A FINITE ELEMENT METHOD FOR NEARLY INCOMPRESSIBLE ELASTICITY PROBLEMS

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Abstract. A finite element method is considered for dealing with nearly incompressible material. In the case of large deformations the nonlinear character of the volumetric contribution has to be taken into account. The proposed mixed method avoids volumetric locking also in this case and is robust for $\lambda \to \infty$ (with λ being the well-known Lamé constant). Error estimates for the L^∞ norm are crucial in the control of the nonlinear terms.

1. Introduction

Isoparametric low-order elements are very popular in solid mechanics due to their simplicity. A major disadvantage, however, is the so-called *locking effect* [2]. Specifically volume locking is encountered when the material is nearly incompressible. In case of linear elasticity it is now well-known how to overcome locking numerically, and several popular methods have been shown to be equivalent [6, 30]. Recently, also the mathematical treatment has been simplified [8].

The situation is less satisfactory in nonlinear elasticity. In the present paper we shall consider a nearly incompressible neo-Hookean material [27, 37]. Let $\boldsymbol{v}$ be a displacement field in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary. The stored energy and the load yield the total energy

$$\mathcal{J}_\lambda(\boldsymbol{v}) := \frac{C_0}{2}\int_\Omega \big(|\mathrm{Id}+\boldsymbol{\nabla v}|^2-2\big)\,d\boldsymbol{x}+\frac{\lambda}{2}\int_\Omega |\det(\mathrm{Id}+\boldsymbol{\nabla v})-1|^2\,d\boldsymbol{x}-\int_\Omega \boldsymbol{f}\boldsymbol{v}\,d\boldsymbol{x}. \tag{1.1}$$

Here C_0 is a positive physical parameter related to the Lamé parameter μ while λ is characteristic for the compressibility. In particular, here we have $\lambda \gg C_0$. For brevity, we let $C_0 \equiv 1$ hereafter. This model (and its analysis) have a characteristic feature of nonlinear theories. We find the determinant $\det(\mathrm{Id}+\boldsymbol{\nabla v})$ in expressions at those places where $\operatorname{div} v$ is encountered in linear theories. The resulting displacement $\boldsymbol{u} = (u_1, u_2)$ is characterized by the minimization of $\mathcal{J}_\lambda(\boldsymbol{v})$, i.e.,

$$\mathcal{J}_\lambda(\boldsymbol{u}) = \min_{\boldsymbol{v}\in\boldsymbol{W}} \mathcal{J}_\lambda(\boldsymbol{v}), \tag{1.2}$$

where $\boldsymbol{W} := \{\boldsymbol{v} \in \boldsymbol{W}_0^{1,4}(\Omega) \mid \det(\mathrm{Id}+\boldsymbol{\nabla v}) > 0 \ a.e. \text{ in } \Omega\}$.

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The above minimization problem is frequently used in industry [23, 29] to model neo-Hookean nearly incompressible material. Natural rubber is a typical example for nearly incompressible material, and materials that undergo plastic deformations may also be considered as nearly incompressible. This model is also encountered when the hard incompressible constraint

$$\det(\mathrm{Id} + \boldsymbol{\nabla v}) = 1$$

is tackled by a penalty term in a way introduced by Ogden [26]. The above constraint reduces to the well-known linear incompressible condition in linear elasticity or incompressible fluid dynamics, i.e.

$$\operatorname{div} \boldsymbol{v} = 0.$$

The model may also be viewed as the 2-D case of the Mooney-Rivlin material; we refer to [12] for examples of other materials. Proceeding along the same line as [22, Theorem II.2] (see also [32]), one can see that $\boldsymbol{u}$ and p converge to the solutions for incompressible neo-Hookean material as λ approaches ∞ if the body force $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$.

An advantage of this model is that it is based upon a displacement-oriented formulation which facilitates the implementation. The model avoids hard constraints. On the other hand, there is a large parameter and the danger of locking. In order to avoid locking in the given nonlinear problem, we will employ a reduced energy functional when the discretization with finite elements of low order is performed. Let $\boldsymbol{X}_h$ be the finite elements space. The solution $\boldsymbol{u}_h$ will minimize the reduced energy functional

$$\begin{aligned}\mathcal{J}_{\lambda,h}(\boldsymbol{v}) :&= \frac{1}{2}\int_\Omega \big(|\mathrm{Id} + \boldsymbol{\nabla v}|^2 - 2\big)\, d\boldsymbol{x} + \frac{\lambda}{2}\int_\Omega |\Pi_0\big(\det(\mathrm{Id} + \boldsymbol{\nabla v}) - 1\big)|^2 d\boldsymbol{x}\\ &\quad - \int_\Omega \boldsymbol{f}\boldsymbol{v}\, d\boldsymbol{x},\end{aligned} \tag{1.3}$$

where Π_0 denotes the L^2-projection onto some finite element space M_h.

The analysis will be based on an equivalent mixed method. To this end the pressure variable

$$p := \lambda(\det(\mathrm{Id} + \boldsymbol{\nabla u}) - 1)$$

is introduced, and a saddle point formulation with a penalty term arises. While such a u-p formulation is equivalent to the method with *enhanced assumed strains* in the linear case [6, 35], there are some differences in the nonlinear case. Instead of using a projection of the determinant of $\mathrm{Id} + \boldsymbol{\nabla u}$, one may for instance consider projections of the four matrix elements. This will be exploited in a forthcoming paper.

The nonlinear problem will be dealt with a homotopy argument. Since the nonlinearities are bounded by L^∞ norms, the L^∞ error has to be controlled for the homotopy. As a consequence, we obtain logarithmic terms with a higher power than in the linear case.

We emphasize that we do not require smallness of the nonlinearities. It will be enough to have a good discretization such that the nonlinear terms with the differences $\boldsymbol{u} - \boldsymbol{u}_h$ and $p - p_h$ can be controlled.

The paper is organized as follows. Some notations are introduced, the main result is stated, and the concept for its proof is outlined in Section 2. In Section 3, we provide some a priori estimates as well as the regularity estimates for the linearized problem. A special Clément interpolant is constructed in Section 4. The regularized Green's functions and their application to the nonlinear problem are detailed in Section 5. Section 6 establishes the final error bounds. Appendix A tackles the finite element approximation of the Green's functions.

In this paper, we will concentrate on a variant of the 4–1 element [36]. The commonly used 9–3 element also fits into our analysis; we will comment on this element in Remark 2.8 and 4.2.

Throughout this paper, C denotes a generic constant that is independent of h and λ.

2. Notations and preliminaries

2.1. **Problem setting.** Let $W^{m,p}(\Omega)$ denote the standard Sobolev space of functions on Ω (see [20]) equipped with the norm and semi-norm

$$\|v\|_{m,p}^p := \sum_{k=0}^{m} |v|_{k,p}^p \quad \text{and} \quad |v|_{k,p}^p := \int_\Omega \sum_{|\boldsymbol{\alpha}|=k} |\mathcal{D}^{\boldsymbol{\alpha}} v|^p \, d\boldsymbol{x}.$$

Here $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is a multi-index whose components α_i are non-negative integers, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$, and $\mathcal{D}^{\boldsymbol{\alpha}} = \partial^{|\boldsymbol{\alpha}|}/\partial x_1^{\alpha_1} x_2^{\alpha_2}$. For brevity, if $p = 2$, the subscript p will be dropped and $W^{m,2}(\Omega)$ is denoted as $H^m(\Omega)$. As usual, $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ functions with vanishing traces on $\Gamma := \partial\Omega$, and $\boldsymbol{H}_0^1(\Omega)$ is the corresponding space of 2-vector-valued functions. More generally, we use boldface letters to denote the vector-valued spaces and operators. Moreover $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$, and $\langle \cdot, \cdot \rangle$ denotes the dual pairing of $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ or of $\boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{H}^{-1}(\Omega)$, respectively. The inner product in $L^2(\Omega)$ is $(\cdot, \cdot)_\Omega$ or for short $(\cdot, \cdot)$.

We set $\boldsymbol{X} := \boldsymbol{H}_0^1(\Omega)$ and $M := L^2(\Omega)$. As usual $\boldsymbol{X}'$ and M' are the dual spaces.

For a vector $\boldsymbol{x} \in \mathbb{R}^2$, we set $\boldsymbol{x}^\perp := (x_2, -x_1)$. For vectors $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2$, $\boldsymbol{x} \otimes \boldsymbol{y}$ is a 2×2 matrix with elements $(\boldsymbol{x} \otimes \boldsymbol{y})_{ij} := x_i y_j$. For a matrix $\boldsymbol{A}$, $\operatorname{adj} \boldsymbol{A} := (\operatorname{Cof} \boldsymbol{A})^T = \operatorname{Cof} \boldsymbol{A}^T$, where $\operatorname{Cof} \boldsymbol{A}$ is the cofactor matrix of $\boldsymbol{A}$. A matrix product is defined by $\boldsymbol{A} : \boldsymbol{B} = \operatorname{tr}(\boldsymbol{A}^T \boldsymbol{B})$. Note that

$$\det(\boldsymbol{A} + \boldsymbol{B}) = \det \boldsymbol{A} + \operatorname{Cof} \boldsymbol{A} : \boldsymbol{B} + \det \boldsymbol{B} \tag{2.1}$$

holds for any 2×2 matrices.

Problem (1.2) still has an undesirable constraint. We consider the minimization of the elastic energy (1.2) over $\boldsymbol{W}_0^{1,4}(\Omega)$ instead of $\boldsymbol{W}$ and the restriction $\det(\mathrm{Id}+\boldsymbol{\nabla v}) > 0$ has to be checked a posteriori. Since zero boundary conditions are assumed, $\int_\Omega \big(|\mathrm{Id}+\boldsymbol{\nabla v}|^2 - 2\big)\, d\boldsymbol{x} = \int_\Omega (\boldsymbol{\nabla v})^2\, d\boldsymbol{x}$. The Euler–Lagrange equation for the minimization of (1.1) is

$$A(\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle, \qquad \forall \boldsymbol{v}\in \boldsymbol{W}_0^{1,4}(\Omega), \tag{2.2}$$

with the nonlinear functional

$$A(\boldsymbol{u},\boldsymbol{v}) := (\boldsymbol{\nabla u},\boldsymbol{\nabla v}) + \lambda(\det(\mathrm{Id}+\boldsymbol{\nabla u}) - 1, \mathrm{Cof}(\mathrm{Id}+\boldsymbol{\nabla u}) : \boldsymbol{\nabla v}).$$

For problems with more general boundary conditions, see [33].

By virtue of the *Piola identity* [12, Theorem 1.7.1],

$$\mathrm{div}\,\mathrm{Cof}\,\boldsymbol{A} = \boldsymbol{0}, \tag{2.3}$$

whenever $\boldsymbol{A}$ is the gradient of a smooth vector field. Therefore $A(\boldsymbol{u},\boldsymbol{v})$ can be rewritten

$$A(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{\nabla u},\boldsymbol{\nabla v}) + \lambda\big(\det(\mathrm{Id}+\boldsymbol{\nabla u}) - 1, \mathrm{div}(\mathrm{adj}(\mathrm{Id}+\boldsymbol{\nabla u})\boldsymbol{v})\big). \tag{2.4}$$

We introduce $p := \lambda(\det(\mathrm{Id}+\boldsymbol{\nabla u}) - 1)$ to put the minimizing problem (1.2) and the weak equations (2.2) into a saddle point formulation.

