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Stability of Saddle Point Problems with Penalty

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We consider saddle point problem with penalty terms. A counterexample shows that penalty terms may have a destabilizing effect. Two extra conditions which guarantee stability are established. A short review of the theory of the Mindlin-Reissner plates is given in this framework.

1. Introduction

In the framework of mixed finite elements the theory of classical saddle point problems is well developed since Brezzi's famous paper [8]. Now saddle point problems with penalty terms occur when some stabilized elements are used or when almost incompressible materials are treated. Moreover, mixed methods with penalty are crucial for the analysis of Mindlin-Reissner plates. These problems with penalty are not so well understood. A reason for the lack of investigations is the folklore that penalty terms always have a stabilizing influence on the saddle point problem.

It has turned out, however, that the situation is not so simple and that some extra conditions are necessary to guarantee stability. One reason is the fact that in actual problems the penalty terms correspond to singular perturbations. We will provide a general theory for saddle point problems in Hilbert spaces which extends Brezzi's result when penalty terms are present.

In this framework the recent development of finite elements for Mindlin-Reissner plates becomes very clear. Some arguments which looked to be technical. appear now to be very natural. For this reason we provide a short review of Mindlin plate theory in the last section.

A part of the results can be found in the book [6] which is written in German.

2. Saddle Point Problems with Penalty in Hilbert Spaces

Let X and M be Hilbert spaces with norms $\|.\|_X$ and $\|.\|_M$, respectively. Moreover let M_c be a dense linear subspace of M. The saddle point problem will be fixed by the definition of three bilinear forms

$$\begin{aligned} a:&\ \ X\times X\to\mathbb{R}, &&a(u,v)=a(v,u)\\ b:&\ \ X\times M\to\mathbb{R}, &&\\ c:&\ M_c\times M_c\to\mathbb{R}, &&c(p,q)=c(q,p),\ \ c(q,q)\geq 0. \end{aligned}\tag{2.1}$$

The forms a and b are assumed to be bounded. On the other hand c gives rise to a seminorm

$$|q| := c(q,q)^{1/2} \quad \text{for } q \in M_c\,. \tag{2.2}$$

Finally t will be considered a small parameter, $0 \le t \le 1$.

Saddle Point Problem with Penalty. (S_t) Given $f \in X'$ and $g \in M_c'$, find $(u,p) \in X \times M_c$ such that

$$\begin{aligned} a(u,v) \ &+ b(v,p) &&= < f, v > \quad &&\text{for } v \in X,\\ b(u,q) \ &- t^2 c(p,q) &&= < g, q > \quad &&\text{for } q \in M_c\,. \end{aligned} \tag{2.3}$$

As usual with this variational problem the bilinear form

$$A(u,p;v,q) := a(u,v) + b(v,p) + b(u,q) - t^2 c(p,q) \tag{2.4}$$

is connected. For the saddle point problem the norm

$$|||(v,q)||| := \|v\|_X + \|q\|_M + t|q|_c \tag{2.5}$$

is the natural one. The problem S_t will be called *stable*, if

$$\sup_{v,q} \frac{A(u,p;v,q)}{|||(v,q)|||} \ge \gamma |||(u,p)||| \tag{2.6}$$

holds with some $\gamma > 0$ for all $(u,p) \in X \times M_c$ and $0 \le t \le 1$.

A natural point of departure is Brezzi's splitting theorem [8].

Theorem A. *The classical saddle point problem (with $t = 0$) is stable, if and only if the following conditions hold:*
1^o*. The bilinear form a is Z-elliptic with*

$$Z := \{v \in X;\ b(v,q) = 0 \textit{ for all } q \in M\},$$

i.e., there is $\alpha > 0$ such that

$$a(v,v) \ge \alpha \|v\|_X^2 \quad \textit{for } v \in Z. \tag{2.7}$$

2^o*. The bilinear form b satisfies an inf-sup condition*

$$\inf_{q \in M} \sup_{v \in X} \frac{b(v,q)}{\|v\|_X \|q\|_M} = \beta > 0. \tag{2.8}$$

In view of the folklore mentioned above we are interested in the influence of the penalty term. To be precise, we put the following

Question. Assume that the classical saddle point problem is stable and that c is a nonnegative symmetric quadratic form. Is the penalized saddle point also stable, i.e., does (2.6) hold with $\gamma > 0$ for all $0 \le t \le 1$?

