$. We denote this solution by $U^\star(x)$.

It proves convenient to introduce the set of constraints that are active respectively inactive for a given $x \in \mathcal{X}_f$. These sets are given by

$$\begin{aligned} \mathcal{A}(x) &= \left\{ i \in \{1, ..., q\} \,\middle|\, G^i\,U^\star(x) - E^i x = w^i \right\}, \\ \mathcal{I}(x) &= \left\{ i \in \{1, ..., q\} \,\middle|\, G^i\,U^\star(x) - E^i x < w^i \right\}, \end{aligned} \tag{6}$$

respectively.

For the remainder of the paper, $x(k)$ and $u(k)$ denote the actual state of the system (1) at time k and the actual input send to the system at time k, respectively (as opposed to the predicted states internal to the MPC problem (3)).

In the classical MPC setup, we compute $U^\star(x(k))$ at time instance k, use the first m elements as inputs, i.e.,

$$u(k) = \Phi\,U^\star(x(k)) \text{ where } \Phi = [I^{m\times m} \;\; 0^{m\times m(N-1)}] \tag{7}$$

and repeat the procedure at time instance $k + 1$ for the new state that actually results at time $k + 1$. Obviously, the optimization problem (4) is solved in every time step.

We intend to reduce the optimization effort by introducing two event-triggered control schemes that do not require to solve (4) in every time step. These control laws will exploit the special structure of the optimal solution $U^\star(x)$, which we briefly summarize in the following paragraph.

Bemporad et al. [9] showed that the optimal solution $U^\star : \mathcal{X}_f \to \mathcal{U}^N$ is continuous and piecewise affine. More precisely, there exist a finite number $\bar{p}$ of polytopes $\mathcal{P}_i$ with pairwise disjoint interiors, gains $K_i \in \mathbb{R}^{mN\times n}$, and biases $b_i \in \mathbb{R}^{mN}$ such that $\bigcup_{i=1}^{\bar{p}} \mathcal{P}_i = \mathcal{X}_f$ and

$$U^\star(x) = \begin{cases} K_1\,x + b_1 & \text{if } x \in \mathcal{P}_1, \\ \quad\vdots & \quad\vdots \\ K_{\bar{p}}\,x + b_{\bar{p}} & \text{if } x \in \mathcal{P}_{\bar{p}}, \end{cases} \tag{8}$$

is continuous. Since a polytope is the intersection of a finite number of halfspaces, there exist, for every $p \in \{1, \dots, \bar{p}\}$, $T_i \in \mathbb{R}^{r_i \times n}$ and $d_i \in \mathbb{R}^{r_i}$ such that

$$\mathcal{P}_i = \{x \in \mathbb{R}^n \,|\, T_i\, x \leq d_i\}\,. \tag{9}$$

We stress we never actually determine optimal control laws of the form (8) explicity. This is an important remark, since explicit laws of the form (8) can only be determined for fairly small problems. We do, however, calculate (7) locally. In fact, it is one of the central ideas of the present paper to infer which of the affine laws $K_i x + b_i$ and polytope $\mathcal{P}_i$ on the r.h.s. of (8) apply for the current $x(k)$, and to use this law, roughly speaking, as long as possible for future $x(k+j)$, $j \geq 0$. Before explaining this idea in detail in Section 3, we collect some established results on the stability of the MPC-controlled system (see, e.g., [7, 10]) in the following Lemma.

Lemma 1: *Let Assumption 1 hold and let P be the positive definite solution of the discrete-time algebraic Riccati equation (DARE). Consider the matrix $K_\infty = -(R + B'PB)^{-1}B'PA$ and let the terminal set $\mathcal{T}$ be such that $(A + B\,K_\infty)\,x \in \mathcal{T}$ and $K_\infty x \in \mathcal{U}$ for every $x \in \mathcal{T}$. Then the origin is an asymptotically stable steady state of the controlled system with domain of attraction $\mathcal{X}_f$. Moreover, the function*

$$V^\star : \mathcal{X}_f \to \mathbb{R},\; V^\star(x) = V(x, U^\star(x)) \tag{10}$$

is a Lyapunov function of the controlled system. In particular, there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\beta > \gamma$ such that

$$\alpha\,\|x\|_2^2 \leq V^\star(x) \leq \quad \beta\,\|x\|_2^2 \qquad \textit{and} \tag{11}$$

$$V^\star(x^+) - V^\star(x) \leq -\gamma\,\|x\|_2^2 \tag{12}$$

for every $x \in \mathcal{X}_f$, where $x^+ = A\,x + B\,\Phi\,U^\star(x)$.

3 Event Triggered MPC

We propose two simple event-triggered MPC schemes that reduce the computational effort of MPC. Essentially, we determine the affine law $K_i x + b_i$ and its domain $\mathcal{P}_i$ (see (8)) from the optimal solution $U^\star(x(k))$ in time step k. Instead of solving the MPC optimization problem (4) in the subsequent time steps $k+1$, $k+2$, etc., we then apply $K_i x + b_i$ as long as possible. The two variants of the proposed method differ w.r.t. the criterion that is used to decide how long $K_i x + b_i$ can be used before going back to solving (4) again.