Problem 2.1. *Find* $(\boldsymbol{u},p)\in \boldsymbol{W}_0^{1,4}(\Omega)\times M$ *such that*

$$\begin{aligned} a((\boldsymbol{u},p),\boldsymbol{v}) &= \langle \boldsymbol{f},\boldsymbol{v}\rangle, && \forall \boldsymbol{v}\in \boldsymbol{W}_0^{1,4}(\Omega),\\ b(\boldsymbol{u},q) - \lambda^{-1}(p,q) &= 0, && \forall q\in M, \end{aligned} \tag{2.5}$$

with

$$\begin{aligned} a((\boldsymbol{u},p),\boldsymbol{v}) &:= (\boldsymbol{\nabla u},\boldsymbol{\nabla v}) + \big(p, \mathrm{div}(\mathrm{adj}(Id+\boldsymbol{\nabla u})\boldsymbol{v})\big),\\ b(\boldsymbol{u},q) &:= (q, \det(Id+\boldsymbol{\nabla u}) - 1). \end{aligned}$$

As above, from the *Piola identity* we have $\mathrm{div}(\mathrm{Cof}(\mathrm{Id}+\boldsymbol{\nabla u})) = 0$ and

$$\big(p, \mathrm{div}(\mathrm{adj}(\mathrm{Id}+\boldsymbol{\nabla u})\boldsymbol{v})\big) = (p, \mathrm{Cof}(\mathrm{Id}+\boldsymbol{\nabla u}) : \boldsymbol{\nabla v}), \tag{2.6}$$

which will be frequently used.

The existence of a solution $(\boldsymbol{u},p)$ of Problem 2.1 can be proven as in [12, section 6.4–6.8]. Following [22], we also assume that

$$\boldsymbol{u}\in \boldsymbol{\mathcal{C}}^2(\overline{\Omega}), \quad p\in \mathcal{C}^1(\overline{\Omega}). \tag{2.7}$$

This regularity is commonly assumed in investigations of nonlinear elasticity, although the assumptions are restrictive. If Ω is a $\mathcal{C}^3$ domain, and the body force $\boldsymbol{f}$ is sufficiently small in $\boldsymbol{W}^{1,p}(\Omega)$ with $p > 2$, one has $(\boldsymbol{u},p)\in \boldsymbol{W}^{3,p}(\Omega)\times W^{2,p}(\Omega)$, and (2.7) follows by the Sobolev imbedding theorem.

Let $\mathcal{T}_h$ be a regular triangulation of Ω into "quadrilaterals" with mesh parameter $h := \max_{K\in\mathcal{T}_h} h_K$. Interelement boundaries are located on straight lines, and elements next to the boundary are transformed/modified to quadrilaterals such that curved boundaries are matched. Moreover, we assume that $\mathcal{T}_h$ is obtained by first constructing a triangulation of Ω with mesh size $2h$, and then dividing each quadrilateral $\overline{K}$ into four sub-quadrilaterals by the lines connecting the mid-points of the opposite edges of $\overline{K}$. To define the isoparametric elements, let $Q_k(K)$ be the space of polynomials of degree $\le k$ in each variable, and set

$$\boldsymbol{X}_h := \{\boldsymbol{v} \in \boldsymbol{X} \mid \boldsymbol{v}_{|K} \in Q_1(K)^2 \ \ \forall K \in \mathcal{T}_h\}, \tag{2.8}$$

$$M_h := \{q \in M \mid q_{|\overline{K}} \in Q_0(\overline{K}) \quad \forall \overline{K} \in \mathcal{T}_{2h}\}, \tag{2.9}$$

for the approximation of the displacement $\boldsymbol{u}$ and the pressure variable p, respectively. In principle, $(\boldsymbol{X}_h, M_h)$ can be any pair of spaces that is stable for the Stokes problem (provided that the interpolation process in Section 4 is adapted); cf. Remark 4.2.

The finite element approximation of Problem 2.1 reads:

Problem 2.2. *Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h \times M_h$ such that*

$$\begin{aligned} a((\boldsymbol{u}_h, p_h), \boldsymbol{v}) &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \quad &\forall \boldsymbol{v} \in \boldsymbol{X}_h, \\ b(\boldsymbol{u}_h, q) - \lambda^{-1}(p_h, q) &= 0, &\forall q \in M_h. \end{aligned} \tag{2.10}$$

Note that functions in M_h are discontinuous, and the pressure variable p_h can be eliminated in finite element computations on an element level.

2.2. **Final assumptions and main problem.** Problems 2.1 and 2.2 will be embedded later into one-parameter families of equations. When applying an implicit function theorem, we make use of the derivatives of the nonlinear form $a((\boldsymbol{u}, p), \boldsymbol{v})$,

$$\begin{aligned} a_{\boldsymbol{u}}(p; \boldsymbol{w}, \boldsymbol{v}) &:= \big(\boldsymbol{\nabla} \boldsymbol{w}, \boldsymbol{\nabla} \boldsymbol{v}\big) + \big(p, \operatorname{div}(\operatorname{adj} \boldsymbol{\nabla} \boldsymbol{w}\, \boldsymbol{v})\big), \\ a_p(\boldsymbol{u}; \boldsymbol{v}, q) &:= \big(q, \operatorname{div}(\operatorname{adj}(\mathrm{Id} + \boldsymbol{\nabla} \boldsymbol{u})\boldsymbol{v})\big). \end{aligned}$$

The derivatives of b can be expressed in terms of a_p and b. Moreover, a_u is symmetrical in the last two arguments.

Let $(\boldsymbol{u}, p)$ be the solution of Problem 2.1, and let $a_{\boldsymbol{u}}(\boldsymbol{w}, \boldsymbol{v})$ and $a_p(\boldsymbol{v}, q)$ be abbreviations of $a_{\boldsymbol{u}}(p; \boldsymbol{w}, \boldsymbol{v})$ and $a_p(\boldsymbol{u}; \boldsymbol{v}, q)$, respectively. We consider the following auxiliary linear problem.

Problem 2.3. *Given $(\boldsymbol{F}, G) \in \boldsymbol{X}' \times M'$, find $(\boldsymbol{w}, r) \in \boldsymbol{X} \times M$ such that*

$$\begin{aligned} a_{\boldsymbol{u}}(\boldsymbol{w}, \boldsymbol{v}) + a_p(\boldsymbol{v}, r) &= \langle \boldsymbol{F}, \boldsymbol{v}\rangle, \quad &\forall \boldsymbol{v} \in \boldsymbol{X}, \\ a_p(\boldsymbol{w}, q) - \lambda^{-1}(r, q) &= (G, q), &\forall q \in M. \end{aligned} \tag{2.11}$$

We define a bilinear form $\mathcal{A}$ as

$$\mathcal{A}(\boldsymbol{w}, r; \boldsymbol{v}, q) := a_{\boldsymbol{u}}(\boldsymbol{w}, \boldsymbol{v}) + a_p(\boldsymbol{v}, r) + a_p(\boldsymbol{w}, q) - \lambda^{-1}(r, q).$$

In particular, (2.11) is now rewritten:

$$\mathcal{A}(\boldsymbol{w}, r; \boldsymbol{v}, q) = \langle \boldsymbol{F}, \boldsymbol{v}\rangle + (G, q), \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{X} \times M. \tag{2.12}$$

Since $\mathcal{A}$ is a bounded operator, it immediately induces a bounded linear operator $\mathcal{L} : \boldsymbol{X} \times M \to \boldsymbol{X}' \times M'$ by the relation $\mathcal{L}(\boldsymbol{w}, r)(\boldsymbol{v}, q) = \mathcal{A}(\boldsymbol{w}, r; \boldsymbol{v}, q)$ for any $(\boldsymbol{w}, r)$ and $(\boldsymbol{v}, q) \in \boldsymbol{X} \times M$. Note that we write $\mathcal{L}(\boldsymbol{w}, r)(\boldsymbol{v}, q)$ rather than $(\mathcal{L}(\boldsymbol{w}, r), (\boldsymbol{v}, q))$.

For the linear problem (2.11), we have

Theorem 2.4. [15, Theorem 1] *Assume that* $p \in \mathcal{C}^1(\overline{\Omega})$ *and* $\boldsymbol{u} \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega})$.

(1) *Let* $C_1 := \frac{1}{2}\|p\|^2_{\mathcal{C}^1(\overline{\Omega})}$. *Then*

$$\frac{1}{2}\|\boldsymbol{\nabla v}\|_0^2 \le a_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{v}) + C_1\|\boldsymbol{v}\|_0^2, \qquad \forall \boldsymbol{v} \in \boldsymbol{X}. \tag{2.13}$$

(2) *There exists a constant* C_2 *which may depend on* $\|\boldsymbol{u}\|_{\boldsymbol{\mathcal{C}}^2(\overline{\Omega})}$ *such that*

$$\sup_{\boldsymbol{v} \in \boldsymbol{X}} \frac{a_p(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \ge C_2\|q\|_0, \qquad \forall q \in M. \tag{2.14}$$

Theorem 2.4 provides only a Gårding inequality for the bilinear form $\mathcal{A}$ in (2.12), and thus the theorem does not exclude the case that Problem 2.3 is not solvable. This is in accordance with the fact that the original problem is nonlinear, and large loads can cause a breakdown of the mechanical system. On the other hand, it is clear that the form $a_u(\cdot, \cdot)$ is coercive for small loads and if the solution (u, p) can also be reached on a path without branch points or singularities. Roughly speaking we have to stay in the region of stability.

After excluding the singularities above and having coercivity in addition to Theorem 2.4, we can verify the following assumption by the general theory of saddle point problems with penalty terms. We recall that the arguments heavily depend on (2.7), and so we prefer the formulation as an assumption.

Assumption R. *Given* $(\boldsymbol{F}, G) \in \boldsymbol{X}' \times M'$, *there exists a unique solution* $(\boldsymbol{w}, r) \in \boldsymbol{X} \times M$ *of Problem 2.3 and*

$$\|\boldsymbol{w}\|_1 + \|r\|_0 \le C(\|\boldsymbol{F}\|_{-1} + \|G\|_0). \tag{2.15}$$

Here, C *may depend on* $\boldsymbol{u}$ *and* p.

The assumption above actually implies that the operator $\mathcal{L}$ is an isomorphism.

Now we are ready to state our main result.

Theorem 2.5. *Let* $(\boldsymbol{u}, p) \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega}) \times \mathcal{C}^1(\overline{\Omega})$ *be the solution of the Problem 2.1, and let Assumption* **R** *hold. Then for sufficiently small* h, *there exists a solution* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h \times M_h$ *of Problem 2.2 in a* $\boldsymbol{W}^{1,\infty}(\Omega) \times L^\infty(\Omega)$*-neighborhood of* $(\boldsymbol{u}, p)$ *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^\infty} + h|\ln h|^{3/2}(\|\boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^\infty} + \|p - p_h\|_{L^\infty}) \le Ch^2|\ln h|^2, \tag{2.16}$$

where C *may depend on* $\|\boldsymbol{u}\|_{\boldsymbol{\mathcal{C}}^2(\overline{\Omega})}$ *and* $\|p\|_{\mathcal{C}^1(\overline{\Omega})}$.

This theorem states that the isolated solution can be approximated with quasi-optimal error bounds. For large data $\boldsymbol{f}$, the uniqueness of the solution may be violated, see [12, section 5.8].

Our result improves a similar one in [15, Theorem 2], since the convergence rate presented there is only of order $\mathcal{O}(h^{2-\epsilon})$ with $\epsilon > 0$.

The main ingredient of our proof is a bound of the regularized Green's functions that will be derived in Section 5. Such kind of estimates cannot be found in the present literature.

Remark 2.6. *Notice that this theorem and its analysis also cover the limit case* $\lambda \to \infty$*, i.e., the incompressible material.*

Remark 2.7. *The error bounds for the displacement with respect to the* $\boldsymbol{W}^{1,\infty}$*-norm and the pressure variable with respect to the* L^∞*-norm are the same with the sharp results for the Stokes problem* [16, Corollary 5.1 with $k = 1$]. *The* $\boldsymbol{L}^\infty$ *error bound for the displacement, however, is polluted by an extra factor of* $|\ln h|$.

Remark 2.8. *Theorem 2.5 is also valid while the finite element spaces are replaced by the* (Q_2, P_1) *pair for* u *and* p*, respectively, called* 9–3 *element by the engineering community* [28, 36].