There are two positive answers. The first refers to the case in which c is a bounded bilinear form (s. [7]).

Theorem 1. *If the classical saddle point problem is stable and if*

$$c(p,q) \le C_0 \|p\|_M \|q\|_M \quad \textit{for all} \;\; p,q \in M, \tag{2.9}$$

then the extended problem is stable for $0 \le t \le 1$.

Another positive result refers to the case that the quadratic form a is elliptic on the whole space X and not only on the kermel Z (s. [13]).

Theorem 2. *If the classical saddle point problem is stable and if*

$$a(v,v) \ge \alpha \|v\|_X^2 \quad \textit{for all} \;\; v \in X \tag{2.10}$$

holds with $\alpha > 0$*, then the extended problem is stable for* $0 \le t \le 1$.

Although the extra conditions in the two theorems refer to very different features, we cannot do without them.

Counterexample. Let $X = M = L_2(\Omega)$ and $M_c = H^1(\Omega)$ and

$$\begin{aligned} a(u,v) \;&:= 0, \\ b(v,q) \;&:= (v,q)_{L_2}, \\ c(p,q) \;&:= (\nabla p, \nabla q)_{L_2}. \end{aligned}$$

The classical version of the problem

$$\begin{aligned} (v,p)_{L_2} \qquad &= (f,v)_{L_2} \qquad \forall v \in L_2\,, \\ (u,q)_{L_2} - t^2 (\nabla p, \nabla q)_{L_2} &= (g,q)_{L_2} \qquad \forall q \in H^1 \end{aligned}$$

is obviously stable. If on the other hand $t > 0$, then a formal solution is

$$p = f, \quad u = g - t^2 \triangle p, \tag{2.11}$$

provided that $f \in H_0^2(\Omega)$. There is no stability with respect to the norm (2.5).

3. Proofs

Originally, the theorems in the preceding section were proved by independent arguments. It was observed by Kirmse [14] that they are corollaries of the following lemma, see also [6. p. 130].

Lemma 3. *Assume that the classical saddle point problem is stable. If, in addition*

$$\frac{a(u,u)}{\|u\|_X} + \sup_{q \in M_c} \frac{b(u,q)}{\|q\|_M + t|q|_c} \ge \alpha \|u\|_X \tag{3.1}$$

holds with $\alpha > 0$*, then the augmented problem is stable for* $0 \le t \le 1$ *and the constant* γ *in (2.6) depends only on* α*,* β*, and* $\|a\|$.

For convenience, the factors in (2.7) and (3.1) are written with the same symbol α. This is possible, since α may be replaced in both inequalities by a smaller value if necessary.

At the first glance, the lemma looks very technical. On the other hand it is equivalent to an inf-sup condition

$$\sup_{v,q} \frac{A(u,0;v,q)}{|||(v,q)|||} \geq \alpha' \|u\|_X \tag{3.2}$$

with some $\alpha' > 0$. Indeed, (3.1) implies (3.2) with $\alpha' < \alpha/2$. On the other hand, we conclude from (3.2) that

$$\begin{aligned}
\alpha' \|u\|_X &\leq \sup_{(v,q)} \frac{a(u,v)+b(u,q)}{\|v\|_X + \|q\|_M + |q|_c} \\
&\leq \sup_v \frac{a(u,v)}{\|v\|_X} + \sup_q \frac{b(u,q)}{\|q\|_M + |q|_c} \\
&\leq \{\|a\|\, a(u,u)\}^{1/2} + \sup_q \frac{b(u,q)}{\|q\|_M + |q|_c}\,.
\end{aligned} \tag{3.3}$$

Now we use the fact that

$$x \leq y + z \quad \text{implies} \quad x \leq 2y + \frac{z^2}{x} \tag{3.4}$$

whenever x, y, and z are positive numbers, and obtain from (3.3)

$$\alpha' \|u\|_X \leq \frac{\|a\|}{\alpha'} \frac{a(u,u)}{\|u\|_X} + 2 \sup_q \frac{b(u,q)}{\|q\|_M + |q|_c}\,.$$

Hence, (3.1) holds with $\alpha \geq {\alpha'}^2/2\|a\|$, and the proof of the equivalence is complete.