We first summarize how to determine the particular affine law $K_i x + b_i$ and polytope $\mathcal{P}_i$ such that $x(k) \in \mathcal{P}_i$ from the optimal solution to (4) for the state $x(k)$ in Lemma 2. The two variants of their use are described in Sections 3.1 and 3.2 subsequently.

Lemma 2: *Let $x \in \mathcal{X}_f$ be arbitrary and assume the quadratic program (4) has been solved for the optimal $U^\star(x)$. Let $\mathcal{A}(x)$ and $\mathcal{I}(x)$ be as in (6) and denote them by $\mathcal{A}$ and $\mathcal{I}$ for brevity. Assume that the matrix $G^{\mathcal{A}}$ has full rank. Let*

$$\begin{aligned}
K^\star &= H^{-1}(G^{\mathcal{A}})'(G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})')^{-1}S^{\mathcal{A}} - H^{-1}F', \\
b^\star &= H^{-1}(G^{\mathcal{A}})'(G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})')^{-1}w^{\mathcal{A}}, \\
T^\star &= \begin{pmatrix} G^{\mathcal{I}}H^{-1}(G^{\mathcal{A}})'(G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})')^{-1}S^{\mathcal{A}} - S^{\mathcal{I}} \\ (G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})')^{-1}S^{\mathcal{A}} \end{pmatrix}, \\
d^\star &= -\begin{pmatrix} G^{\mathcal{I}}H^{-1}(G^{\mathcal{A}})'(G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})')^{-1}w^{\mathcal{A}} - w^{\mathcal{I}} \\ (G^{\mathcal{A}}H^{-1}(G^{\mathcal{A}})')^{-1}w^{\mathcal{A}} \end{pmatrix},
\end{aligned}$$

where $S = E + GH^{-1}F'$, $S \in \mathbb{R}^{q\times n}$. *Let*

$$\mathcal{P}^\star = \{x \in \mathbb{R}^n \,|\, T^\star x \leq d^\star\}. \tag{13}$$

Then $U^\star(x) = K^\star x + b^\star$, $x \in \mathcal{P}^\star$, *and* $K^\star x + b^\star$ *is the optimal control law* (8) *for all states in the polytope* $\mathcal{P}^\star$, *i.e.*

$$U^\star(\tilde{x}) = K^\star \tilde{x} + b^\star \text{ for all } \tilde{x} \in \mathcal{P}^\star. \tag{14}$$

Proof. The Karush-Kuhn-Tucker (KKT) conditions for (4) read

$$HU + F'x + G'\lambda = 0, \tag{15}$$
$$\lambda^i \left(G^i U - E^i x - w^i\right) = 0, \quad i = 1, \dots, q \tag{16}$$
$$\lambda^i \geq 0, \quad i = 1, \dots, q$$
$$GU - Ex - w \leq 0, \tag{17}$$

As a preparation recall the Lagrange multipliers of the inactive constraints are zero, i.e. $\lambda^{\mathcal{I}} = 0$, which follows from combining (16) and $G^iU - E^ix - w^i < 0$ for all $i \in \mathcal{I}$, where the latter statements holds according to the definition of $\mathcal{I}$ in (6).

Since H is invertible, (15) can be solved for U. This yields

$$\begin{aligned} U &= -H^{-1}F'x - H^{-1}G'\lambda \\ &= -H^{-1}F'x - H^{-1}G^{\mathcal{A}\prime}\lambda^{\mathcal{A}}, \end{aligned} \tag{18}$$

where we used $G'\lambda = G^{\mathcal{A}\prime}\lambda^{\mathcal{A}}$, which holds because $\lambda^{\mathcal{I}} = 0$. Since the active set $\mathcal{A}$ is known by assumption, we can split (17) into those rows that are fulfilled with equality and those that are fulfilled with inequality. The former ones read

$$G^{\mathcal{A}}U - E^{\mathcal{A}}x - w^{\mathcal{A}} = 0. \tag{19}$$

Substituting (18) yields

$$-G^{\mathcal{A}}H^{-1}G^{\mathcal{A}\prime}\lambda^{\mathcal{A}} - G^{\mathcal{A}}H^{-1}F'x - E^{\mathcal{A}}x - w^{\mathcal{A}} = 0$$

or equivalently

$$-G^{\mathcal{A}}H^{-1}G^{\mathcal{A}\prime}\lambda^{\mathcal{A}} - S^{\mathcal{A}}x - w^{\mathcal{A}} = 0. \tag{20}$$

The matrix $G^{\mathcal{A}}H^{-1}G^{\mathcal{A}\prime}$ is invertible since $H \succ 0$ and the matrix $G^{\mathcal{A}}$ has full rank by assumption. Thus, (20) can be solved for $\lambda^{\mathcal{A}}$. This yields

$$\lambda^{\mathcal{A}} = -\left(G^{\mathcal{A}}H^{-1}G^{\mathcal{A}\prime}\right)^{-1}\left(S^{\mathcal{A}}x + w^{\mathcal{A}}\right). \tag{21}$$