2.3. **Embedding Problem 2.2.** We will consider a one-parameter family of problems with solutions in a neighborhood of $(\boldsymbol{u}, p)$. For any $h > 0$ we define a mesh-dependent norm on $\boldsymbol{W}^{1,\infty}(\Omega) \times L^\infty(\Omega)$,

$$\|\boldsymbol{v}, q\|_h := \|\boldsymbol{v}\|_{L^\infty} + h|\ln h|^{3/2}(\|\boldsymbol{\nabla v}\|_{L^\infty} + \|q\|_{L^\infty}). \tag{2.17}$$

For $\gamma > 0$ which will be fixed later, set

$$U_\gamma := \{(\boldsymbol{v}, q) \in \boldsymbol{W}^{1,\infty}(\Omega) \times L^\infty(\Omega) \mid \|\boldsymbol{v} - \boldsymbol{u}, q - p\|_h \leq \gamma\}.$$

Next, we introduce a homotopy

$$\mathcal{H}\big(t, (\boldsymbol{w}, q)\big) : [0,1] \times \big((\boldsymbol{X}_h \times M_h) \cap U_\gamma\big) \to \boldsymbol{X}'_h \times M'_h, \tag{2.18}$$

between the original nonlinear problem and a linear one. When applied to $(\boldsymbol{v}, q) \in \boldsymbol{X}_h \times M_h$, the operator $\mathcal{H}\big(t, (\boldsymbol{w}, r)\big)$ takes the form

$$\mathcal{H}\big(t, (\boldsymbol{w}, r)\big)(\boldsymbol{v}, q) = \begin{cases} t[a((\boldsymbol{w}, r), \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v}\rangle] \\ \quad +(1-t)[a_{\boldsymbol{u}}(p; \boldsymbol{w} - \boldsymbol{u}, \boldsymbol{v}) + a_p(\boldsymbol{u}; \boldsymbol{v}, r - p)], \\ t[b(\boldsymbol{w}, q) - \lambda^{-1}(r, q)] \\ \quad +(1-t)[a_p(\boldsymbol{u}; \boldsymbol{w} - \boldsymbol{u}, q) - \lambda^{-1}(r - p, q)]. \end{cases}$$

$\mathcal{H}(t, (\boldsymbol{w}, r))$ is continuously differentiable with respect to $(\boldsymbol{w}, r)$ with derivatives

$$\mathcal{D}_{(\boldsymbol{w},r)}\mathcal{H}\big(t, (\boldsymbol{w}, r)\big)\big((\boldsymbol{\phi}, s), (\boldsymbol{v}, q)\big) = \begin{cases} t[a_{\boldsymbol{u}}(r; \boldsymbol{\phi}, \boldsymbol{v}) + a_p(\boldsymbol{w}; \boldsymbol{v}, s)] \\ \quad +(1-t)[a_{\boldsymbol{u}}(p; \boldsymbol{\phi}, \boldsymbol{v}) + a_p(\boldsymbol{u}; \boldsymbol{v}, s)], \\ t\, a_p(\boldsymbol{w}; \boldsymbol{\phi}, q) \\ \quad +(1-t)a_p(\boldsymbol{u}; \boldsymbol{\phi}, q) - \lambda^{-1}(s, q). \end{cases}$$

We will omit the subscript of $\mathcal{D}_{(\boldsymbol{w},q)}\mathcal{H}\big(t, (\boldsymbol{w}, q)\big)$ when no confusion can occur.

Now we are prepared for the embedding of Problem 2.2.

Problem 2.9. *Find* $\big(\boldsymbol{u}_h^t, p_h^t\big) \in (\boldsymbol{X}_h \times M_h) \cap U_\gamma$ *for some* $\gamma > 0$ *such that*

$$\mathcal{H}\big(t, (\boldsymbol{u}_h^t, p_h^t)\big)(\boldsymbol{v}, q) = 0, \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{X}_h \times M_h. \tag{2.19}$$

Here t is the deformation parameter. The problem is linear for $t = 0$, and for $t = 1$ we end with the original nonlinear problem 2.2. In particular, $\big(\boldsymbol{u}_h^0, p_h^0\big)$ can be understood as the Fortin interpolant of $\big(\boldsymbol{u}, p\big)$ in $X_h \times M_h$ for the mapping given by Problem 2.3. We will exploit the continuity method [19, 31] to prove existence and to derive the error bounds simultaneously.

The linearized expressions are easily separated from the nonlinear variational form by using (2.5) and (2.10)

$$\begin{aligned}
a((\boldsymbol{u}_h^t, p_h^t), \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v}\rangle \;&=\; a((\boldsymbol{u}_h^t, p_h^t), \boldsymbol{v}) - a((\boldsymbol{u}, p), \boldsymbol{v}) \\
&= (\boldsymbol{\nabla}(\boldsymbol{u}_h^t - \boldsymbol{u}), \boldsymbol{v}) + (p_h^t, \operatorname{div}(\operatorname{adj} \boldsymbol{\nabla} \boldsymbol{u}_h^t \boldsymbol{v})) - (p, \operatorname{div}(\operatorname{adj} \boldsymbol{\nabla} \boldsymbol{u} \boldsymbol{v})) \\
&= (\boldsymbol{\nabla}(\boldsymbol{u}_h^t - \boldsymbol{u}), \boldsymbol{v}) + (p, \operatorname{div}(\operatorname{adj} \boldsymbol{\nabla}(\boldsymbol{u}_h^t - \boldsymbol{u})\boldsymbol{v})) \\
&\quad + (p_h^t - p, \operatorname{div}(\operatorname{adj} \boldsymbol{\nabla} \boldsymbol{u}\, \boldsymbol{v})) + (p_h^t - p, \operatorname{div}(\operatorname{adj} \boldsymbol{\nabla}(\boldsymbol{u}_h^t - \boldsymbol{u})\boldsymbol{v})) \\
&= a_{\boldsymbol{u}}(p; \boldsymbol{u} - \boldsymbol{u}_h^t, \boldsymbol{v}) + a_p(\boldsymbol{u}; \boldsymbol{v}, p - p_h^t) + (p_h^t - p; \operatorname{div} \operatorname{adj} \boldsymbol{\nabla}(\boldsymbol{u}_h^t - \boldsymbol{u})\boldsymbol{v})).
\end{aligned}$$

Similarly, by using in addition (2.1), we have

$$\begin{aligned}
&(\det(\mathrm{Id} + \boldsymbol{\nabla} \boldsymbol{u}_h^t) - 1, q) - \lambda^{-1}(p_h^t, q) \\
=\;& (\det(\mathrm{Id} + \boldsymbol{\nabla} \boldsymbol{u}) - 1, q) + (\mathrm{Cof}(\mathrm{Id} + \boldsymbol{\nabla} \boldsymbol{u}) : \boldsymbol{\nabla}(\boldsymbol{u}_h^t - \boldsymbol{u}), q) \\
& + (\det \boldsymbol{\nabla}(\boldsymbol{u}_h^t - \boldsymbol{u}), q) - \lambda^{-1}(p_h^t, q) \\
=\;& \lambda^{-1}(p, q) + \big(\operatorname{div}(\operatorname{adj}(\mathrm{Id} + \boldsymbol{\nabla} \boldsymbol{u})(\boldsymbol{u}_h^t - \boldsymbol{u})), q\big) + (\det \boldsymbol{\nabla}(\boldsymbol{u}_h^t - \boldsymbol{u}), q) - \lambda^{-1}(p_h^t, q) \\
=\;& a_p(\boldsymbol{u}; \boldsymbol{u}_h^t - \boldsymbol{u}, q) - \lambda^{-1}(p_h^t - p, q) + (\det \boldsymbol{\nabla}(\boldsymbol{u}_h^t - \boldsymbol{u}), q).
\end{aligned}$$

We insert these expressions in (2.19) to obtain an equation for the error:

$$\begin{aligned}
a_{\boldsymbol{u}}(\boldsymbol{u} - \boldsymbol{u}_h^t, \boldsymbol{v}) + a_p(\boldsymbol{v}, p - p_h^t) &= t(p - p_h^t, \operatorname{div}(\operatorname{adj} \boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{u}_h^t)\boldsymbol{v})), \quad \forall \boldsymbol{v} \in \boldsymbol{X}_h, \\
a_p(\boldsymbol{u} - \boldsymbol{u}_h^t, q) - \lambda^{-1}(q, p - p_h^t) &= t(\det \boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{u}_h^t), q), \qquad\qquad \forall q \in M_h.
\end{aligned} \tag{2.20}$$

When performing the homotopy method we define a subset of the interval $[0, 1]$ by

$$\begin{aligned}
\Xi_h(\mathcal{C}) := \{\, t \in [0, 1] \mid \text{Problem 2.9 has a solution } \big(\boldsymbol{u}_h^t, p_h^t\big) \in \boldsymbol{X}_h \times M_h \\
\text{such that } \|\boldsymbol{u} - \boldsymbol{u}_h^t, p - p_h^t\|_h \le \mathcal{C} h^2 |\ln h|^2 \,\}.
\end{aligned} \tag{2.21}$$

The set $\Xi_h(\mathcal{K}_1)$ will be shown to be nonvoid, open and closed with respect to the interval $[0, 1]$ for sufficiently small h and some $\mathcal{K}_1 > 0$. Hence, it coincides with $[0, 1]$. The cornerstones are an a priori estimate and the applicability of an inverse function theorem. They are provided in the following two lemmas that will be proven in Section 6.

Lemma 2.10. *Let* $(\boldsymbol{u}, p) \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega}) \times \mathcal{C}^1(\overline{\Omega})$ *be the solution of Problem 2.1, and assume that Assumption* **R** *holds. Then there exist positive constants* h_1 *and* $\mathcal{K}_1$ *with the following property: If* $\big(\boldsymbol{u}_h^t, p_h^t\big) \in U_\gamma$ *is a solution of Problem 2.9, with* $\gamma := 2\mathcal{K}_1 h^2 |\ln h|^2$, $0 < h < h_1$, *and* $t \in [0, 1]$, *then we have*

$$\big(\boldsymbol{u}_h^t, p_h^t\big) \in U_{\gamma/2}. \tag{2.22}$$

Furthermore, $(\boldsymbol{u}_h^0, p_h^0) \in U_{\gamma/2}$*, i.e., the inclusion holds for* $t = 0$*.*

Actually, we will show a stronger result. It is sufficient to have $\big(\boldsymbol{u}_h^t, p_h^t\big) \in U_{\gamma_0}$ for some γ_0 of the order $h|\ln h|^{1/2}$ instead of $h^2|\ln h|^2$.

Lemma 2.11. *Let* $(\bar{\boldsymbol{u}}, \bar{p}) \in (\boldsymbol{X}_h \times M_h) \cap U_{\mathcal{K}_1 h^2 |\ln h|^2}$*. If* h *is sufficiently small, then the operator* $\mathcal{DH}\big(t, (\bar{\boldsymbol{u}}, \bar{p})\big)$ *is a regular mapping from* $\boldsymbol{X}_h \times M_h$ *into itself for all* $t \in [0, 1]$*.*

3. A priori Estimates and Regularity

First, we consider the regularity of Problem 2.3.

Theorem 3.1. *For any* $(\boldsymbol{F}, G) \in \boldsymbol{L}^2(\Omega) \times H^1(\Omega)$*, the solution* $(\boldsymbol{w}, r)$ *of Problem 2.3 belongs to* $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ *and*

$$\|\boldsymbol{w}\|_2 + \|r\|_1 \leq C(\|\boldsymbol{F}\|_0 + \|G\|_1). \tag{3.1}$$

Here C *is independent of* λ*.*

Proof. If we restrict λ to a bounded set, a proof of the theorem is essentially included in [15, Lemma 3.5]. The key point is that (2.11) is elliptic in the sense of ADN [1] as was observed by Le Dret [21].

