Finite Elements for Kirchhoff and Mindlin-Reissner Plates

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Abstract. The Kirchhoff plate model leads to a differential equation of fourth order, while the Mindlin-Reissner plate is modelled by a system of differential equations of second order. Nevertheless, the latter is more difficult. Recently finite elements have been established which provide convergence uniformly in the thickness. The main tool is a (discrete) Helmholtz decomposition of L_2.

§1. Introduction

In the theory of plates usually two different limit cases are considered: the Kirchhoff and the Mindlin-Reissner plate. When you ask different people which model is more easily handled, you will get different answers. Engineers will usually give you the intuitive answer that the Mindlin plate is the easier one since C^0-elements are conforming for this model. If you treat the Kirchhoff plate with C^0-elements, then you need the more advanced theory of nonconforming elements.

In plate theory, however, the right answers are often counterintuitive. The Mindlin plate is a problem with a small parameter, and so the mathematian will prefer the Kirchhoff model if he wants a simple treatment. He will prefer, however, the other one if he wants an intriguing and challenging problem.

In the last years several groups have contributed to the problem. In this paper we want to report on this progress from the viewpoint of the question at the beginning.

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§2. The Mixed Formulations for the Two Plate Models

The starting point for the treatment of the clamped Mindlin plate is the following variational problem:
Determine *the transverse displacement* $w \in H_0^1(\Omega)$
and *the rotation* $\theta = (\theta_1, \theta_2) \in H_0^1(\Omega)^2$
such that the energy

$$\Pi[\theta, w] := \frac{1}{2}a(\theta,\theta) + \frac{t^{-2}}{2}\|\nabla w - \theta\|_0^2 - (f, w)_0 \tag{2.1}$$

is minimal. As usual, $(.,.)_0$ is the usual L_2-inner product and $\|\cdot\|_0$ the associated L_2-norm. The terms on the right hand side of (2.1) are the bending energy, the shear energy and the load, resp. The parameter t is essentially the thickness of the plate, some factors are also absorbed. So t is a small parameter. For completeness, the bilinear form a is given

$$a(\theta,\psi) := \frac{E}{12(1-\nu^2)}\Big[\sum_{i,j}(1-\nu)(\varepsilon_{ij}(\theta), \varepsilon_{ij}(\psi))_0 + \nu(\div\theta, \div\psi)_0\Big],$$

where $\varepsilon_{ij}(\theta) = \frac{1}{2}(\partial_i\theta_j + \partial_j\theta_i)$ is the symmetrical derivative. From Korn's inequality it follows that a is elliptic, and for our reasons it may be replaced by the simpler form

$$\tilde{a}(\theta,\psi) := (\nabla\theta, \nabla\psi)_0.$$

Here only first derivatives are found in the energy. One may get the impression that the problem can be treated with standard finite elements.

The crucial term is the shear term which depends on $\nabla w - \theta$, *i.e.*, on the deviation of the normal of the new surface from the position which the original normals have after the deformation.

In the limit case $t = 0$ we have $\theta = \nabla w$, *i.e.*, the rotations are fixed by the Kirchhoff postulate that normal fibers are also normal fibers in the deformed state. When θ is eliminated by setting $\theta = \nabla w$, a functional with second derivatives is obtained

$$J[w] = \frac{1}{2}(\nabla^2 w, \nabla^2 w)_0 - (f, w)_0.$$

This is the energy of the Kirchhoff plate.

The two variational forms look quite different. In order to make the similarities more apparent, we will first consider analogous problems in n-space. The Kirchhoff model corresponds to a minimization problem with an explicit restriction

$$\begin{gathered}\frac{1}{2}x'Ax - bx \to \min! \\ Bx = 0\end{gathered} \tag{2.2}$$