Obviously, (3.2) is a special case of (2.6) and is therefore also a necessary condition. The essential difference between (3.1) and the corresponding relation in the classical case is the additional term $|q|_c$ in the denominator.

Proof of Lemma 3. First we note that

$$t|p|_c \leq [a(u,u) + t^2|p|_c^2]^{1/2} = A(u,p;u,-p)^{1/2}. \tag{3.5}$$

We distinguish two cases:
Case 1. Let $\|u\|_X \leq \frac{\beta}{2\|a\|}\{\|p\|_M + t|p|_c\}$. From the inf-sup condition (2.8) it follows that in this case

$$\begin{aligned}
\beta \|p\|_M &\leq \sup_v \frac{b(v,p)}{\|v\|_X} = \sup_v \frac{A(u,p;v,0) - a(u,v)}{\|v\|_X} \\
&\leq \sup_{v,q} \frac{A(u,p;v,q)}{|||(v,q)|||} + \|a\|\,\|u\|_X \\
&\leq \sup_{v,q} \frac{A(u,p;v,q)}{|||(v,q)|||} + \frac{1}{2}\beta\|p\|_M + \frac{1}{2}\beta t|p|_c\,.
\end{aligned}$$

The second term in the last expression can be absorbed by spending a factor 2. We set $\beta_1 = \min\{\|a\|, \beta\}$ and obtain

$$
\begin{aligned}
|||(u,p)||| &\le \beta_1^{-1}(\|a\| \cdot \|u\|_X + \beta\|p\|_M + \beta t|p|_c) \\
&\le \frac{3}{2}\beta_1^{-1}(\beta\|p\|_M + \beta t|p|_c) \\
&\le \beta_1^{-1}(3 \sup_{v,q} \frac{A(u,p;v,q)}{|||(v,q)|||} + \frac{5}{2}\beta t|p|_c) \\
&\le 3\beta_1^{-1} \sup_{v,q} \frac{A(u,p;v,q)}{|||(v,q)|||} + \frac{5}{2}\,\beta\beta_1^{-1} A(u,p;u,-p)^{1/2}.
\end{aligned}
\tag{3.4}
$$

Referring to (3.4) we obtain

$$
\begin{aligned}
|||(u,p)||| &\le 6\beta_1^{-1} \sup_{v,q} \frac{A(u,p;v,q)}{|||(v,q)|||} + \frac{25}{4}\,\beta^2\beta_1^{-2} \frac{A(u,p;u,-p)}{|||(u,-p)|||} \\
&\le 7\beta_1^{-1}(1+\beta^2\beta_1^{-1}) \sup_{v,q} \frac{A(u,p;v,q)}{|||(v,q)|||}.
\end{aligned}
$$

Case 2. Let $\|u\|_X \ge \frac{\beta}{2\|a\|}\{\|p\|_M + t|p|_c\}$. This implies $|||(u,p)||| \le M\|u\|_X$ with $M := 1 + 2\|a\|/\beta$. From assumption (3.1) and (3.5) we conclude

$$
\begin{aligned}
\alpha M^{-1}|||(u,p)||| &\le \alpha\|u\|_X \\
&\le \frac{a(u,u)}{\|u\|_X} + \sup_q \frac{A(u,p;0,q) + t^2 c(p,q)}{\|q\|_M + t|q|_c} \\
&\le \frac{A(u,p;u,-p)}{M^{-1}|||(u,p)|||} + \sup_q \frac{A(u,p;0,q)}{|||(0,q)|||} + t|p|_c \\
&\le (1+M) \sup_{v,q} \frac{A(u,p;v,q)}{|||(v,q)|||} + A(u,p;u,-p)^{1/2}.
\end{aligned}
$$

We multiply by $\alpha^{-1}M$ and refer once more to (3.4)

$$
\begin{aligned}
|||(u,p)||| &\le 2\alpha^{-1}M(1+M) \sup_{v,q} \frac{A(u,p;v,q)}{|||(v,q)|||} + (\alpha^{-1}M)^2 \frac{A(u,p;u,-p)}{|||(u,-p)|||} \\
&\le \alpha^{-1}M(2+2M+\alpha^{-1}M) \sup_{v,q} \frac{A(u,p;v,q)}{|||(v,q)|||}.
\end{aligned}
$$

Therefore, in each case the inf-sup condition for $A(u,p;v,q)$ holds. □

Theorems 1 and 2 are now immediate.