Therefore, we only need to consider Problem 2.3 for large λ, and we assume that $\lambda \geq \lambda_0$, with λ_0 to be specified later. We consider an auxiliary problem without penalty term. *Find* $(\boldsymbol{w}_1, r_1) \in \boldsymbol{X} \times M$ *such that*

$$\begin{array}{rcll} a_{\boldsymbol{u}}(\boldsymbol{w}_1, \boldsymbol{v}) + a_p(\boldsymbol{v}, r_1) & = & \langle \boldsymbol{F}, \boldsymbol{v} \rangle & \forall \boldsymbol{v} \in \boldsymbol{X}, \\ a_p(\boldsymbol{w}_1, q) & = & (G, q) & \forall q \in M. \end{array} \tag{3.2}$$

In view of [15, Lemma 3.5], we have

$$\|\boldsymbol{w}_1\|_2 + \|r_1\|_1 \leq C_3(\|\boldsymbol{F}\|_0 + \|G\|_1). \tag{3.3}$$

Next we define a mapping T from $H^1(\Omega)$ into itself. Given r, let $Tr := \bar{r}$ be defined by

$$\begin{array}{rcll} a_{\boldsymbol{u}}(\bar{\boldsymbol{w}}, \boldsymbol{v}) + a_p(\boldsymbol{v}, \bar{r}) & = & \langle \boldsymbol{F}, \boldsymbol{v} \rangle & \forall \boldsymbol{v} \in \boldsymbol{X}, \\ a_p(\bar{\boldsymbol{w}}, q) & = & (G + \lambda^{-1} r, q) & \forall q \in M. \end{array}$$

Given $r_1, r_2 \in H^1(\Omega)$, it follows from (3.3) that $\|\bar{r}_1 - \bar{r}_2\|_1 \leq \lambda^{-1} C_3 \|r_1 - r_2\|_1 \leq \frac{1}{2}\|r_1 - r_2\|_1$ whenever $\lambda \geq \lambda_0 := 2C_3$. Hence, T is a contraction and it has a fixed point $r \in H^1(\Omega)$. Let $\boldsymbol{w} := \bar{\boldsymbol{w}}$ be the other component of the solution with this r. From

$$\|\boldsymbol{w}\|_2 + \|r\|_1 \leq C_3(\|\boldsymbol{F}\|_0 + \|G\|_1) + \frac{1}{2}\|r\|_1,$$

we conclude that (3.1) holds with $C := 2C_3$, and the proof is complete. □

A duality argument [24], yields the following fact.

Lemma 3.2. *Let* $(\boldsymbol{F}, G) \in \boldsymbol{H}^2(\Omega)' \times H^1(\Omega)'$*. Then the solution* $(\boldsymbol{w}, r)$ *of Problem 2.3 belongs to* $\boldsymbol{L}^2(\Omega) \times H^{-1}(\Omega)$ *and*

$$\|\boldsymbol{w}\|_0 + \|r\|_{-1} \le C(\|\boldsymbol{F}\|_{-2} + \|G\|_{-1}). \tag{3.4}$$

Proof. Given $(\boldsymbol{\phi}, s) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{H}_0^1(\Omega)$, consider the auxiliary problem: *Find* $(\overline{\boldsymbol{w}}, \overline{r}) \in \boldsymbol{X} \times M$ *such that*

$$\mathcal{A}(\boldsymbol{v}, q; \overline{\boldsymbol{w}}, \overline{r}) = (\boldsymbol{\phi}, \boldsymbol{v}) + (s, q), \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{X} \times M. \tag{3.5}$$

Theorem 3.1 implies that $\|\overline{\boldsymbol{w}}\|_2 + \|\overline{r}\|_1 \le C(\|\boldsymbol{\phi}\|_0 + \|s\|_1)$. By setting $\boldsymbol{v} := \boldsymbol{w}$ and $q := r$ in (3.5) it follows that

$$\begin{aligned}
|(\boldsymbol{w}, \boldsymbol{\phi}) + (r, s)| = |\mathcal{A}(\boldsymbol{w}, r; \overline{\boldsymbol{w}}, \overline{r})| &= |\langle \boldsymbol{F}, \overline{\boldsymbol{w}} \rangle + (G, \overline{r})| \\
&\le (\|\boldsymbol{F}\|_{-2} + \|G\|_{-1})(\|\overline{\boldsymbol{w}}\|_2 + \|\overline{r}\|_1) \\
&\le C(\|\boldsymbol{F}\|_{-2} + \|G\|_{-1})(\|\boldsymbol{\phi}\|_0 + \|s\|_1),
\end{aligned}$$

which implies the bounds of $\|\boldsymbol{w}\|_0$ and $\|r\|_{-1}$ as stated in (3.4). $\square$

To achieve L^∞ estimates, we consider error estimates for weighted Sobolev norms with the weight function

$$\sigma(\boldsymbol{x}) := (|\boldsymbol{x} - \boldsymbol{x}_0|^2 + \theta^2)^{1/2}, \quad \boldsymbol{x}, \boldsymbol{x}_0 \in \Omega. \tag{3.6}$$

Here $\theta = Kh$, with $K \ge 1$ being a constant to be specified later. For $\alpha \in \mathbb{R}$ and k a nonnegative integer, we define the weighted semi-norms

$$\|\mathcal{D}^k q\|_{\sigma^\alpha}^2 := \sum_{|\beta| = k} \int_\Omega |\partial^\beta q(\boldsymbol{x})|^2 \sigma(\boldsymbol{x})^\alpha \, d\boldsymbol{x}, \quad q \in H^k(\Omega). \tag{3.7}$$

The same notation will be used for vector-valued functions. We list some properties of σ (see [9, p. 170] for more general cases),

$$\max_{\boldsymbol{x} \in K} \sigma(\boldsymbol{x}) \le C \min_{\boldsymbol{x} \in K} \sigma(\boldsymbol{x}), \qquad \forall K \in \mathcal{T}_h, \tag{3.8}$$

$$|\mathcal{D}^j \sigma(\boldsymbol{x})^\alpha| \le C(j, \alpha) \sigma^{\alpha - j}(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \Omega, \ \alpha \in \mathbb{R}. \tag{3.9}$$

By [11, Theorem 3.3.2], if θ is small enough,

$$\int_\Omega \sigma^{-(2+\alpha)}(\boldsymbol{x}) \, d\boldsymbol{x} \le \begin{cases} C\theta^{-\alpha}, & \text{for } \alpha > 0, \\ C|\ln \theta|, & \text{for } \alpha = 0. \end{cases} \tag{3.10}$$

Theorem 3.3. *For any* $(\boldsymbol{F}, G) \in \boldsymbol{L}^2(\Omega) \times H^1(\Omega)$*, the solution* $(\boldsymbol{w}, r)$ *of Problem 2.3 satisfies*

$$\begin{aligned}
\|\mathcal{D}^2 \boldsymbol{w}\|_{\sigma^2} + \|\boldsymbol{\nabla} r\|_{\sigma^2} &\le C\theta(\|\boldsymbol{F}\|_0 + \|G\|_1) \\
&\quad + C(\|\boldsymbol{F}\|_{\sigma^2} + \|\boldsymbol{\nabla} G\|_{\sigma^2} + \|\boldsymbol{F}\|_{-1} + \|G\|_0).
\end{aligned} \tag{3.11}$$

Proof. As in [11, Chapter 3, Section 3.3], let $\mu_j = x^j - x_0^j$, $(j = 1, 2)$. From $\mu_j \boldsymbol{\nabla} r = \boldsymbol{\nabla}(\mu_j r) - (\boldsymbol{\nabla} x_j) r$ and a similar formula for $\boldsymbol{w}$ it follows that

$$\|\mu_j \mathcal{D}^2 \boldsymbol{w}\|_0 + \|\mu_j \boldsymbol{\nabla} r\|_0 \le C(\|\mathcal{D}^2(\mu_j \boldsymbol{w})\|_0 + \|\boldsymbol{\nabla}(\mu_j r)\|_0 + \|\boldsymbol{\nabla} \boldsymbol{w}\|_0 + \|r\|_0).$$

We only need to establish a bound of the first two terms on the right-hand side since the other terms are covered by (2.15). Observe that the strong form of Problem 2.3 reads

$$\begin{aligned} -\triangle \boldsymbol{w} - (\operatorname{adj} \boldsymbol{\nabla} \boldsymbol{w}) \boldsymbol{\nabla} p - \operatorname{Cof}(\mathrm{Id} + \boldsymbol{\nabla} \boldsymbol{u}) \boldsymbol{\nabla} r &= \boldsymbol{F}, \\ \operatorname{adj}(\mathrm{Id} + \boldsymbol{\nabla} \boldsymbol{u}) : \boldsymbol{\nabla} \boldsymbol{w} - \lambda^{-1} r &= G. \end{aligned}$$

A simple computation yields

$$\begin{aligned} -\triangle(\mu_j \boldsymbol{w}) - (\operatorname{adj} \boldsymbol{\nabla}(\mu_j \boldsymbol{w})) \boldsymbol{\nabla} p - \operatorname{Cof}(\mathrm{Id} + \boldsymbol{\nabla} \boldsymbol{u}) \boldsymbol{\nabla}(\mu_j r) &= \overline{\boldsymbol{F}}, \\ \operatorname{adj}(\mathrm{Id} + \boldsymbol{\nabla} \boldsymbol{u}) : \boldsymbol{\nabla}(\mu_j \boldsymbol{w}) - \lambda^{-1} \mu_j r &= \overline{G}, \end{aligned}$$

where the modified right-hand sides differ from the original ones only by terms of lower order

$$\begin{aligned} \overline{\boldsymbol{F}} &= \mu_j \boldsymbol{F} - 2 \boldsymbol{\nabla} \mu_j \boldsymbol{\nabla} \boldsymbol{w} - \boldsymbol{\nabla} p \boldsymbol{\nabla} \mu_j^{\perp} \otimes \boldsymbol{w}^{\perp} - \operatorname{Cof}(\mathrm{Id} + \boldsymbol{\nabla} \boldsymbol{u}) \boldsymbol{\nabla} \mu_j r, \\ \overline{G} &= \mu_j G + \operatorname{adj}(\mathrm{Id} + \boldsymbol{\nabla} \boldsymbol{u}) : \boldsymbol{w} \otimes \boldsymbol{\nabla} \mu_j. \end{aligned}$$

By virtue of Theorem 3.1 and Assumption **R** it follows that

$$\begin{aligned} \|\mathcal{D}^2(\mu_j \boldsymbol{w})\|_0 + \|\boldsymbol{\nabla}(\mu_j r)\|_0 &\le C(\|\mu_j \boldsymbol{F}\|_0 + \|\mu_j \boldsymbol{\nabla} G\|_0 + \|\boldsymbol{\nabla} \boldsymbol{w}\|_0 + \|r\|_0) \\ &\le C(\|\mu_j \boldsymbol{F}\|_0 + \|\mu_j \boldsymbol{\nabla} G\|_0 + \|\boldsymbol{F}\|_{-1} + \|G\|_0). \end{aligned} \tag{3.12}$$

Next,

$$\|\mathcal{D}^2 \boldsymbol{w}\|_{\sigma^2}^2 = \theta^2 \|\mathcal{D}^2 \boldsymbol{w}\|_0^2 + \sum_{j=1}^{2} \|\mu_j \mathcal{D}^2 \boldsymbol{w}\|_0^2.$$

Combining this with (3.12) and the analogous estimates of $\boldsymbol{\nabla} r$ we obtain (3.11). □

Proceeding along the same line and using Lemma 3.2, we obtain

Lemma 3.4. *For any* $(\boldsymbol{F}, G) \in \boldsymbol{L}^2(\Omega) \times H^1(\Omega)$, *the solution* $(\boldsymbol{w}, r)$ *of Problem 2.3 satisfies*

$$\begin{aligned} \|\boldsymbol{\nabla} \boldsymbol{w}\|_{\sigma^2} + \|r\|_{\sigma^2} &\le C\theta(\|\boldsymbol{F}\|_{-1} + \|G\|_0) \\ &\quad + C(\sum_{j=1}^{2} \|\mu_j \boldsymbol{F}\|_{-1} + \|G\|_{\sigma^2}) + C(\|\boldsymbol{F}\|_{-2} + \|G\|_{-1}). \end{aligned} \tag{3.13}$$

4. An Interpolant of Clément Type

In this section, we provide some approximation properties of an interpolant of Clément type. Consider a macroelement $\mathcal{M}$ of $\mathcal{T}_{2h}$; see Figure 1. Denote its four edges by $\mathcal{E}_i$ $(1 \le i \le 4)$, and the midpoint of $\mathcal{E}_i$ by a_i. Let b_0 be the center of $\mathcal{M}$, and b_i $(i = 1, 2, 3, 4)$ be the corner nodes labeled anticlockwise from the lower-left corner, and the four corresponding squares are denoted by K_i $(i = 1, 2, 3, 4)$.