Substituting this result into (18) gives

$$\begin{aligned} U = &\left(H^{-1}G^{\mathcal{A}\prime}\left(G^{\mathcal{A}}H^{-1}G^{\mathcal{A}\prime}\right)^{-1}S^{\mathcal{A}} - H^{-1}F'\right)x \\ &+ H^{-1}G^{\mathcal{A}\prime}\left(G^{\mathcal{A}}H^{-1}G^{\mathcal{A}\prime}\right)^{-1}w^{\mathcal{A}}, \end{aligned}$$

which proves (14). The relation $T^\star x \leq d^\star$, i.e. the defining relation in (13), can be shown analogously by substituting (18) and (21) into $G^{\mathcal{I}}U - E^{\mathcal{I}}x - w^{\mathcal{I}} < 0$ instead of into (19). This results in the first rows in $T^\star$ and $d^\star$. The respective second rows result from $\lambda^{\mathcal{A}} > 0$ for $\lambda^{\mathcal{A}}$ from (21). ■

We assumed that $G^{\mathcal{A}}$ has full rank in Lemma 2 for simplicity only. If this is not the case, there still exists an affine control law $K^\star x + b^\star$ that is optimal on a polytope $\mathcal{P}^\star$ such that Lemma 2 holds. The Lagrange multipliers are not uniquely defined in this case, however, and the linear algebra required to solve the KKT conditions for U becomes more complicated. We omit this case in the present paper due to space limitations and refer to [9, Section 4.1.1] or [11, Section II.C].

3.1 Optimal Event-Triggered MPC

Since $K^\star x + b^\star$ defined in Lemma 2 is the optimal control law on $\mathcal{P}^\star$, it is obviously safe to use it as long as the state remains in $\mathcal{P}^\star$. The following corollary therefore is a direct consequence of Lemma 2.

Corollary 1: *Let $x \in \mathcal{X}_f$ and let $U^\star(x)$ be the solution of (4). Moreover, let $K^\star$, $b^\star$, and $\mathcal{P}^\star$ be defined as in Lemma 2. Consider the successor state $x^+ = A\,x + B\,\Phi\,U^\star(x)$. If $x^+ \in \mathcal{P}^\star$, then*

$$U^\star(x^+) = K^\star\,x^+ + b^\star. \tag{22}$$

Lemma 2 suggests to introduce a time-variant control law and an event-triggered update rule. We first introduce the control law and the update rule and comment on them immediately below. The time varying control law reads (23)

$$u(k) = \Phi\left(K^\dagger(k)\,x(k) + b^\dagger(k)\right), \tag{23}$$

$K^\dagger(k)$, $b^\dagger(k)$ and the polytope on which $K^\dagger x(k) + b^\dagger$ is used are updated according to the following event-triggered rule

$$\begin{pmatrix} K^\dagger(k) \\ b^\dagger(k) \\ \mathcal{P}^\dagger(k) \end{pmatrix} = \begin{cases} \begin{pmatrix} K^\star \\ b^\star \\ \mathcal{P}^\star \end{pmatrix} & \begin{array}{l}\text{if } k = 0 \text{ or} \\ x(k) \notin \mathcal{P}^\dagger(k-1),\end{array} \\ \begin{pmatrix} K^\dagger(k-1) \\ b^\dagger(k-1) \\ \mathcal{P}^\dagger(k-1) \end{pmatrix} & \text{otherwise,} \end{cases} \tag{24}$$

where $K^\star$, $b^\star$ and $\mathcal{P}^\star$ are defined as in Lemma 2 for $x = x(k)$. Rule (24) essentially states to determine the optimal $K^\star x + b^\star$ and its domain $\mathcal{P}^\star$ with Lemma 2 for the current state $x(k)$ if $x(k)$ has left the most recent polytope (case $x(k) \notin \mathcal{P}^\dagger(k-1)$ in (24)). In this case the QP (4) and Lemma 2 must be applied to obtain $U^\star(x(k))$ and $K^\star$, $b^\star$ and $\mathcal{P}^\star$ for the state $x(k)$. On the other hand, the optimal $K^\star x + b^\star$ can be reused as long as the state remains in $\mathcal{P}^\star$ (case *otherwise* in (24)). No calculations beyond evaluating the affine control law are required in this case, in particular no QP (4) needs to be solved.

The event-triggered control law (23), (24) is equal to the classical MPC control law (7) that requires solving the optimal control problem (3) (or equivalently (4)) in every time step. This is stated more precisely in the following proposition.