Ellipticity on the whole space (2.10) means that (3.1) holds even if we ignore the second term on the left hand side. This proves theorem 2.

To prove theorem 1 we start with an estimate of the pair $(u,0)$ by the inf-sup condition for the classical problem and recall that $\|q\|_M + t|q|_c \le (1+\|c\|)\|q\|_M$:

$$
\gamma\|u\|_X \le \sup_{(v,q)} \frac{a(u,v)+b(u,q)}{\|v\|_X + \|q\|_M} \le (1+\|c\|) \sup_{(v,q)} \frac{a(u,v)+b(u,q)}{\|v\|_X + \|q\|_M + |q|_c}\,.
$$

Hence, (3.2) holds with $\alpha = \gamma/(1+\|c\|)$. □

We note that the proof of stability is much easier if the quadratic form c is assumed to be also elliptic on M, i.e., $c(q,q) \ge c_2(\|q\|_M^2 + |q|_c^2)$. For this we refer to [1], [12, p. 65] and [3]. In order to see that Theorem 2 in [3] is a special case of Theorem 2, one has to identify M_c with W and M with the completion of W with respect to the norm $\|q\|_M := \sup_X\{b(v,q)/\|v\|_X\}$.

4. Review of Mindlin-Reissner-Plate Theory

The Reissner-Mindlin plate is a model which is difficult due to a small parameter which is related to the thickness of the plate. In particular the small parameter implies that simple finite element discretizations are not successful. In the last years there has been much progress by treating the Mindlin plate as a saddle point problem with penalty. This model is attractive also since the corresponding classical saddle point problem refers to the Kirchhoff plate.

The saddle point formulation of the Mindlin plate which shows its neighborhood to the Kirchhoff plate does not satisfy the assumptions of theorem 1 or 2. Now the successful investigations of the last decade can be understood as a search for a possibility to reduce the problem to another one which can be covered by these theorems. Under this viewpoint the recent treatment of the finite element discretizations look very natural and are by no means technical.

For this reason we present a short review of the stability theory for the plate model. In this framework we disregard the questions which are concerned with the good approximation of boundary layers, see e.g. [5].

The Variational Formulation

Let $\Omega \subset \mathbb{R}^2$ denote the region in $\mathbb{R}^2$ which is occupied by the plate when it is projected onto 2-space. The deformation of the plate at each $x \in \Omega$ is specified by the vertical displacement w and the rotation $\theta = (\theta_1, \theta_2)$ of the fibers normal to its midsection. Moreover, let t be the thickness of the plate.

The variational problem for the clamped plate may be written as follows.
Find $(w,\theta) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$ such that

$$a(\theta,\psi) + \lambda t^{-2}(\nabla w - \theta, \nabla v - \psi) = (f,v) \qquad \forall (v,\psi) \in H_0^1(\Omega) \times H_0^1(\Omega) \tag{4.1}$$

with

$$\begin{aligned} a(\theta,\psi) := \frac{E}{12(1-\nu^2)} \int_\Omega \Big[\Big(\frac{\partial\theta_1}{\partial x} + \nu\frac{\partial\theta_2}{\partial y}\Big)\frac{\partial\psi_1}{\partial x} + \Big(\nu\frac{\partial\theta_1}{\partial x} + \frac{\partial\theta_2}{\partial y}\Big)\frac{\partial\psi_2}{\partial y} \\ + \frac{1-\nu}{2}\Big(\frac{\partial\theta_1}{\partial y} + \frac{\partial\theta_2}{\partial x}\Big)\Big(\frac{\partial\psi_1}{\partial y} + \frac{\partial\psi_2}{\partial x}\Big)\Big]. \end{aligned} \tag{4.2}$$

Here, E is Young's modulus, ν the Poisson ratio, k the shear correction factor, and $\lambda = Ek/2(1+\nu)$. Moreover, $(.,.)$ refers to the inner product in $L_2(\Omega)$.