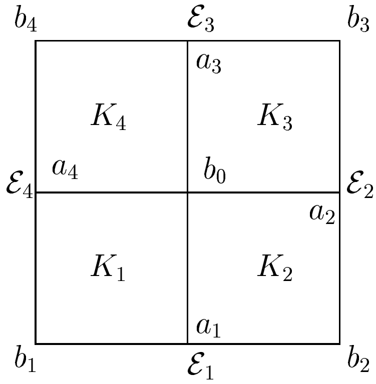

Fig. 1. Macroelement $\mathcal{M}$

We start with the Clément type interpolation operator R_h defined in [3]. It will be modified following [4] in order to preserve the integrals over the edges. If we modify an interpolant v_h at the midpoints a_1, a_2, a_3, and a_4, only the change of $v_h(a_i)$ will contribute to $\int_{\mathcal{E}_i} v_h ds$. In this way we obtain an interpolation operator

$$\Pi_h : H^1(\Omega) \to \{v \in H^1(\Omega) \mid v_{|K} \in Q_1(K)\ \forall K \in \mathcal{T}_h\}$$

that is given by

$$\begin{aligned} (\Pi_h)_{|K_i} &: H^1(\mathcal{M}) \to Q_1(K_i), \quad i = 1, 2, 3, 4, \\ \Pi_h v(b_i) &:= R_h v(b_i), \quad i = 0, 1, 2, 3, 4, \\ \int_{\mathcal{E}_i} &(v - \Pi_h v)\, d\boldsymbol{s} = 0, \quad 1 \le i \le 4. \end{aligned} \tag{4.1}$$

The interpolation process preserves the nullity of traces due to $(4.1)_2$ and $(4.1)_3$. For a vector function $\boldsymbol{v} = (v_1, v_2) \in \boldsymbol{H}_0^1(\Omega)$, we define the interpolant $\boldsymbol{\Pi}_h \boldsymbol{v}$ by the components,

$$\boldsymbol{\Pi}_h \boldsymbol{v} := (\Pi_h v_1, \Pi_h v_2). \tag{4.2}$$

Similarly $\boldsymbol{R}_h \boldsymbol{v} := (R_h v_1, R_h v_2)$.

We note that the construction of a similar operator in [4] is based on a Ritz projector. The quasi-local character of the operator (4.1) enables us, however, to establish the estimates (4.6) and (4.7) below.

Theorem 4.1. *The interpolation operator* $\boldsymbol{\Pi}_h$ *defined by* (4.1) *and* (4.2) *has the following properties:*

(1) *For any matrix* $\boldsymbol{A}$ *whose elements are piecewise constant on each* K *of* $\mathcal{T}_{2h}$, *we have*

$$\sum_{\mathcal{M}\in\mathcal{T}_{2h}} \int_{\mathcal{M}} \operatorname{div}\big(\boldsymbol{A}(\boldsymbol{v}-\boldsymbol{\Pi}_h\boldsymbol{v})\big) q\, d\boldsymbol{x} = 0, \quad \forall q \in M_h. \tag{4.3}$$

(2) *There exists a constant* C *such that*

$$\|\mathcal{D}^k(\boldsymbol{v}-\boldsymbol{\Pi}_h\boldsymbol{v})\|_{L^p} \le Ch^{j-k}\|\mathcal{D}^j\boldsymbol{v}\|_{L^p}, \quad k=0,1 \text{ and } j=1,2, \;\; p\in[1,\infty], \tag{4.4}$$

$$\|\mathcal{D}^k(\boldsymbol{v}-\boldsymbol{\Pi}_h\boldsymbol{v})\|_{\sigma^2} \le Ch^{j-k}\|\mathcal{D}^j\boldsymbol{v}\|_{\sigma^2}, \quad k=0,1 \;\text{ and }\; j=1,2. \tag{4.5}$$

(3) *There exists a constant* C *such that*

$$\|\boldsymbol{\nabla}\big(\sigma^2\boldsymbol{v}-\boldsymbol{\Pi}_h(\sigma^2\boldsymbol{v})\big)\|_{\sigma^{-2}} \le Ch\|\boldsymbol{v}\|_{\sigma^{-2}} + Ch\|\boldsymbol{\nabla}\boldsymbol{v}\|_0, \quad \forall \boldsymbol{v}\in\boldsymbol{X}_h. \tag{4.6}$$

(4) *There exists a constant* C *such that for all* $p\in[1,\infty]$

$$\|\boldsymbol{\nabla}\boldsymbol{\Pi}_h\boldsymbol{v}\|_{L^p} \le C\|\boldsymbol{\nabla}\boldsymbol{v}\|_{L^p} \quad \text{and} \quad \|\boldsymbol{\nabla}\boldsymbol{\Pi}_h\boldsymbol{v}\|_{\sigma^2} \le C\|\boldsymbol{\nabla}\boldsymbol{v}\|_{\sigma^2}\,. \tag{4.7}$$

Here and in the proof, $\mathcal{D}^2\boldsymbol{v}$ *is understood in a piecewise manner for any* $\boldsymbol{v}\in\boldsymbol{X}_h$, *so does* $\|\mathcal{D}^2\boldsymbol{v}\|_{L^p}$ *and* $\|\mathcal{D}^2\boldsymbol{v}\|_{\sigma^2}$.

A similar bound for the first part of (4.7) with $p=2$ is included in [4, Theorem 3.1]. The main ingredient of the following proof is the scaled trace inequality as well as an error estimate of $\boldsymbol{R}_h$.

Proof. (1) Let $\boldsymbol{A}$ be a matrix whose elements are constant on each $\mathcal{M}$ in $\mathcal{T}_{2h}$. By $(4.1)_3$ we obtain for the normal components $\int_{\mathcal{E}_i}\boldsymbol{A}(\boldsymbol{\Pi}_h\boldsymbol{v}-\boldsymbol{v})\cdot\boldsymbol{n}\,ds=0$ on each edge $\mathcal{E}_i$ of the $2h$-grid. Gauss' theorem yields now (4.3).

(2) We restrict ourselves to $p<\infty$ since the case $p=\infty$ follows the same lines.

Referring to Figure 1, we select K_1 as a typical element in $\mathcal{T}_h$. A simple scaling argument yields

$$\int_{K_1}|(\boldsymbol{\Pi}_h-\boldsymbol{R}_h)\boldsymbol{v}|^p\,d\boldsymbol{x} \le Ch_{K_1}^2\sum_{i=1}^4|(\boldsymbol{\Pi}_h-\boldsymbol{R}_h)\boldsymbol{v}(x_i)|^p,$$

where $x_1=b_1$, $x_2=a_1$, $x_3=b_0$, $x_4=a_4$. Since $(\boldsymbol{\Pi}_h-\boldsymbol{R}_h)\boldsymbol{v}$ vanishes at b_0 and b_1, we obtain contributions only from a_1 and a_4. Similar estimates hold for the integrals on K_2, K_3 and K_4. Summing up, we have on the macro-element $\mathcal{M}\in\mathcal{T}_{2h}$,

$$\int_{\mathcal{M}}|(\boldsymbol{\Pi}_h-\boldsymbol{R}_h)\boldsymbol{v}|^p\,d\boldsymbol{x} \le Ch_{\mathcal{M}}^2\sum_{i=1}^4|(\boldsymbol{\Pi}_h-\boldsymbol{R}_h)\boldsymbol{v}(a_i)|^p. \tag{4.8}$$

Next, since $(\boldsymbol{\Pi}_h-\boldsymbol{R}_h)\boldsymbol{v}$ is piecewise linear on $\mathcal{E}_i$ and vanishes at the vertices of $\mathcal{M}$, we have in view of $(4.1)_3$,

$$(\boldsymbol{\Pi}_h-\boldsymbol{R}_h)\boldsymbol{v}(a_i) = \frac{1}{|\mathcal{E}_i|}\int_{\mathcal{E}_i}(\boldsymbol{\Pi}_h-\boldsymbol{R}_h)\boldsymbol{v}\,ds = \frac{1}{|\mathcal{E}_i|}\int_{\mathcal{E}_i}(\boldsymbol{v}-\boldsymbol{R}_h\boldsymbol{v})\,d\boldsymbol{s}. \tag{4.9}$$

Combining (4.8) with (4.9) and using the scaled trace inequality [20, Theorem 1.5.1.10] and Hölder's inequality we obtain for any $j = 1, 2$,

$$\begin{aligned}
\int_{\mathcal{M}} |(\boldsymbol{\Pi}_h - \boldsymbol{R}_h)\boldsymbol{v}|^p \, d\boldsymbol{x} &\le C h_{\mathcal{M}} \sum_{i=1}^{4} \int_{\mathcal{E}_i} |\boldsymbol{v} - \boldsymbol{R}_h \boldsymbol{v}|^p \, d\boldsymbol{s} \\
&\le C h_{\mathcal{M}} \left(h_{\mathcal{M}}^{-1} \int_{\mathcal{M}} |\boldsymbol{v} - \boldsymbol{R}_h \boldsymbol{v}|^p \, d\boldsymbol{x} + h_{\mathcal{M}}^{p-1} \int_{\mathcal{M}} |\boldsymbol{\nabla}(\boldsymbol{v} - \boldsymbol{R}_h \boldsymbol{v})|^p \, d\boldsymbol{x} \right) \\
&\le C h_{\mathcal{M}}^{jp} \int_{\omega(\mathcal{M})} |\mathcal{D}^j \boldsymbol{v}|^p \, d\boldsymbol{x}, \qquad (4.10)
\end{aligned}$$

where $\omega(\mathcal{M}) := \{\mathcal{M}' \in \mathcal{T}_{2h} \mid \mathcal{M}' \cap \mathcal{M} \neq \emptyset\}$. An application of the *inverse inequality* leads to

$$\|\boldsymbol{\nabla}(\boldsymbol{\Pi}_h - \boldsymbol{R}_h)\boldsymbol{v}\|_{L^p(\mathcal{M})} \le C h_{\mathcal{M}}^{(j-1)p} \|\mathcal{D}^j \boldsymbol{v}\|_{L^p(\omega(\mathcal{M}))}, \quad j = 1, 2. \tag{4.11}$$

Summing up all $\mathcal{M} \in \mathcal{T}_{2h}$ in (4.10) and (4.11), we obtain

$$\|\mathcal{D}^k(\boldsymbol{\Pi}_h v - \boldsymbol{R}_h \boldsymbol{v})\|_{L^p} \le C h^{j-k} \|\mathcal{D}^j \boldsymbol{v}\|_{L^p}, \quad k = 0, 1 \text{ and } j = 1, 2, \;\; p \in [1, \infty).$$

This inequality together with the well-known estimates for $\boldsymbol{R}_h$ [3], namely

$$\|\mathcal{D}^k(\boldsymbol{v} - \boldsymbol{R}_h \boldsymbol{v})\|_{L^p} \le C h^{j-k} \|\mathcal{D}^j \boldsymbol{v}\|_{L^p}, \quad k = 0, 1 \text{ and } j = 1, 2, \;\; p \in [1, \infty),$$

leads to (4.4) for all j and k, which together with (3.8) immediately yields (4.5).