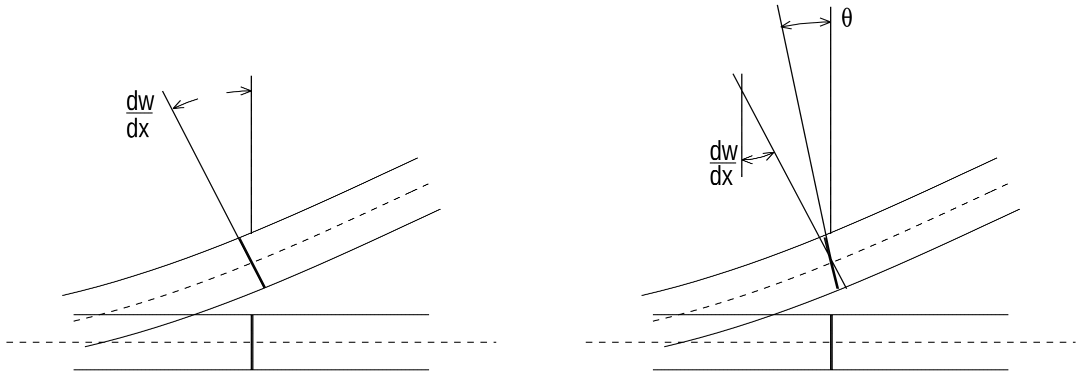

Fig. Cross-section of Kirchhoff and Mindlin plate

The solution may be characterized by the equations

$$\begin{aligned} Ax \; + B'\lambda &= b, \\ Bx \qquad &= 0. \end{aligned} \tag{2.3}$$

where λ is the Lagrangian parameter which is associated to the restrictions.

If, on the other hand, the restriction is replaced by a penalty term in the energy, we have

$$\frac{1}{2}x'Ax + \frac{t^{-2}}{2}\|Bx\|^2 - bx \to \text{min!} \tag{2.4}$$

Obviously the minimum is given by the equation

$$Ax + t^{-2}B^TBx = b. \tag{2.5}$$

The two problems still look quite different. Now things change, when we set $\lambda = t^{-2}Bx$. Adding this relation to (2.5) we obtain

$$\begin{aligned} Ax \; + B'\lambda &= b, \\ Bx \; - t^2\lambda &= 0. \end{aligned} \tag{2.6}$$

Here t enters as a *small* parameter and (2.6) may be understood as a perturbation of (2.3). We only note that the quantity λ has a different meaning in both cases: It is a Lagrangian parameter in the first case while it refers to a normalized value of the restriction in the second case.

We also note that with (2.6) we have returned to a saddle point problem although the penalty method (2.4) had the intention to circumvent saddle point problemes. It is known (see e.g. Brezzi and Fortin [6]) that (2.4) cannot be solved by iterative methods due to the large quantity t^{-2}. Roughly speaking we have the same effect with the analysis of the Mindlin plate.

Moreover, if we find a method for solving (2.6) which is uniformly good for any small t, it is expected to be good also for $t = 0$. This is also a hint that the Mindlin plate is probably not easier than the Kirchhoff model.

§3. Mixed Problems with Penalty

Before we treat the mixed formulations for the plates in the right function spaces, we will recall some results from the abstract theory. We note that the forms with penalty terms are not only encountered with Mindlin plates and the Timoshenko beam, but also in the treatment of materials for which the Poisson ratio is very close to $\frac{1}{2}$.

Now let X and M be Hilbert spaces and M_c be a dense subset of M. Having the bilinear forms

$$\begin{aligned} a &: X \times X \to \mathbb{R}, \\ b &: X \times M \to \mathbb{R}, \\ c &: M_c \times M_c \to \mathbb{R}, \end{aligned}$$

we consider the following problem: Given $f \in X'$ and $g \in M'$, find $u \in X$ and $p \in M_c$ such that

$$\begin{aligned} a(u,v) \ &+ b(v,p) &&= \langle f, v \rangle \quad \text{for } v \in X, \\ b(u,q) \ &- t^2 c(p,q) &&= \langle g, q \rangle \quad \text{for } q \in M_c. \end{aligned}$$

Of course, we will assume that this mixed problem satisfies the Babuška-Brezzi conditions if $t = 0$:

1) The bilinear form a is continuous and elliptic (coercive) on the kernel

$$Z := \{v \in X;\ b(v,q) = 0 \quad \text{for all } q \in M\}.$$

2) The bilinear form b is continuous and satisfies the inf-sup-condition

$$\inf_{q \in M} \sup_{v \in X} \frac{b(v,q)}{\|v\|_X \cdot \|q\|_M} = \beta > 0.$$

The penalty term c will be assemed to be positive definite, i.e.

$$c(q,q) \geq 0 \quad \text{for } q \in M_c.$$

As a consequence the quantity $|q|_c := c(q,q)^{\frac{1}{2}}$ defines a seminorm.

Now we are in a position to put a question which is very natural in this context.

Question: Assume that the standard mixed problem with $t = 0$ is stable for the norm

$$\|u\|_X + \|p\|_M. \tag{3.1}$$

Is the problem stable with respect to the norm

$$\|u\|_X + \|p\|_M + t|p|_c \tag{3.2}$$

uniformly for $0 \leq t \leq 1$?

One can hear sometimes that a positive penalty term always induces a stabilization, i.e., an affirmatve answer is expected.. Unfortunately, this "folklore" is not true in general.

The following answers show that some extra conditions are necessary.

Answers. (1) If the bilinear form c is continuous on M, then stability holds with respect to the norm (3.1).

The continuity implies that the norms (3.1) and (3.2) are equivalent. Therefore the situation is simple, cf. Braess and Blömer [3].

(2) If the form a is elliptic not only on the kernel Z but on the whole space X, then stability holds with respect to the norm (3.2), see e.g. Huang [10].

(3) Consider the (counter-)example from [3]

$$X = M = L_2, \quad M_c = H^1(\Omega).$$

$$a(u,v) := 0, \quad b(v,q) := (v,q)_0, \quad c(p,q) = (\nabla p, \nabla q)_0.$$

The mixed problem

$$\begin{aligned} (p,v)_0 \qquad\qquad &= \langle f, v \rangle \\ (u,q)_0 - t^2(\nabla p, \nabla q)_0 &= 0 \end{aligned}$$

has the formal solution

$$p = f, \quad u = -t^2 \Delta f.$$

There is no stability with respect to (3.2).

§4. Mindlin Plates

The solution of the variational problem in the displacement formulation with the energy (2.1) is characterized by

$$a(\theta,\psi) + t^{-2}(\nabla w - \theta, \nabla v - \psi)_0 = (f,\psi)_0 \quad \text{for } \psi \in H_0^1(\Omega)^2, v \in H_0^1(\Omega).$$

Following the same procedure as that which led to (2.6), one is led to the introduction of the shear term

$$\gamma = t^{-2}(\nabla w - \theta).$$

In this way one gets the mixed formulation

$$\begin{aligned} a(\theta,\psi) \ + (\nabla v - \psi, \gamma) &= (f,v) \qquad \psi \in H_0^1(\Omega)^2, v \in H_0^1(\Omega), \\ (\nabla w - \theta, \eta) \ - t^2(\gamma,\eta) \quad &= 0 \qquad\qquad \eta \in L_2(\Omega)^2. \end{aligned} \tag{4.1}$$

The inf-sup-condition holds for $t = 0$ if the following pairing in chosen [5]

$$\begin{aligned} X &= H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ M &= H^{-1}(\div, \Omega). \end{aligned}$$

Here, H_0^1 is endowed with the Sobolev norm $\|.\|_1$ and $H^{-1}(\div,\Omega)$ with

$$\|\gamma\|_{H-1(\div)} = \|\gamma\|_{-1} + \|\div\gamma\|_{-1}.$$

The penalty term $c(\gamma,\eta) = (\geq,\eta)_0$ lives on $M_c = L_2(\Omega)$. Note that

$$L_2(\Omega) \neq H^{-1}(\div,\Omega),$$

and c is not continuous on $H^{-1}(\div,\Omega)$. Moreover,

$$a \text{ is not elliptic on } X,$$

since it depends only on the rotation variables, but not on the transverse displacement. The general results for mixed methods with penalty terms are not applicable. None of the two conditions which admitted an affirmative answer in the last section, is satisfiel.