By Korn's inequality, $a(\theta, \psi)$ is an inner product on $H_0^1(\Omega)^2$ which is equivalent to

$$(\nabla\theta, \nabla\psi). \tag{4.3}$$

For convenience, we will ignore the factor λ since it can be a eliminated from (4.1) by a simple rescaling of a and of the load f.

The displacement formulation of the plate (4.1) is changed into a mixed formulation by the introduction of the shear strain vector

$$\gamma := t^{-2}(\nabla w - \theta). \tag{4.4}$$

Equs. (4.1) and (4.4) are equivalent to

$$\begin{aligned} a(\theta,\psi) + (\nabla v - \psi, \gamma) &= (f, v) \quad && \forall (v,\psi) \in H_0^1(\Omega) \times H_0^1(\Omega)^2, \\ (\nabla w - \theta, \eta) - t^2(\gamma, \eta) &= 0 && \forall \eta \in L_2(\Omega)^2. \end{aligned} \tag{4.5}$$

This formulation has the advantage that t enters now as a small parameter. Moreover for $t = 0$ we have the mixed formulation of the Kirchhoff plate where the Kirchhoff hypothesis $\theta - \nabla w = 0$ is strictly satisfied. If $t = 0$, then γ is understood as a Lagrangian parameter and is not defined by (4.4).

The equs. (4.5) fit into the general framework of (2.3). To see this, following [4], one sets.

$$\begin{aligned} X &:= H_0^1(\Omega) \times H_0^1(\Omega)^2, \qquad && a(w,\theta,v,\psi) := a(\theta,\psi), \\ M &:= H^{-1}(\mathrm{div}, \Omega), && b(v,\psi,\eta) := (\nabla v - \psi, \eta), \\ M_c &:= L_2(\Omega)^2, && c(\gamma,\eta) := (\gamma,\eta). \end{aligned}$$

Here, $H^{-1}(\mathrm{div}, \Omega)$ is endowed with the norm

$$\|\gamma\|^2_{H^{-1}(\mathrm{div})} := \|\gamma\|^2_{-1} + \|\,\mathrm{div}\,\gamma\|^2_{-1}.$$

A simple application of Green's formula shows that b is a bounded bilinear form on $X \times M$

$$\begin{aligned} b(v,\psi,\eta) = (\nabla v - \psi, \eta) &= -(v, \mathrm{div}\,\eta) - (\psi, \eta) \\ &\le \|v\|_1 \|\,\mathrm{div}\,\eta\|_{-1} + \|\psi\|_1 \|\eta\|_{-1} \\ &\le (\|v\|_1 + \|\psi\|_1)\, \|\eta\|_{H^{-1}(\mathrm{div})}. \end{aligned}$$

One cannot choose a stronger norm for M if an inf-sup condition is to be satisfied. Since, $L_2(\Omega) \neq H^{-1}(\mathrm{div}, \Omega)$, the penalty term c corresponds to a singular perturbation.

On the other hand, the bilinear form a does only depend on the rotations and not on the transversal displacements. It is elliptic only on the kernel, i.e., on the subset of functions which satisfy the Kirchhoff condition.

The mixed formulation (4.5) cannot be treated by the theorems in Section 2.

The Helmholtz Decomposition

Of course, the problem (4.5) can be treated by the above theory if $X \times M$ can be split into two factors such that the subproblem for the first factor satisfies the assumptions of theorem 1 while the second one satisfies the assumptions of theorem 2. Brezzi and Fortin [10] found even a better splitting.

By Helmholtz' theorem each vector field on a domain Ω with smooth boundary can be decomposed into a divergence free field and a rotation free field:

$$L_2(\Omega)^2 = \operatorname{grad} H_0^1(\Omega) + \operatorname{curl} H^1(\Omega)/\mathbb{R}. \tag{4.6}$$

If the shear terms are decomposed in this spirit:

$$\gamma = \nabla r + \operatorname{curl} p,$$
$$\eta = \nabla z + \operatorname{curl} q,$$

then the following equations are obtained from (4.5).