(3) Since the triangulation $\mathcal{T}_h$ is quasi-uniform, we can derive in the same way as (3.8),

$$\max_{\boldsymbol{x} \in K} \sigma(\boldsymbol{x}) \le C \min_{\boldsymbol{x} \in \omega(\mathcal{M})} \sigma(\boldsymbol{x}), \qquad \forall K \subset \mathcal{M} \in \mathcal{T}_{2h}\,. \tag{4.12}$$

We recall (4.11) for $p = 2$, i.e., $\|\boldsymbol{\nabla}(\boldsymbol{w} - \boldsymbol{\Pi}_h \boldsymbol{w})\|_{L^2(K)} \le C h_{\mathcal{M}} \|\mathcal{D}^2 \boldsymbol{w}\|_{L^2(\omega(\mathcal{M}))}$, for $K \subset \mathcal{M} \in \mathcal{T}_{2h}$. Together with (4.12) we have

$$\int_K \sigma(\boldsymbol{x})^{-2} |\boldsymbol{\nabla}(\boldsymbol{w} - \boldsymbol{\Pi}_h \boldsymbol{w})| \, d\boldsymbol{x} \le C h_{\mathcal{M}}^2 \int_{\omega(\mathcal{M})} \sigma(\boldsymbol{x})^{-2} |\mathcal{D}^2 \boldsymbol{w}|^2 \, d\boldsymbol{x}.$$

Summing up for all $K \in \mathcal{T}_h$ gives

$$\|\boldsymbol{\nabla}(\boldsymbol{w} - \boldsymbol{\Pi}_h \boldsymbol{w})\|_{\sigma^{-2}} \le C h \|\mathcal{D}^2(\sigma^2 \boldsymbol{w})\|_{\sigma^{-2}}.$$

For any $\boldsymbol{v} \in \mathcal{T}_h$, let $\boldsymbol{w}$ be $\sigma^2 \boldsymbol{v}$ in the above inequality, we get

$$\|\boldsymbol{\nabla}\big(\sigma^2 \boldsymbol{v} - \boldsymbol{\Pi}_h(\sigma^2 \boldsymbol{v})\big)\|_{\sigma^{-2}} \le C h \|\mathcal{D}^2(\sigma^2 \boldsymbol{v})\|_{\sigma^{-2}} \le C(h\|\boldsymbol{v}\|_{\sigma^{-2}} + h\|\boldsymbol{\nabla} \boldsymbol{v}\|_0 + h^2 \|\mathcal{D}^2 \boldsymbol{v}\|_0).$$

An inverse inequality for finite element spaces applied to the last term yields (4.6).

(4) Finally, by applying (4.4) and (4.5) with $k = j = 1$ and using the triangle inequality we obtain the bound (4.7). $\square$

Remark 4.2. *The property* (4.3) *is similar to the key assumption in* [16, (1.8)], *where the assumption was written for* $\boldsymbol{A}$ *being the identity matrix. The essential point is that the pair* $(\boldsymbol{X}_h, M_h)$ *is stable for the Stokes problem and there is a locally constructed operator with properties like the* Fortin interpolant.

Similar constructions can be performed for other pairs of finite element spaces; see [16, Section 5]. *In particular the aforementioned* (Q_2, P_1) *element (see Remark 2.8), is easily treated by combining the procedure above with techniques in* [7] *and* [10].

A standard scaling argument together with an inverse inequality yields the following approximation and superapproximation properties that refer to the other function in the variational problem.

Theorem 4.3. *Let* $\mathcal{Q}_h$ *be the* L^2 *projection onto* M_h.

(1) *There exists a constant* C *such that*

$$\|q - \mathcal{Q}_h q\|_{L^p} \leq Ch\|\boldsymbol{\nabla} q\|_{L^p}, \qquad \forall p \in [1, \infty], \tag{4.13}$$

$$\|q - \mathcal{Q}_h q\|_{\sigma^2} \leq Ch\|\boldsymbol{\nabla} q\|_{\sigma^2}. \tag{4.14}$$

(2) $\mathcal{Q}_h$ *admits the following superapproximation property*

$$\|\sigma^2 q - \mathcal{Q}_h(\sigma^2 q)\|_{\sigma^{-2}} \leq Ch\|q\|_0, \qquad \forall q \in M_h. \tag{4.15}$$

5. Regularized Green's Functions and L^∞ Estimates

The proof of Lemma 2.10 will be based on L^∞ estimates of the terms on the right-hand side of (2.20). They will be obtained by the technique proposed by Frehse and Rannacher [18] and refer to the regularized Green's functions and the weighted Sobolev norm estimates introduced by Natterer [25]. Our approach is similar to that in [16]; the concept is even more transparent in the scalar case [9, Chapter 7]. In the present situation, there is, however, an extra difficulty. We have only a *Gårding inequality* instead of the usual coercivity of $a_{\boldsymbol{u}}(\cdot,\cdot)$. Thus the error bounds with respect to the energy norm are interwoven with an L^2 estimate, and the latter requires an appropriate duality argument.

Lemma 2.10 could also be proven by weighted Sobolev norm estimates as in [15], but only an order of $h^{2-\epsilon}$ would be achieved in (2.16). Moreover, the estimates for the regularized Green's functions are of independent interest.

We start with a general argument following [18]. Let $\chi = \chi_i$ $(1 \leq i \leq 3)$ be a piecewise polynomial, and let $|\chi|$ attain its maximum at some $\boldsymbol{x}_i \in K_i$. Since the mesh $\mathcal{T}_h$ is regular, we can find a ball $\mathcal{B}_i \subset K_i$ with radius αh, such that its center $\boldsymbol{y}_i$ satisfies $|\boldsymbol{x}_i - \boldsymbol{y}_i| \leq C\alpha h$. Here α is assumed to be sufficiently small depending on the shape parameter of the triangulation $\mathcal{T}_h$. Next, let $\delta_i \in \mathcal{C}_0^\infty(\Omega)$ satisfy

$$\operatorname{supp} \delta_i \subset \mathcal{B}_i, \tag{5.1}$$

$$\int_\Omega \delta_i \, d\boldsymbol{x} = 1, \quad \delta_i \geq 0, \tag{5.2}$$

$$\|\mathcal{D}^j \delta_i\|_{L^\infty} \leq Ch^{-2-j}, \quad j = 0, 1. \tag{5.3}$$

The mean value theorem, combined with (5.2), yields

$$\int_\Omega \chi\delta\, d\boldsymbol{x} = \chi(\boldsymbol{z}) \quad \text{for some} \quad \boldsymbol{z}\in\mathcal{B}_i.$$

Using an inverse inequality for polynomials leads to

$$\|\chi\|_{L^\infty(\Omega)} = |\chi(\boldsymbol{x}_i)| \le |\chi(\boldsymbol{z})| + C\alpha h\|\boldsymbol{\nabla}\chi\|_{L^\infty(K_1)} \le C\alpha\|\chi\|_{L^\infty(\Omega)} + \left|\int_\Omega \chi\delta\, d\boldsymbol{x}\right|.$$

Choosing α such that $C\alpha = 1/2$, we may rewrite the previous inequality as

$$\|\chi\|_{L^\infty(\Omega)} \le 2\left|\int_\Omega \chi\delta\, d\boldsymbol{x}\right|. \tag{5.4}$$

Therefore the required bound can be obtained from an estimate of a functional.

We define three families of regularized Green's functions and describe simultaneously their relation to L^∞ estimates. By duality first their L^1 norms enter that in turn will be bounded by weighted L^2 norms.

Lemma 5.1. *Let $(\boldsymbol{G}_1,\zeta_1)\in\boldsymbol{X}\times M$ and the corresponding regularized Green's functions $(\boldsymbol{G}_1^h,\zeta_1^h)\in\boldsymbol{X}_h\times M_h$ satisfy*

$$\mathcal{A}(\boldsymbol{G}_1,\zeta_1;\boldsymbol{v},q) = (\boldsymbol{\delta}_1,\boldsymbol{v}), \quad \forall(\boldsymbol{v},q)\in\boldsymbol{X}\times M, \tag{5.5}$$

$$\mathcal{A}(\boldsymbol{G}_1^h,\zeta_1^h;\boldsymbol{v},q) = (\boldsymbol{\delta}_1,\boldsymbol{v}), \quad \forall(\boldsymbol{v},q)\in\boldsymbol{X}_h\times M_h, \tag{5.6}$$

respectively. Here $\boldsymbol{\delta}_1 = (\delta_1,0)$ or $\boldsymbol{\delta}_1=(0,\delta_1)$. Then we have

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h^t\|_{L^\infty} \le{}& 2\|\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u}\|_{L^\infty} + C|\ln h|^{1/2}\big(\|\boldsymbol{\nabla}(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u})\|_{L^\infty} + \|p-\mathcal{Q}_hp\|_{L^\infty}\big)\\ &\times\big(\|\boldsymbol{\nabla}(\boldsymbol{G}_1-\boldsymbol{G}_1^h)\|_{\sigma_1^2} + \|\zeta_1-\zeta_1^h\|_{\sigma_1^2}\big)\\ &+ Ct|\ln h|^{1/2}\|\boldsymbol{\nabla}(\boldsymbol{u}-\boldsymbol{u}_h^t)\|_{L^\infty}^2\|\zeta_1^h\|_{\sigma_1^2}\\ &+ Ct|\ln h|^{1/2}\|\boldsymbol{\nabla}(\boldsymbol{u}-\boldsymbol{u}_h^t)\|_{L^\infty}\|p-p_h^t\|_{L^\infty}\|\boldsymbol{\nabla}\boldsymbol{G}_1^h\|_{\sigma_1^2}.\end{aligned} \tag{5.7}$$

Proof. It is enough to establish a bound of $\|\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h^t\|_{L^\infty}$. To this end, let $\boldsymbol{x}_1\in\Omega$ be a point where the maximum is attained

$$\|\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h^t\|_{L^\infty} = \max_{1\le i\le 2}|((\boldsymbol{\Pi}_h\boldsymbol{u})^i - \boldsymbol{u}_h^{t,i})(\boldsymbol{x}_1)|.$$

In view of (5.4) and (5.6), we have

$$\begin{aligned}\|\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h^t\|_{L^\infty} &\le 2\left|(\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h^t,\boldsymbol{\delta}_1)\right|\\ &= 2\left|a_{\boldsymbol{u}}(\boldsymbol{G}_1^h,\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h^t) + a_p(\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h^t,\zeta_1^h)\right|.\end{aligned}$$

By (2.20) together with (5.5) and (5.6), the above expressions can be expanded as

$$
\begin{aligned}
&(\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h^t,\boldsymbol{\delta}_1)\\
&=a_{\boldsymbol{u}}(\boldsymbol{G}_1^h,\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h^t)+a_p(\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u}_h^t,\zeta_1^h)\\
&=a_{\boldsymbol{u}}(\boldsymbol{G}_1^h-\boldsymbol{G}_1,\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u})+a_p(\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u},\zeta_1^h-\zeta_1)+a_p(p-\mathcal{Q}_hp,\boldsymbol{G}_1-\boldsymbol{G}_1^h)\\
&\quad+\lambda^{-1}(p-\mathcal{Q}_hp,\zeta_1^h-\zeta_1)+a_{\boldsymbol{u}}(\boldsymbol{G}_1,\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u})+a_p(\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u},\zeta_1)\\
&\quad+t(p-p_h^t,\operatorname{div}(\operatorname{adj}\boldsymbol{\nabla}(\boldsymbol{u}-\boldsymbol{u}_h^t)\boldsymbol{G}_1^h))+t(\det\boldsymbol{\nabla}(\boldsymbol{u}-\boldsymbol{u}_h^t),\zeta_1^h)\\
&=:I_1+\cdots+I_8.
\end{aligned}
$$

Now we estimate I_1 to I_8. An application of Hölder's inequality yields a bound of I_1 to I_4

$$
\begin{aligned}
|I_1|+\cdots+|I_4|\le\ &C(\|\boldsymbol{\nabla}(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u})\|_{L^\infty}+\|p-\mathcal{Q}_hp\|_{L^\infty})\\
&\times(\|\boldsymbol{\nabla}(\boldsymbol{G}_1-\boldsymbol{G}_1^h)\|_{L^1}+\|\zeta_1-\zeta_1^h\|_{L^1}).
\end{aligned}
\tag{5.8}
$$

Recalling (5.5) we have

$$
\begin{aligned}
|I_5+I_6|&=|a_{\boldsymbol{u}}(\boldsymbol{G}_1,\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u})+a_p(\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u},\zeta_1)|\\
&=|(\boldsymbol{\delta}_1,\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u})|\le\|\boldsymbol{\delta}_1\|_{L^1}\|\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u}\|_{L^\infty}=\|\boldsymbol{\Pi}_h\boldsymbol{u}-\boldsymbol{u}\|_{L^\infty}.
\end{aligned}
\tag{5.9}
$$

I_7 and I_8 are dealt with Hölder's inequality as

$$|I_7|\le t\|p-p_h^t\|_{L^\infty}\|\boldsymbol{\nabla}(\boldsymbol{u}-\boldsymbol{u}_h^t)\|_{L^\infty}\|\boldsymbol{\nabla}\boldsymbol{G}_1^h\|_{L^1},$$

and

$$|I_8|\le Ct\|\boldsymbol{\nabla}(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^\infty}^2\|\zeta_1^h\|_{L^1}.$$

From (3.10) and Hölder's inequality we obtain

$$\|\boldsymbol{\nabla}(\boldsymbol{G}_1-\boldsymbol{G}_1^h)\|_{L^1}\le C|\ln h|^{1/2}\|\boldsymbol{\nabla}(\boldsymbol{G}_1-\boldsymbol{G}_1^h)\|_{\boldsymbol{\sigma}_1^2},$$

as well as similar inequalities for $\boldsymbol{\nabla}\boldsymbol{G}_1^h$, $\zeta_1-\zeta_1^h$, and $\boldsymbol{\nabla}\zeta_1^h$. Summing up all the above estimates we complete the proof of (5.7). □

Similarly, we define another two families of regularized Green's functions and their discrete versions.