Proposition 1: *The control law* (7) *and the control law* (23) *with the event-triggered update rule* (24) *are equivalent in the sense that the resulting system states $x(k+1)$ with initial condition $x(0) = x_0 \in \mathcal{X}_f$ and the associated inputs $u(k)$ are identical for both controllers and every $k \in \mathbb{N}$.*

Proof. The proof can be carried out by induction. Consider the time instance $k = 0$. We have $u(0) = \Phi\,(K^\star\,x(0) + b^\star)$ and consequently $x(1) = A\,x(0) + B\,u(0)$ for both controllers, where $K^\star$ and $b^\star$ are defined as in Lemma 2 for $x = x(0)$. Now consider any $\tilde{k} \in \mathbb{N}$ with $\tilde{k} > 0$ and assume the states $x(k+1)$ and inputs $u(k)$ are identical for both controllers for all $k \in \{0, \ldots, \tilde{k}-1\}$. Let $K^\star$, $b^\star$ and $\mathcal{P}^\star$ be defined as in Lemma 2 for $x = x(\tilde{k})$. The classical MPC law (7) evaluates to

$$u(\tilde{k}) = \Phi\,(K^\star\,x(\tilde{k}) + b^\star). \tag{25}$$

For the event-triggered controller (23), (24), we have to distinguish the two cases (i) $x(\tilde{k}) \notin \mathcal{P}^\dagger(\tilde{k}-1)$ and (ii) $x(\tilde{k}) \in \mathcal{P}^\dagger(\tilde{k}-1)$ from one another. In case (i), the same input $u(\tilde{k})$ as (25) results according to (24). In case (ii), we have $\mathcal{P}^\dagger(\tilde{k}-1) = \mathcal{P}^\star$ and, consequently, $K(k) = K^\star$, $b(k) = b^\star$ and the same $u(\tilde{k})$ as (25) results. ■

The stability properties of the classical MPC control law (7) carry over to the event-triggered control law (23), (24).

Corollary 2: *Let the weighting matrix P and the terminal set $\mathcal{T}$ be defined as in Lemma 1. Then, the control law* (23) *with update rule* (24) *asymptotically stabilizes the origin with domain of attraction $\mathcal{X}_f$. Moreover, the function $V^\star : \mathcal{X}_f \to \mathbb{R}$ is a Lyapunov function of the controlled system.*

Proof. Since they are equal according to Proposition 1, the control law (23) inherits the stability properties of (7) stated in Lemma 1. ■

3.2 Suboptimal Event-Triggered MPC

It is the very point of the control law (23), (24) introduced in the previous section to reduce the computational effort of MPC by avoiding to solve the QP (4) whenever possible. We show in this section that even fewer QPs (4) need to be solved if a suboptimal control law is accepted. While it is suboptimal, the control law proposed below does maintain asymptotic stability.

Instead of reusing the locally optimal $K^\star x + b^\star$ as long as it remains optimal, we reuse it here as long as it results in a decreasing cost function and satisfies the constraints. This can be achieved as follows. Let $\lambda \in (0, 1)$ be arbitrary and consider the time-variant control law (23) with the new event-triggered update rule

$$\begin{pmatrix} K^\ddagger(k) \\ b^\ddagger(k) \\ V^\ddagger(k) \end{pmatrix} = \begin{cases} \begin{pmatrix} K^\star \\ b^\star \\ V^\star(x(k)) \end{pmatrix} & \begin{array}{l} \text{if } k = 0 \text{ or} \\ \hat{V} > \lambda V^\ddagger(k-1) \text{ or} \\ G\hat{U} - E\,x(k) > w, \end{array} \\ \begin{pmatrix} K^\ddagger(k-1) \\ b^\ddagger(k-1) \\ \hat{V} \end{pmatrix} & \text{otherwise}, \end{cases} \tag{26}$$

where $K^\star$ and $b^\star$ are defined as in Lemma 2 for $x = x(k)$, $V^\star(x)$ is as defined in (10), and

$$\begin{aligned} \hat{U} &= K^\ddagger(k-1)\,x(k) + b^\ddagger(k-1), \\ \hat{V} &= V(x(k), \hat{U}). \end{aligned} \tag{27}$$

Note that $\hat{U}$ is the input sequence that results from evaluating the control law $K^\ddagger(k-1)x + b^\ddagger(k-1)$ used in the previous time step for the current $x(k)$. $\hat{V}$ is the corresponding, generally suboptimal, value of the cost function. Essentially, (26) states to solve the MPC problem (4) and to determine $K^\star$, $b^\star$ with Lemma 2 in the current time step k, if the most recent control law results in a infeasible input (case $G\hat{U} - E\,x(k) > w$), or the most recent control law does not result in a sufficient decrease of the cost function (case $\hat{V} > \lambda V^\ddagger(k-1)$). In contrast, the most recent control law is reused, if it results in a feasible input and a sufficient decrease (case *otherwise*).

Before turning to the stability properties of the new control law, we need to confirm that (26) always results in a feasible input. Lemma 3 below states that a feasible U exists, even if the most recent control law results in an infeasible one (case $G\hat{U} - E\,x(k) > w$ in (26) with $\hat{U}$ from (27)). Moreover, Lemma 3 states that there exists a feasible U that results in a certain decrease of the cost function $V(x, U)$ of (4).