The equations (4.1) show that singular perturbation exist. The existence of boundary layers is also a hint that the singular perturbations cannot be eliminated by a simple trick. On the other hand the general theory could be applied if there is an affirmative answer to the following question:

Is it possible to split the spaces such that in one component we have ellipticity on the whole space while in the other component one has a continuous (regular) perturbation?

Indeed, Brezzi and Fortin [5] solved the problem by a splitting process. They consider the Helmholtz decomposition of the shear term

$$\begin{aligned} L_2(\Omega)^2 &= \operatorname{grad} H_0^1(\Omega) \oplus \operatorname{curl}(H^1(\Omega)/\mathbb{R}), \\ \gamma &= \nabla r + \operatorname{curl} p, \\ \eta &= \nabla z + \operatorname{curl} q. \end{aligned} \tag{4.2}$$

Note that ∇r is the rotation free part and $\operatorname{curl} p$ is divergence free.

When inserting (4.2) into (4.1) one gets the four equations [5]

$$\begin{aligned} (\nabla r, \nabla v) &= (f, v) && \text{for all } v \in H_0^1(\Omega) \\ a(\theta,\psi) - (p, \operatorname{rot}\psi) &= (\nabla r, \psi) && \text{for all } \psi \in [H_0^1(\Omega)]^2 \\ -(\operatorname{rot}\theta, q) - t^2(\operatorname{curl} p, \operatorname{curl} q) &= 0 && \text{for all } q \in H^1(\Omega)/\mathbb{R} \\ (\nabla w, \nabla z) &= (\theta, \nabla z) + t^2(f, z) && \text{for all } z \in H_0^1(\Omega). \end{aligned} \tag{4.3}$$

The problem can be treated by first solving the first equation, then both the second and the third one are solved together. Finally, the fourth one require the solution of a simple Poisson equation.

The mixed problem given by the second and the third equation, is the interesting part. It is a singularly perturbed mixed problem, specifically a perturbed Stokes problem. On the other hand, the form a is now elliptic on the whole space $H_0^1(\Omega)^2$, and the problem is stable with respect to the norm

$$\|\theta\|_1 + \|p\|_0 + t\|\operatorname{curl} p\|_0.$$

§5. Discretization

In the numerical solution of the Mindlin plate, it is not enough to replace the Sobolev spaces by some finite element subspaces, say $\Theta_h \subset H_0^1(\Omega)^2$ and $W_h \subset H_0^1(\Omega)$, where h is the mesh parameter. In order the prevent the scheme from locking, often some reduced integration is used when the shear energy is evaluated. For this purpose a linear reduction operator

$$R : H_0^1(\Omega)^2 \to \Gamma_h \tag{5.1}$$

is introduced by Brezzi, Bathe and Fortin [4]. This operator makes the shear terms belong to a third finite element space Γ_h. The discretized version of the variational problem with the energy (2.1) has the form:

$$\frac{1}{2}a(\theta_h, \theta_h) + \frac{t^{-2}}{2}\|\nabla w_h - R\theta_h\|_0^2 - (f, w_h) \to \min! \tag{5.2}$$

The analysis of the finite element approximation of the scheme (4.1) is simplified if one has a Helmholtz decomposition of Γ_h into two finite element spaces. In this way Arnold and Falk [1] investigated a nonconforming method. It is our aim to analyse the MITC-elements of Bathe, Brezzi et al. (see Bathe and Dvorkin [2], Brezzi, Bathe, and Fortin [4]).

Although a decomposition is not given in the strong sense, a certain step in this direction was done when Brezzi et al. [4] introduced the following five properties.