Lemma 5.2. *Let* $(\boldsymbol{G}_2,\zeta_2)\in\boldsymbol{X}\times M$ *and the corresponding regularized Green's function* $(\boldsymbol{G}_2^h,\zeta_2^h)\in\boldsymbol{X}_h\times M_h$ *satisfy*

$$\mathcal{A}(\boldsymbol{G}_2,\zeta_2;\boldsymbol{v},q)=(\mathcal{D}\boldsymbol{\delta}_2,\boldsymbol{v}),\quad\forall(\boldsymbol{v},q)\in\boldsymbol{X}\times M,\tag{5.10}$$

$$\mathcal{A}(\boldsymbol{G}_2^h,\zeta_2^h;\boldsymbol{v},q)=(\mathcal{D}\boldsymbol{\delta}_2,\boldsymbol{v}),\quad\forall(\boldsymbol{v},q)\in\boldsymbol{X}_h\times M_h,\tag{5.11}$$

respectively. Here $\mathcal{D}\boldsymbol{\delta}_2$ stands for any direction derivatives of either $\boldsymbol{\delta}_2 = (\delta_2, 0)$ or $\boldsymbol{\delta}_2 = (0, \delta_2)$. Then we have

$$\begin{aligned}\|\boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty} \le 2\|\boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u})\|_{L^\infty}\\ + C|\ln h|^{1/2}\big(\|\boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u})\|_{L^\infty} + \|p - \mathcal{Q}_hp\|_{L^\infty}\big)\\ \times \big(\|\boldsymbol{\nabla}(\boldsymbol{G}_2 - \boldsymbol{G}_2^h)\|_{\sigma_2^2} + \|\zeta_2 - \zeta_2^h\|_{\sigma_2^2}\big)\\ + Ct|\ln h|^{1/2}\|\boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty}^2\|\zeta_2^h\|_{\sigma_2^2}\\ + Ct|\ln h|^{1/2}\|\boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty}\|p - p_h^t\|_{L^\infty}\|\boldsymbol{\nabla}\boldsymbol{G}_2^h\|_{\sigma_2^2}\,.\end{aligned}$$

Proof. The proof of this lemma is almost the same with Lemma 5.1, the only difference is that we choose $\boldsymbol{x}_2$ to maximize $|\boldsymbol{\nabla}(\boldsymbol{\Pi}_h\boldsymbol{u} - \boldsymbol{u}_h^t)|$. By using (5.4) again we obtain

$$\begin{aligned}\|\boldsymbol{\nabla}(\boldsymbol{\Pi}_h\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty} &\le 2\left|(\boldsymbol{\nabla}(\boldsymbol{\Pi}_h\boldsymbol{u} - \boldsymbol{u}_h^t), \boldsymbol{\delta}_2)\right| = 2\left|(\boldsymbol{\Pi}_h\boldsymbol{u} - \boldsymbol{u}_h^t, \mathcal{D}\boldsymbol{\delta}_2)\right|\\ &= 2\left|a_{\boldsymbol{u}}(\boldsymbol{G}_2^h, \boldsymbol{\Pi}_h\boldsymbol{u} - \boldsymbol{u}_h^t) + a_p(\boldsymbol{\Pi}_h\boldsymbol{u} - \boldsymbol{u}_h^t, \zeta_2^h)\right|.\end{aligned}$$

The rest of the proof proceeds like the proof of Lemma 5.1. □

As in the preceding lemmas, we get

Lemma 5.3. *Let $(\boldsymbol{G}_3, \zeta_3) \in \boldsymbol{X} \times M$ and the corresponding regularized Green's function $(\boldsymbol{G}_3^h, \zeta_3^h) \in \boldsymbol{X}_h \times M_h$ satisfy*

$$\mathcal{A}(\boldsymbol{G}_3, \zeta_3; \boldsymbol{v}, q) = (\delta_3, q), \quad \forall(\boldsymbol{v}, q) \in \boldsymbol{X} \times M, \tag{5.12}$$

$$\mathcal{A}(\boldsymbol{G}_3^h, \zeta_3^h; \boldsymbol{v}, q) = (\delta_3, q), \quad \forall(\boldsymbol{v}, q) \in \boldsymbol{X}_h \times M_h, \tag{5.13}$$

respectively. Then we have

$$\begin{aligned}\|p - p_h^t\|_{L^\infty} \le 2\|p - \mathcal{Q}_hp\|_{L^\infty} + C|\ln h|^{1/2}\big(\|\boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u})\|_{L^\infty} + \|p - \mathcal{Q}_hp\|_{L^\infty}\big)\\ \times \big(\|\boldsymbol{\nabla}(\boldsymbol{G}_3 - \boldsymbol{G}_3^h)\|_{\sigma_3^2} + \|\zeta_3 - \zeta_3^h\|_{\sigma_3^2}\big)\\ + Ct|\ln h|^{1/2}\|\boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty}^2\|\zeta_3^h\|_{\sigma_3^2}\\ + Ct|\ln h|^{1/2}\|\boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty}\|p - p_h^t\|_{L^\infty}\|\boldsymbol{\nabla}\boldsymbol{G}_3^h\|_{\sigma_3^2}\,.\end{aligned}$$

When applying the three lemmas above, we need bounds of $\boldsymbol{G}_i$, $\boldsymbol{G}_i^h$, ζ_i, ζ_i^h, and their $\|\cdot\|_{\sigma_i^2}$ norms. The bounds are related to L^∞ estimates of linear problems. Therefore, the proofs are postponed to the appendix.

Lemma 5.4. *There exists a constant C which may depend on $\|\boldsymbol{u}\|_{\boldsymbol{C}^2(\overline{\Omega})}$ and $\|p\|_{C^1(\overline{\Omega})}$ such that*

$$\|\mathcal{D}^2\boldsymbol{G}_1\|_{\sigma_i^2} + \|\boldsymbol{\nabla}\zeta_1\|_{\sigma_i^2} \le \begin{cases} C|\ln h|^{1/2}, & i = 1,\\ Ch^{-1}, & i = 2, 3,\end{cases} \tag{5.14}$$

$$\|\boldsymbol{\nabla}\boldsymbol{G}_i\|_{\sigma_i^2} + \|\zeta_i\|_{\sigma_i^2} \le C|\ln h|^{1/2}, \qquad i = 1, 2, 3. \tag{5.15}$$

Theorem 5.5. *Let $\boldsymbol{G}_i$ and $\boldsymbol{G}_i^h$ be defined as before. If h is small enough,*

$$\|\nabla(\boldsymbol{G}_i - \boldsymbol{G}_i^h)\|_{\sigma_1^2} + \|\zeta_i - \zeta_i^h\|_{\sigma_1^2} \le \begin{cases} Ch|\ln h|^{1/2}, & \text{for } i = 1, \\ C, & \text{for } i = 2, 3. \end{cases} \tag{5.16}$$

6. Proof of the Main Result

In this section, we present the proof of Theorem 2.5. As a preparation, we first prove Lemma 2.10 and Lemma 2.11.

Proof of Lemma 2.10. Expressions with the discrete Green's functions will be estimated by applying the triangle inequality to $\nabla \boldsymbol{G}_i^h = \nabla \boldsymbol{G}_i + (\nabla \boldsymbol{G}_i^h - \nabla \boldsymbol{G}_i)$ and $\zeta_i^h = \zeta_i + (\zeta_i^h - \zeta_i)$. Inserting (5.16) into Lemma 5.2 and Lemma 5.3, respectively, and recalling (5.15) we obtain the a priori estimates

$$\begin{aligned} \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty} + \|p - p_h^t\|_{L^\infty} \le Ch|\ln h|^{1/2} + Ct|\ln h|\, \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty} \\ \times \big(\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty} + \|p - p_h^t\|_{L^\infty}\big). \end{aligned} \tag{6.1}$$

Similarly, inserting (5.16) into Lemma 5.1, we have

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h^t\|_{L^\infty} &\le Ch^2|\ln h| \\ &+ Ct|\ln h|\, \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty}\big(\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty} + \|p - p_h^t\|_{L^\infty}\big). \end{aligned} \tag{6.2}$$

Without loss of generality we may assume that $C \ge 1$ and $h \le e^{-1}$. Setting

$$A := \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h^t)\|_{L^\infty} + \|p - p_h^t\|_{L^\infty}, \qquad B := \|\boldsymbol{u} - \boldsymbol{u}_h^t\|_{L^\infty}$$

the two inequalities above may be rewritten

$$\begin{aligned} A &\le Ch|\ln h|^{1/2} + C|\ln h|A^2, \\ B &\le Ch^2|\ln h| + C|\ln h|A^2. \end{aligned}$$

Referring to the norm $\|\cdot\|_h$ that has been defined in (2.17) we assume that

$$\|\boldsymbol{u} - \boldsymbol{u}_h^t, p - p_h^t\|_h = h|\ln h|^{3/2}A + B \le \gamma := \frac{1}{2C}h|\ln h|^{1/2}. \tag{6.3}$$

From the inequality for A and the restriction (6.3) we conclude that

$$\begin{aligned} A &\le Ch|\ln h|^{1/2} + C|\ln h|Ah^{-1}|\ln h|^{-3/2}\gamma \\ &\le Ch|\ln h|^{1/2} + \frac{1}{2}A \le 2Ch|\ln h|^{1/2}. \end{aligned} \tag{6.4}$$

Combining the inequality for B with (6.4) we obtain

$$B \le Ch^2|\ln h| + C|\ln h|\, 4C^2h^2|\ln h| \le 5C^3h^2|\ln h|^2. \tag{6.5}$$

Finally we assume that

$$h \le h_1, \quad \text{where} \quad h_1|\ln h_1|^{3/2} \le \frac{1}{28C^4}. \tag{6.6}$$

Hence,

$$h|\ln h|^{3/2}A + B \le 2Ch^2|\ln h|^2 + 5C^3h^2|\ln h|^2 \le 7C^3h^2|\ln h|^2$$
$$\le \frac{1}{4C}h|\ln h|^{1/2} \le \frac{1}{2}\gamma.$$

After setting $\mathcal{K}_1 := \max(2C, 7C^3) = 7C^3$, the proof of the first assertion is complete. Note that for $t = 0$,

$$\|\boldsymbol{u} - \boldsymbol{u}_h^0, p - p_h^0\|_h \le 2Ch^2|\ln h|^2 < \mathcal{K}_1 h^2|\ln h|^2.$$