Lemma 3: *Let the weighting matrix P and the terminal set $\mathcal{T}$ be defined as in Lemma 1. Let $x \in \mathcal{X}_f$ and let $\hat{U} \in \mathbb{R}^{mN}$ be feasible for (4), i.e., $G\,\hat{U} - E\,x \leq w$. Consider the successor state $x^+ = A\,x + B\,\Phi\,\hat{U}$. Then, there exists a $U^+ \in \mathbb{R}^{mN}$ such that $G\,U^+ - E\,x^+ \leq w$ and*

$$V^*(x^+) \leq V(x^+, U^+) \leq V(x, \hat{U}) - \lambda_{\min}(Q)\,\|x\|_2^2. \tag{28}$$

Proof. Let $\hat{X} = (\hat{x}'(1), \ldots, \hat{x}'(N))'$ be the trajectory associated with the input sequence $\hat{U} = (\hat{u}'(0), \ldots, \hat{u}'(N-1))'$ and the initial state $\hat{x}(0) = x$. Since $\hat{U}$ is feasible for (4), $\hat{X}$ and $\hat{U}$ are feasible for the optimal control problem (3). Now let K_∞ be defined as in Lemma 1 and consider the input sequence $U^+ = (\tilde{u}'(0), \ldots, \tilde{u}'(N-1))'$ with $\tilde{u}(i) = \hat{u}(i+1)$ for $i = 0, \ldots, N-2$ and $\tilde{u}(N-1) = K_\infty\,\hat{x}(N)$. Let $X^+ = (\tilde{x}'(1), \ldots, \tilde{x}'(N))'$ be the trajectory associated with U^+ and the initial state $\tilde{x}(0) = x^+$. Note that $\tilde{x}'(i) = \hat{x}'(i+1)$ for $i = 1, \ldots, N-1$ and $\tilde{x}(N) = (A + B\,K_\infty)\hat{x}(N)$. Since $\hat{x}(N) \in \mathcal{T}$, we have $\tilde{u}(N-1) = K_\infty\,\hat{x}(N) \in \mathcal{U}$ and $\tilde{x}(N) = (A + B\,K_\infty)\hat{x}(N) \in \mathcal{T}$ according to Lemma 1. Thus, X^+ and U^+ are feasible for (3) at x^+ and U^+ is feasible for (4) at x^+, i.e., $G\,U^+ - E\,x^+ \leq w$.

To prove (28), first note that $V^*(x^+) \leq V(x^+, U^+)$ holds by optimality. The second relation in (28) can be proven as follows. The difference $V(x^+, U^+) - V(x, \hat{U})$ evaluates to

$$\begin{aligned}
&V(x^+, U^+) - V(x, \hat{U}) \\
&= \hat{x}'(N)((A + B\,K_\infty)'P(A + B\,K_\infty) - P)\,\hat{x}(N) \\
&\qquad - x'Q\,x - \hat{u}'(0)R\,\hat{u}(0), \\
&= -\hat{x}'(N)(Q + K_\infty' R K_\infty)\,\hat{x}(N) - x'Q\,x - \hat{u}'(0)R\,\hat{u}(0), \\
&\leq -x'Q\,x \leq -\lambda_{\min}(Q)\,\|x\|_2^2,
\end{aligned}$$

where the first equation holds by definition of $V(x, U)$ (see the objective function in (3)). The second equation results from the fact that $(A + B\,K_\infty)'P(A + B\,K_\infty) - P + Q + K_\infty' R K_\infty = 0$. Finally, the third relation holds since the matrices Q and R are positive definite by Assumption 1 and since $Q + K_\infty' R K_\infty$ is positive definite by construction. Rewriting $V(x^+, U^+) - V(x, \hat{U}) \leq -\lambda_{\min}(Q)\,\|x\|_2^2$ yields (28). ∎

We can now proceed to state the stability properties that result with the new event-triggered update rule (26).

Proposition 2: *Let the weighting matrix P and the terminal set $\mathcal{T}$ be defined as in Lemma 1 and let $\lambda \in (0, 1)$. Apply the control law (23) with event-triggered update rule (26) to the system (1). The origin is an asymptotically stable steady state of the controlled system with domain of attraction $\mathcal{X}_f$, i.e., for every $\epsilon > 0$ there exist $\delta > 0$ and $\tilde{k} \in \mathbb{N}$ such that*

$$\|x_0\|_2 < \delta \implies \qquad \|x(k)\|_2 < \epsilon, \quad \forall\, k \in \mathbb{N} \quad \textit{and} \tag{29}$$

$$x_0 \in \mathcal{X}_f \implies \qquad \|x(k)\|_2 < \epsilon, \quad \forall\, k \geq \tilde{k}, \tag{30}$$

where $x(k)$ denotes the state of the controlled system at time $k \in \mathbb{N}$. Moreover, we have $x(k) \in \mathcal{X}_f$ for every $k \in \mathbb{N}$ and every $x_0 \in \mathcal{X}_f$.