Assume that besides

$$\begin{aligned} W_h &\subset H_0^1(\Omega) && \text{(transverse displacements)} \\ \Theta_h &\subset H_0^1(\Omega)^2 && \text{(rotations)} \end{aligned}$$

two additional spaces

$$\begin{aligned} \Gamma_h &\subset H(\mathrm{rot}, \Omega) && \text{(shear strain)} \\ Q_h &\subset L_2(\Omega)/\mathbb{R} && \text{(pressure in the divergence free term)} \end{aligned}$$

and a reduction operator R as in (4.1) are given such that the follogwing properties hold:

P_1) $\nabla W_h \subset \Gamma_h$, i.e. the discrete shear

$$\gamma_h := t^{-2}(\nabla w_h - R\theta_h)$$

belongs to Γ_h.

P_2) $\mathrm{rot}\,\Gamma_h \subset Q_h$.

P_3) The pair of spaces (Θ_h, Q_h) satisfies the inf-sup condition:

$$\inf_{q_h \in Q_h} \sup_{\psi_h \in \Theta_h} \frac{(\mathrm{rot}\,\psi_h, q_h)}{\|\psi_h\|_1 \|q_h\|_0} =: \beta > 0$$

with β being independent of the mesh size h.

P_4) *Commuting diagram property.* Let P_h be the L_2-projection onto Q_h. Then

$$\operatorname{rot} R\,\eta = P_h \operatorname{rot}\eta \quad \text{for all } \eta \in [H_0^1(\Omega)]^2,$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} [H_0^1(\Omega)]^2 & \xrightarrow{\text{rot}} & L_2(\Omega) \\ R\big\downarrow & & \big\downarrow P_h \\ \Gamma_h & \xrightarrow{\text{rot}} & Q_h. \end{array}$$

P_5) *Completeness of decomposition.* If $\delta_h \in \Gamma_h$ and $\operatorname{rot}\delta_h = 0$ then $\delta_h \in \nabla W_h$.

As an example of elements which satisfy the properties we mention the MITC7 elements. It is assumed that there is a triangulation $\mathcal{T}_h$. The restriction of the functions to each triangle T is a polynomial of some degree k, i.e. belongs to $\mathcal{P}_k$ for some k.

$$\begin{aligned} W_h &:= \{w \in H_0^1(\Omega);\ w_{|T} \in \mathcal{P}_2(T) \quad \text{for all } T \in \mathcal{T}_h\}; \\ \Theta_h &:= \{\psi \in [H_0^1(\Omega)]^2;\ \psi_{|T} \in \mathcal{P}_2(T)^2 \oplus B_3(T)^2 \quad \text{for all } T \in \mathcal{T}_h\}, \\ Q_h &:= \{q \in L_2(\Omega)/\mathbb{R};\ q_{|T} \in \mathcal{P}_1(T) \ \text{ for all } T \in \mathcal{T}_h\}; \\ \Gamma_h &:= \{\gamma \in H_0(\text{rot},\Omega);\ \gamma_{|T} \in RT \ \text{ for all } T \in \mathcal{T}_h\}, \end{aligned}$$

Here,

$$\begin{aligned} B_3(T) &:= \{p \in P_3(T);\ p \text{ vanishes on the edges of } T\}, \\ RT &:= \left\{ \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + p_3 \begin{pmatrix} y \\ -x \end{pmatrix};\ p_i \in \mathcal{P}_1 \right\} \end{aligned} \quad .$$

Furthermore, the operator R is defined by

$$\begin{aligned} \int_e (\gamma - R\gamma)\tau p_{k-1} ds = 0 &\quad \text{for each edge } e \text{ and } p_1 \in \mathcal{P}_1(e), \\ \int_T (\gamma - R\gamma) p_{k-2} dx = 0 &\quad \text{for each } T \in \mathcal{T}_h \text{ and } p_0 \in \mathcal{P}_0(T). \end{aligned}$$

There are lower order elements with the specified properties although lower order elements require extra considerations [8].

In 1989 Brezzi, Bathe, and Fortin used the five properties for the analysis of the plate in the limit case $t = 0$. In the meantime Brezzi, Fortin and Stenberg [7] have also completed the theory for positive t.

Simultaneously, we have analyzed the plate by establishing a discrete version of the Helmholtz decomposition. The first step is a discrete version of the curl operator.