Find $(r, \theta, p, w) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \times H^1(\Omega)/\mathbb{R} \times H_0^1(\Omega)$ *such that*

$$(\nabla r, \nabla v) = (f, v) \qquad \forall\, v \in H_0^1(\Omega), \tag{4.7}$$

$$\begin{aligned} a(\theta, \psi) - (p, \operatorname{rot} \psi) &= (\nabla r, \psi) && \forall\, \psi \in H_0^1(\Omega)^2, \\ -(\operatorname{rot} \theta, q) - t^2(\operatorname{curl} p, \operatorname{curl} q) &= 0 && \forall\, q \in H^1(\Omega)/\mathbb{R}, \end{aligned} \tag{4.8}$$

$$(\nabla w, \nabla z) = (\theta, \nabla z) + t^2(f, z) \quad \forall\, z \in H_0^1(\Omega). \tag{4.9}$$

Note that (4.7) is a simple Poisson equation, (4.8) a saddle point problem with penalty, and (4.9) is again a simple Poisson equation.

The crucial point is that (4.8) is a *good* problem. Since $(\psi, \operatorname{curl} q) = (\operatorname{rot} \psi, q)$ due to boundary conditions, one can set

$$X = H_0^1(\Omega)^2, \quad M = L_2(\Omega)/\mathbb{R}, \quad M_c = H(\operatorname{rot}, \Omega)/\mathbb{R}.$$

Now, the quadratic form a is elliptic on X, and the singular perturbation by $c(p, q) := (\operatorname{curl} p, \operatorname{curl} q)$ is no longer a drawback. Stability with respect to

$$\|w\|_1 + \|\theta\|_1 + \|r\|_1 + \|p\|_{L_2(\Omega)/\mathbb{R}} + t\|\operatorname{curl} p\|_0$$

holds, and the stability with respect to the components θ and p is provided by Theorem 2.

Moreover, one has convergence for $t \to 0$ to the Kirchhoff plate for $\|w\|_1$, $\|\theta\|_1$, and $\|\gamma\|_{H^{-1}(\mathrm{div})}$.

The analysis of the finite element discretization can be performed in this spirit, if one has a Helmholtz decomposition for the finite element spaces

$$\Gamma_h = \operatorname{grad} W_h \oplus \operatorname{curl} Q_h, \tag{4.10}$$

if T_h is the finite element space for the shear term, Q_h for the generator of the divergence free part, and W_h for the vertical displacement, respectively. This was done by Arnold and Falk [4] for a nonconforming method. The analysis of the family of the MITC element was obtained by [11] and [15] by using relaxed versious of (4.10).

An Alternative Decomposition of the Shear Term

Recently, Arnold and Brezzi [2] found a much simpler stable method which contains some similarity with the method of the augmented Lagrangian. They observed that the original quadratic form in (4.1) is elliptic on the whole space $H_0^1(\Omega) \times H_0^1(\Omega)^2$, and they modified the transformation which yields the mixed form (4.5) such that this elliptic is saved.

Specifically, they set

$$\frac{1}{t^2} = 1 + \frac{1}{t'^2}, \quad \text{i.e.,} \quad t'^2 = \frac{t^2}{1-t^2},$$

split the shear term, and put only the second part into the shear parameter γ:

$$\gamma := t'^{-2}(\nabla w - \theta). \tag{4.11}$$

With this one gets instead of (4.5)

$$\begin{aligned} a(\theta,\psi) + (\nabla w - \theta, \nabla v - \psi) + (\nabla v - \psi, \gamma) &= (f, v) \quad && \forall (v,\psi) \in H_0^1(\Omega) \times H_0^1(\Omega)^2, \\ (\nabla w - \theta, \eta) \qquad\qquad - {t'}^2(\gamma, \eta) &= 0 && \forall \eta \in L_2(\Omega)^2. \end{aligned} \tag{4.12}$$

The bilinear form

$$a(\theta,\psi) + (\nabla w - \theta, \nabla v - \psi)$$

is elliptic on $H_0^1(\Omega) \times H_0^1(\Omega)^2$ and not only on the subspace on which the Kirchhoff law holds. Theorem 2 yields the stability of the variational equations (4.12).

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