Proof. To prove $x(k) \in \mathcal{X}_f$, we show the relation

$$G\,(K^\ddagger(k)\,x(k) + b^\ddagger(k)) - E\,x(k) \leq w \tag{31}$$

holds for every $k \in \mathbb{N}$ and every $x_0 \in \mathcal{X}_f$, which implies feasibility of $x(k)$. Clearly, (31) holds for $k = 0$ since $x_0 \in \mathcal{X}_f$ and since $K^\ddagger(0) = K^\star$ and $b(0) = b^\star$, where $K^\star$ and $b^\star$ are defined as in Lemma 2 for $x = x(0)$. Now consider any $k > 0$ and assume (31) holds for $k - 1$. We either have

(i) $G\,(K^\ddagger(k-1)\,x(k)+b^\ddagger(k-1))-E\,x(k)\leq w$ and $V(x(k),K^\ddagger(k-1)\,x(k)+b^\ddagger(k-1))\leq \lambda V^\ddagger(k-1)$,

(ii) $G\,(K^\ddagger(k-1)\,x(k)+b^\ddagger(k-1))-E\,x(k)\leq w$ and $V(x(k),K^\ddagger(k-1)\,x(k)+b^\ddagger(k-1))> \lambda V^\ddagger(k-1)$, or

(iii) $G\,(K^\ddagger(k-1)\,x(k)+b^\ddagger(k-1))-E\,x(k)> w$.

In case (i), (31) holds since we have $K^\ddagger(k)=K^\ddagger(k-1)$ and $b^\ddagger(k)=b^\ddagger(k-1)$ according to (26). In case (ii) and (iii), feasibility of $x(k-1)$ guarantees feasibility of $x(k)$ according to Lemma 3. Thus, (31) holds with $K^\ddagger(k)=K^\star$ and $b^\ddagger(k)=b^\star$, where $K^\star$ and $b^\star$ are defined as in Lemma 2 for $x=x(k)$. In summary, $x(k-1)\in\mathcal{X}_f$ implies $x(k)\in\mathcal{X}_f$. Hence, $x(k)\in\mathcal{X}_f$ holds for every $k\in\mathbb{N}$ and every $x_0\in\mathcal{X}_f$ by induction.

It remains to prove relations (29) and (30). First note that $V^\ddagger(0)=V^\star(x(0))$. For all other steps k, the figure $V^\ddagger(k)$ in (26) represents the cost function value of a feasible or optimal solution to (4) for $x=x(k)$. Thus,

$$V^\ddagger(k)\geq V^*(x(k))\quad\text{for every}\quad k\in\mathbb{N}. \tag{32}$$

For any two consecutive values $V^\ddagger(k)$ and $V^\ddagger(k+1)$, we either have

$$V^\ddagger(k+1)\leq\lambda\,V^\ddagger(k) \tag{33}$$

by construction of (26) or

$$V^\ddagger(k+1)=V^\star(x(k+1))\leq V^\ddagger(k)-\lambda_{\min}(Q)\,\|x(k)\|_2^2 \tag{34}$$

according to Lemma 3. Since $\lambda\in(0,1)$ and $\lambda_{\min}(Q)>0$, we obtain $V^\ddagger(k+1)\leq V^\ddagger(k)$ in both cases. Thus

$$V^\ddagger(k)\leq V^\ddagger(0)=V^\star(x(0))\quad\text{for every}\quad k\in\mathbb{N}. \tag{35}$$

Combining Equations (11), (32), and (35) yields

$$\|x(k)\|_2^2\leq\frac{V^\star(x(k))}{\alpha}\leq\frac{V^\ddagger(k)}{\alpha}\leq\frac{V^\star(x_0)}{\alpha}\leq\frac{\beta}{\alpha}\|x_0\|_2^2. \tag{36}$$

From (36), we infer the choice $\delta=\sqrt{\frac{\alpha}{\beta}}\,\epsilon$ is such that (29) holds. To prove (30), let $\tilde{\gamma}=\min\{\alpha\,(1-\lambda),\lambda_{\min}(Q)\}$ and choose $\tilde{k}\in\mathbb{N}$ such that

$$\tilde{k}>\frac{\beta\,r^2}{\tilde{\gamma}\,\delta^2}-\frac{\alpha}{\tilde{\gamma}}, \tag{37}$$

where $r\in\mathbb{R}_+$ is such that $\|x\|_2\leq r$ for every $x\in\mathcal{X}_f$. Note that $\tilde{\gamma}>0$. Further note that a suitable choice of r is always possible since $\mathcal{X}_f\subseteq\mathcal{X}$ and since $\mathcal{X}$ is compact. Instead of proving (30) directly, we show that

$$x_0\in\mathcal{X}_f\quad\Longrightarrow\quad\exists k\in\{0,\dots,\tilde{k}\}:\quad\|x(k)\|_2<\delta. \tag{38}$$

Clearly, (38) in combination with (29) implies (30). To prove (38), assume the relation does not hold and show that a contradiction results. Clearly, if (38) is false, we have $\|x(k)\|_2\geq\delta$ for every $k\in\{0,\dots,\tilde{k}\}$. Thus

$$V^\ddagger(\tilde{k})\geq V^\star(x(\tilde{k}))\geq\alpha\,\|x(\tilde{k})\|_2^2\geq\alpha\,\delta^2. \tag{39}$$