Formally, it is introduced like a distributional derivative on a finite element space, and the test functions are also taken from a finite dimensional space.

Definition and Theorem (Peisker and Braess [11]).
Let the operator

$$\mathrm{curl}_h : Q_h \to \Gamma_h$$

be defined by

$$(\mathrm{curl}_h\, q_h, \eta)_0 = (q_h, \mathrm{rot}\, \eta)_0 \quad \textit{for all } \eta \in \Gamma_h.$$

Moreover assume that P_1, P_2 and P_5 hold. Then an L_2-orthogonal decomposition is given by

$$\Gamma_h = \nabla W_h \oplus \mathrm{curl}_h\, Q_h.$$

Proof: (1) From the definition of $curl_h$ and P_1 it follows that

$$\nabla W_h \oplus \mathrm{curl}_h\, Q_h \subset \Gamma_h.$$

Furthermore, given $q_h \in Q_h$ and $w_h \in W_h$ it follows that

$$(\mathrm{curl}_h\, q_h, \nabla w_h) = (q_h, \mathrm{rot}\, \nabla w_h) = 0.$$

Therefore, the functions $\mathrm{curl}_h\, q_h$ and ∇w_h are L_2-orthogonal.

(2) Given $\gamma_h \in \Gamma_h$, let η_h be the L_2-projection onto $\mathrm{curl}_h\, Q_h$. Then η_h is characterized by

$$(\gamma_h - \eta_h, \mathrm{curl}_h\, q_h) = 0 \quad \text{for all } q_h \in Q_h.$$

From the definition we conclude that $(\mathrm{rot}(\gamma_h - \eta_h), q_h) = 0$ for all $q_h \in Q_h$, and P_2 implies that

$$\mathrm{rot}(\gamma_h - \eta_h) = 0.$$

Therefore, P_5 asserts that $\gamma_h - \eta_h \in \nabla W_h$, so that, by construction, $\gamma_h \in \nabla W_h \oplus \mathrm{curl}_h\, Q_h$. ■

Now the variational problem (5.2) can be treated in the same way as we know from the derivation of (4.3). Specifically, following [5] we obtain

$$\begin{aligned}
(\nabla r_h, \nabla v_h) &= (f, v_h) && \text{for all } v_h \in W_h, \\
a(\theta_h, \psi_h) - (p_h, \mathrm{rot}\, \psi_h) &= (\nabla r_h, R\psi_h) && \text{for all } \psi_h \in \Theta_h, \\
-(\mathrm{rot}\, \theta_h, q_h) - t^2(\mathrm{curl}_h\, p_h, \mathrm{curl}_h\, q_h) &= 0 && \text{for all } q_h \in Q_h, \\
(\nabla w_h, \nabla z_h) &= (R\theta_h, \nabla z_h) + t^2(f, z_h) && \text{for all } z_h \in W_h.
\end{aligned} \tag{5.3}$$

This problem is stable and we obtain an error estimate which is expected for the elements. The extra term which is typical for nonconforming theories arises, since the variational form uses selected reduced integration.

Theorem 5.2. *[11] Assume that the properties P_1 to P_5 hold. Let (θ, p) and (θ_h, p_h) be the solutions of (2.6) and (3.5), resp. Then the following error bound*

$$\begin{aligned}\|\theta-\theta_h\|_1 + \|p-p_h\|_0 + t\,\|\operatorname{curl} p - \operatorname{curl}_h p_h\|_0 \\ \le C\Big\{\inf_{\psi_h\in\Theta_h}\|\theta-\psi_h\|_1 + \inf_{q_h\in Q_h}\|p-q_h\|_0 + t\,\|R\operatorname{curl} p - \operatorname{curl} p\|_0 \\ + \|r-r_h\|_0 + \sup_{\psi_h\in\Theta_h}\frac{|(\nabla r_h, R\psi_h-\psi_h)|}{\|\psi_h\|_1}\Big\}\end{aligned}$$

holds with C being independent of the mesh size h and of the parameter t.