On the other hand, $V^{\ddagger}(\tilde{k})$ can be written as

$$V^{\ddagger}(\tilde{k}) = V^{\ddagger}(0) + \sum_{k=0}^{\tilde{k}-1} V^{\ddagger}(k+1) - V^{\ddagger}(k). \tag{40}$$

In this context, we recognize that (33) implies

$$\begin{aligned} V^{\ddagger}(k+1) - V^{\ddagger}(k) &\leq -(1-\lambda)\, V^{\ddagger}(k), \\ &\leq -(1-\lambda)\, V^{\star}(x(k), \\ &\leq -\alpha\,(1-\lambda)\, \|x(k)\|_2^2, \end{aligned}$$

where the first and the second relation result from (32) and (11), respectively. In combination with (34), we obtain

$$\begin{aligned} V^{\ddagger}(k+1) - V^{\ddagger}(k) &\leq -\min\{\alpha\,(1-\lambda), \lambda_{\min}(Q)\}\, \|x(k)\|_2^2, \\ &= -\tilde{\gamma}\, \|x(k)\|_2^2, \end{aligned}$$

which holds for every $k \in \mathbb{N}$. Overestimating (40) yields

$$\begin{aligned} V^{\ddagger}(\tilde{k}) &\leq V^{\star}(x_0) - \sum_{k=0}^{\tilde{k}-1} \tilde{\gamma}\, \|x(k)\|_2^2 \\ &\leq \beta\, \|x_0\|_2^2 - \sum_{k=0}^{\tilde{k}-1} \tilde{\gamma}\, \delta^2 \\ &\leq \beta\, r^2 - \tilde{k}\, \tilde{\gamma}\, \delta^2 \end{aligned} \tag{41}$$

Clearly, (39) and (41) can only be valid at the same time, if

$$\alpha\, \delta^2 \leq \beta\, r^2 - \tilde{k}\, \tilde{\gamma}\, \delta^2. \tag{42}$$

Relation (42), however, contradicts (37). ■

4 Examples

We apply both approaches to two different examples.

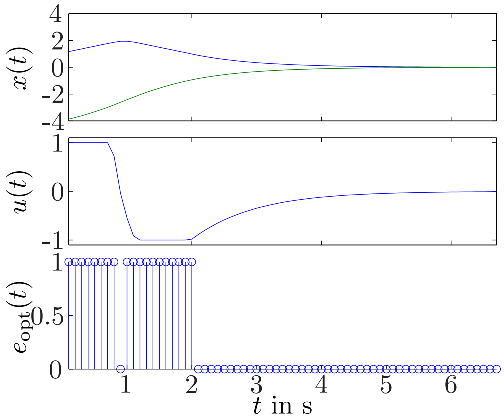
Double Integrator. Consider the double integrator system that results from discretizing $\ddot{y} = u$ with zero-order hold (ZOH) and a sample time of $T_s = 0.1s$. We impose the state and input constraints $-5 \leq x_1(k) \leq 5$, $-5 \leq x_2(k) \leq 5$ and $-1 \leq u(k) \leq 1$, respectively. We set P to the solution of the DARE, and $R = 0.1$, $Q = I^{n \times n}$. Furthermore, we chose the horizons to $N = 10$ and $\lambda = 0.99$.

MIMO system. We consider the state space system that results from discretizing the continuous-time transfer function

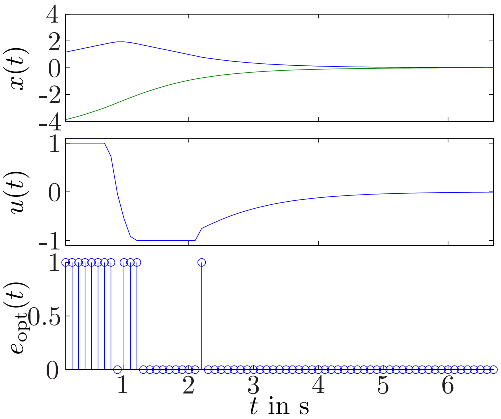
$$G(s) = \begin{pmatrix} \frac{-5s+1}{36s^2+6s+1} & \frac{0.5s}{8s+1} & 0 \\ 0 & \frac{0.1(-10s+1)}{s(8s+1)} & \frac{-0.1}{(64s^2+6s+1)s} \\ \frac{-2s+1}{12s^2+3s+1} & 0 & \frac{2(-5s+1)}{16s^2+2s+1} \end{pmatrix}, \tag{43}$$

with ZOH and $T_s = 1$s. The state and input constraints read $-10 \leq x_i(k) \leq 10$ for $i = 1, \ldots, 10$ and $-1 \leq u_j(k) \leq 1$ for $j = 1, \ldots, 3$, respectively. The matrix P is set to the solution of the DARE, and the weighting matrices on the states and inputs are $Q = I^{n \times n}$

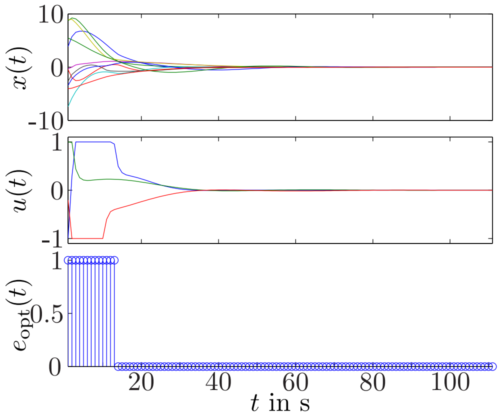
and $R = 0.25 I^{m \times m}$, respectively. The horizon is chosen to be $N = 30$. We pre-stabilize the system with the LQR controller proposed in [12] to obtain a well-conditioned optimization problem. Again, we chose $\lambda = 0.99$ for our numerical experiments. Terminal sets $\mathcal{T}$ are constructed with the algorithms from [13, 14] for both examples.