Appendix: DKT-Elements

The nonconforming elements for the Kirchhoff plate are less complicated than the MITC-elements considered before. Nevertheless, they also contain some traps. Since it is clear now how to circumvent the traps, one can fed safe. We will report on this. For details the reader is referred to Pithäranta [1988].

The starting point is that finite element caculations are obvionsly simpler than with C^1-elements, if the Kirchhoff hypothesis $\nabla w = 0$ is no postulated in the whole domain Ω, but only at the vertices of a triangulation. This has lead to the triangular elements with discrete Kirchhoff constraints (for short DKT-elements).

The two interesting DKT-elements refer to the reduced cubic polynomials. Let a_1, a_2, a_3 be the vertices of a triangle T and a_0 its center of mass.

$$\mathcal{P}_{3,red} := \Big\{p\in\mathcal{P}_3;\ 6p(a_0) - \sum_{i=1}^{3}[2p(a_i) - \nabla p(a_i)(a_i-a_0)] = 0\Big\}.$$

Specifically, the bubble function has been eliminated by the linear constraint. The polynomials in $P_{3,red}(T)$ have 9 degress of freedom and are determined by the function values and the first derivatives at the three vertices.

The DKT-element due to Batoz, Bathe und Ho [1980] refers to the two spaces

$$\begin{aligned} W_h &:= \{w\in H_0^1(\Omega);\ \nabla w \text{ is continuous at all vertices of } \mathcal{T}_h,\\ &\qquad w_{|T}\in\mathcal{P}_{3,red} \text{ for } T\in\mathcal{T}_h\},\\ \Theta_h &:= \{\theta\in H_0^1(\Omega)^2;\ \theta_{|T}\in(\mathcal{P}_2)^2 \text{ and } \theta\cdot n\in\mathcal{P}_1(e)\\ &\qquad \text{for each edge } e\in\partial T \text{ and } T\in\mathcal{T}_h\}.\end{aligned} \tag{A.1}$$

As usual, the normal vector n and the tangential vector t are defined on the boundary of T. Furthermore, $a_{ij} = \frac{1}{2}(a_i + q_j)$. Now a rotation is associated to each bending by a *discrete Kirchhoff condition*:

$$w_h \longmapsto \theta_h(w_h): \quad \begin{aligned} \theta_h(a_i) &= \nabla w_h(a_i) && \text{for the vertices } a_i, \\ \theta_h(a_{ij}) \cdot t &= \nabla w_h(a_{ij}) \cdot t && \text{for the midpoints } a_{ij}. \end{aligned} \tag{A.2}$$

With this an h-dependent energy is defined

$$\Pi_n(w_h) := \frac{1}{2} a(\theta_h(w_h), \theta_h(w_h) - (f, w_h)_0. \tag{A.3}$$

The convergence of the associated finite element solution to the solution of the Kirchhoff plate model is usually proved via mesh-dependent norms, see [,].

The situation is more complicated when a DKT-element is designed on the basis of the Zienkiewicz triangle. Then a pairing with a smaller space for the rotations is chosen:

$$\begin{aligned} W_h &:= \{w \in H_0^1(\Omega);\ \nabla w \text{ is continuous at all vertices of } \mathcal{T}_h, \\ &\qquad\qquad w_{|T} \in \mathcal{P}_{3,red} \ \text{ for } T \in \mathcal{T}_h\}, \\ \Theta_h &= \{\theta \in H_0^1(\Omega)^2;\ \theta_{|T} \in (P_1)^2 \text{ for } T \in \mathcal{T}_h\}. \end{aligned}$$

Also in this case a rotation is associated to a bending by a discrete Kirchhoff condition

$$w_h \longmapsto \theta_h(w_h): \quad \theta_h(a_i) = \nabla w_h(a_i) \quad \text{for the vertices } a_i.$$

Here h_T is the diameter of the element T. Note that usually we have $h_T \gg t$ for the parameter in the penalty term.

It is an advantage of this element that the stabilization is satisfactory from the mathematical viewpoint as well as for engineers.

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