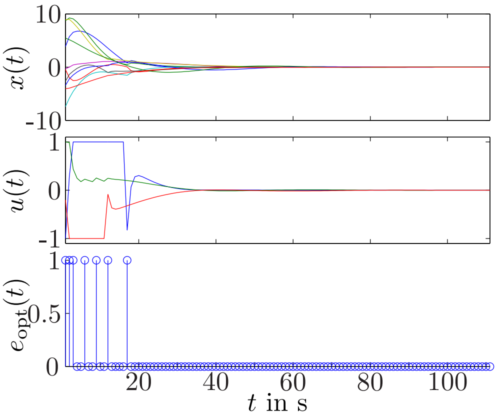
(a) Optimal triggered MPC, double integrator



(b) Suboptimal triggered MPC, double integrator



(c) Optimal triggered MPC, MIMO system



(d) Suboptimal triggered MPC, MIMO system

Figure 1: Results of the optimal event-triggered MPC (a,c) and the suboptimal event-triggered MPC (b,d) for both examples.

Results for both examples and both the optimal and suboptimal variant of the proposed method are shown in Figure 1 for a random initial point. The figures show the trajectories of the states $x(t)$, the inputs $u(t)$ and the trigger functions $e_{\text{opt}}(t)$ and $e_{\text{sub}}(t)$, which indicates if a trigger event occurs. Specifically,

$$e_{\text{opt}}(t) = \begin{cases} 1 & \text{if } k = 0 \text{ or } x(k) \notin \mathcal{P}^{\dagger}(k-1) \\ 0 & \text{otherwise} \end{cases},$$

and

$$e_{\text{sub}}(t) = \left\{ \begin{array}{cc} 1 & \begin{array}{c} \text{if } k = 0 \text{ or } \hat{V} > \lambda V^{\ddagger}(k-1) \text{ or} \\ G\hat{U} - E\,x(k) > w \end{array} \\ 0 & \text{otherwise} \end{array} \right. ,$$

respectively. We calculated the trajectories of the closed-loop system until $\|x(k)\|_2 \leq 10^{-2}$.

We stress that the results shown in Figure 1 are only meant to corroborate the theoretical findings presented in the previous sections. In particular the shown data is obviously not statistically relevant. Nevertheless, the results shown in Figure 1 illustrate that the proposed event-triggered MPC controllers work as anticipated. Consider the results for the optimal event-triggering according to rule (24) first. In the double integrator example, the MPC problem is solved for all but one time ($t = 1$) for the first two seconds. Then the MPC optimization is never triggered again afterwards (cf. Fig. 1a). For the second example, the MPC optimization is triggered in every time step up to a certain point, and never triggered again afterwards (cf. Fig. 1c). In both cases, optimal event-triggered MPC detects that no MPC problem (3) needs to be solved once a certain neighborhood of the origin has been reached. (Specifically, 19 trigger events occur with (24) and thus 19 QPs (4) are solved for the double integrator example for the simulated time period of 6.6s. Since 6.6s result in 66 sampling periods, 66 QPs are solved by the classical MPC controller. For the MIMO example, 13 trigger events occur with (24) and thus 13 QPs are solved. The classical MPC controller solves 111 QPs, which result from a simulation time of 111s and $\Delta T = 1$s.)

The suboptimal rule (26) results in a nontrivial triggering, see Figures 1b, d. Even for the very simple double integrator example, a period exists in which a suboptimal control law is selected for a certain period. For the MIMO example, this happens repeatedly. The suboptimal triggering rule results in trajectories that obviously are suboptimal (cp. the trajectories in Figure 1a to those in b and the trajectories in c to those in d) but drive the system to the origin as anticipated. (We note that in 12 out of 66 and in 7 out of 111 sampling times the suboptimal rule (26) triggers a QP for the double integrator and MIMO example, respectively, while classical MPC solves 66 QPs and 111 QPs for the double integrator and MIMO example, respectively.)

5 Conclusion and Outlook

We proposed two event-triggered variants of MPC, showed that they result in asymptotically stable closed-loop systems, and illustrated them with two simple examples. Future research has to investigate the impact of the suboptimality parameter λ, and the extension to nonlinear MPC.

Acknowledgment

Support by the Deutsche Forschungsgemeinschaft (DFG) under grant MO 1086/11-1 is gratefully acknowledged.

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