Thus the second assertion follows. □

Proof of Lemma 2.11. Let $(\bar{\boldsymbol{u}}, \bar{p}) \in U_{\mathcal{K}_1 h^2|\ln h|^2}$. We want to show that the problem

$$\mathcal{DH}\big(t, (\bar{\boldsymbol{u}}, \bar{p})\big)\big((\boldsymbol{w}_h, r_h), (\boldsymbol{\phi}, s)\big) = 0, \quad \forall (\boldsymbol{\phi}, s) \in \boldsymbol{X}_h \times M_h \tag{6.7}$$

has only the trivial solution $(\boldsymbol{w}_h, r_h) = 0$ whenever h is sufficiently small. Note that for any $(\boldsymbol{\phi}, s) \in \boldsymbol{X}_h \times M_h$,

$$\mathcal{DH}\big(t, (\bar{\boldsymbol{u}}, \bar{p})\big)\big((\boldsymbol{w}_h, r_h), (\boldsymbol{\phi}, s)\big) = \mathcal{A}(\boldsymbol{w}_h, r_h; \boldsymbol{\phi}, s) + tM(\boldsymbol{w}_h, r_h; \boldsymbol{\phi}, s),$$

where $M(\boldsymbol{w}_h, r_h; \boldsymbol{\phi}, s)$ is defined as

$$M(\boldsymbol{w}_h, r_h; \boldsymbol{\phi}, s) := \big(\bar{p} - p, \operatorname{div}(\operatorname{adj} \boldsymbol{\nabla} \boldsymbol{w}_h \boldsymbol{\phi})\big) + \big(r_h, \operatorname{div}(\operatorname{adj} \boldsymbol{\nabla}(\boldsymbol{u} - \bar{\boldsymbol{u}})\boldsymbol{\phi})\big)$$
$$+ \big(s, \operatorname{div}(\operatorname{adj} \boldsymbol{\nabla}(\boldsymbol{u} - \bar{\boldsymbol{u}})\boldsymbol{w}_h)\big).$$

By Theorem A.3 we have sufficiently small h

$$\sup_{(\boldsymbol{\phi}, s) \in \boldsymbol{X}_h \times M_h} \frac{\mathcal{DH}\big(t, (\bar{\boldsymbol{u}}, \bar{p})\big)\big((\boldsymbol{w}_h, r_h), (\boldsymbol{\phi}, s)\big)}{\|\boldsymbol{\phi}\|_1 + \|s\|_0}$$
$$\ge C(\|\boldsymbol{w}_h\|_1 + \|r_h\|_0)$$
$$- Ct(\|\boldsymbol{\nabla}(\boldsymbol{u} - \bar{\boldsymbol{u}})\|_{L^\infty} + \|p - \bar{p}\|_{L^\infty})(\|\boldsymbol{w}_h\|_1 + \|r_h\|_0)$$
$$\ge C(\|\boldsymbol{w}_h\|_1 + \|r_h\|_0). \tag{6.8}$$

This together with (6.7) leads to $(\boldsymbol{w}_h, r_h) = 0$, and the proof is complete. □

Now we are ready to prove the main theorem.

Proof of Theorem 2.5. We will show that for sufficiently small h the set $\Xi_h(\mathcal{K}_1)$ is nonvoid, open and closed with respect to the interval $[0, 1]$. Hence, it coincides with $[0, 1]$.

Obviously, $0 \in \Xi_h(\mathcal{K}_1)$ by virtue of the last assertion of Lemma 2.10.

We claim that $\Xi_h(\mathcal{K}_1)$ is open in $[0, 1]$. If $\tau \in \Xi_h(\mathcal{K}_1)$, we have $\mathcal{H}\big(\tau, (\boldsymbol{u}_h^\tau, p_h^\tau)\big) = 0$ and $\mathcal{DH}\big(\tau, (\boldsymbol{u}_h^\tau, p_h^\tau)\big)$ is regular by Lemma 2.11. By the implicit function theorem [14, Theorem A] there exists a neighborhood $\mathcal{O}_1(\tau) \subset [0, 1]$ such that for all $t \in \mathcal{O}_1(\tau)$,

(1) The nonlinear equation $\mathcal{H}\big(t, (\boldsymbol{u}, p)\big) = 0$ has a solution $\big(\boldsymbol{u}_h^t, p_h^t\big)$.
(2) $(\boldsymbol{u}_h^t, p_h^t)$ is continuous in t with respect to the norm $\|\cdot\|_h$.

Since $(\boldsymbol{u}_h^\tau, p_h^\tau) \in U_{\mathcal{K}_1 h^2|\ln h|^2}$, by (2) there exists a neighborhood $\mathcal{O}_2(\tau)$ of τ such that for all $t \in \mathcal{O}_2(\tau)$, we have

$$(\boldsymbol{u}_h^t, p_h^t) \in U_{2\mathcal{K}_1 h^2|\ln h|^2}. \tag{6.9}$$

An application of Lemma 2.10 yields

$$(\boldsymbol{u}_h^t, p_h^t) \in U_{\mathcal{K}_1 h^2|\ln h|^2}.$$

Thus $t \in \Xi_h(\mathcal{K}_1)$ for all $t \in \mathcal{O}_1(\tau) \cap \mathcal{O}_2(\tau)$, and $\Xi_h(\mathcal{K}_1)$ is open.

The set $\Xi_h(\mathcal{K}_1)$ also is closed. Indeed, let $t_k \in \Xi_h(\mathcal{K}_1)$ be parameters tending to $\tau \in [0,1]$ and let $(\boldsymbol{u}_h^{t_k}, p_h^{t_k})$ be the corresponding solutions of Problem 2.9. By virtue of the compactness of $U_{\mathcal{K}_1 h^2|\ln h|^2} \cap \big(\boldsymbol{X}_h \times M_h\big)$, a subsequence converges to some $(\boldsymbol{u}_h^\tau, p_h^\tau) \in U_{\mathcal{K}_1 h^2|\ln h|^2} \cap \big(\boldsymbol{X}_h \times M_h\big)$. Since $\mathcal{H}\big(t_k, (\boldsymbol{u}_h^{t_k}, p_h^{t_k})\big) = 0$, it follows from the continuity that $\mathcal{H}\big(\tau, (\boldsymbol{u}_h^\tau, p_h^\tau)\big) = 0$. Applying Lemma 2.10 once more, we see that the solutions $(\boldsymbol{u}_h^\tau, p_h^\tau)$ satisfy the strict inequality in (2.21). This proves $\tau \in \Xi_h(\mathcal{K}_1)$, and $\Xi_h(\mathcal{K}_1)$ is closed.

Consequently, we have $\Xi_h(\mathcal{K}_1) = [0,1]$ which proves Theorem 2.5. □

Remark 6.1. *Until now we have disregarded the orientation preserving condition. This is justified by the fact that the finite element solution $\boldsymbol{u}_h$ is orientation preserving, i.e.,*

$$\det(Id + \boldsymbol{\nabla}\boldsymbol{u}_h) > 0 \quad \textit{in } \Omega$$

for sufficiently small h, provided that the solution of the continuous problem $\boldsymbol{u}$ has this property.

Indeed, since $\boldsymbol{u} \in \boldsymbol{C}^1(\Omega)$, the determinant is continuous and bounded from below. It follows that

$$\begin{aligned}\det(Id + \boldsymbol{\nabla}\boldsymbol{u}_h) &= \det(Id + \boldsymbol{\nabla}\boldsymbol{u}) + \mathrm{Cof}(Id + \boldsymbol{\nabla}\boldsymbol{u}) : \boldsymbol{\nabla}(\boldsymbol{u}_h - \boldsymbol{u}) + \det \boldsymbol{\nabla}(\boldsymbol{u}_h - \boldsymbol{u}) \\ &\geq \det(Id + \boldsymbol{\nabla}\boldsymbol{u}) - Ch|\ln h|^{1/2} - Ch^2|\ln h| \geq \frac{1}{2}\det(Id + \boldsymbol{\nabla}\boldsymbol{u}) > 0\end{aligned}$$

if h is small, and the deformation is orientation preserving. □

Appendix A. Discretization Error of Green's Functions

The aim of this appendix is the proof of Theorem 5.5, i.e., we are concerned with the estimates of $\boldsymbol{G}_i - \boldsymbol{G}_i^h$ and $\zeta_i - \zeta_i^h$ in the weighted norms. The Green's functions are defined by linear problems and, in essence, we have to perform those steps which make the crucial part of L^∞ *error estimates for the linear variational problem 2.3.*

As already mentioned, Problem 2.3 has some special features. It is a saddle-point problem with a penalty term. Fortunately that term belongs to a bounded operator; cf. [5]. Moreover there is only a Gårding inequality for $a_u(.,.)$ and no coercivity. Based on the observation in [34] we will restrict ourselves to small h.

The formulation of $\boldsymbol{G}_i^h$ and ζ_i^h can be embedded into the Problem 2.3. Since the following considerations are independent of i, the subscript i is dropped whenever possible.

The stability of Problem 2.3 is now established via the *discrete LBB condition*, a *Gårding inequality*, and a compactness argument. To this end another norm will be used. For any $(\boldsymbol{v}, q) \in \boldsymbol{X}_h \times M_h$, we define

$$\|\boldsymbol{v}, q\|^2 := \|\boldsymbol{v}\|_1^2 + \|q\|_0^2.$$

Lemma A.1. *There exists a constant $C > 0$ such that for sufficiently small h,*

$$\sup_{\boldsymbol{v}\in\boldsymbol{X}_h} \frac{a_p(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \geq C\|q\|_0, \quad \forall q \in M_h. \tag{A.1}$$

Proof. The proof of this *discrete LBB inequality* is just a combination of the *LBB inequality* (2.14) and the interpolant $\boldsymbol{\Pi}_h$ (4.2). The latter serves as a Fortin-like operator [17], [7, p.136]. For a matrix $\boldsymbol{A}$, $\mathrm{adj}_h \boldsymbol{A}$ is defined by

$$(\mathrm{adj}_h \boldsymbol{A})_{ij} := (\boldsymbol{\mathcal{Q}}_h \,\mathrm{adj}\, \boldsymbol{A})_{ij} = \mathcal{Q}_h(\mathrm{adj}\, A)_{ij},$$

and $\mathrm{Cof}_h \boldsymbol{A}$ is defined in the same manner.

Given $q \in M_h$, by the *LBB inequality* (2.14), there exists $\boldsymbol{v} \in \boldsymbol{X}$ such that

$$a_p(\boldsymbol{v}, q) = \|q\|_0^2 \quad \text{and} \quad \|\boldsymbol{v}\|_1 \leq C_b\|q\|_0. \tag{A.2}$$

By definition of a_p we have

$$\begin{aligned} a_p(\boldsymbol{\Pi}_h\boldsymbol{v}, q) = a_p(\boldsymbol{v}, q) &+ (q, \mathrm{div}(\mathrm{adj}_h(\mathrm{Id} + \boldsymbol{\nabla u})(\boldsymbol{\Pi}_h\boldsymbol{v} - \boldsymbol{v}))) \\ &+ (q, \mathrm{div}((\mathrm{adj} - \mathrm{adj}_h)(\mathrm{Id} + \boldsymbol{\nabla u})(\boldsymbol{\Pi}_h\boldsymbol{v} - \boldsymbol{v}))). \end{aligned} \tag{A.3}$$

Here, we understand

$$(q, \mathrm{div}(\mathrm{adj}_h(\mathrm{Id} + \boldsymbol{\nabla u})\boldsymbol{\Pi}_h\boldsymbol{v})) = \sum_{K\in\mathcal{T}_{2h}} (q, \mathrm{div}(\mathrm{adj}_h(\mathrm{Id} + \boldsymbol{\nabla u})\boldsymbol{\Pi}_h\boldsymbol{v}))_K.$$

This is justified by the fact that $\mathrm{adj}_h(\mathrm{Id} + \boldsymbol{\nabla u})$ is a piecewise constant matrix, and due to $(4.1)_3$ no terms with inner boundaries arise when Green's formula is applied. From (4.3) we conclude that the second term on the right-hand side of (A.3) vanishes and using the *Piola identity* (2.6)

$$\begin{aligned} a_p(\boldsymbol{\Pi}_h\boldsymbol{v}, q) &= a_p(q, \boldsymbol{v}) + \big(q, (\mathrm{Cof} - \mathrm{Cof}_h)(\mathrm{Id} + \boldsymbol{\nabla u}) : \boldsymbol{\nabla}(\boldsymbol{\Pi}_h\boldsymbol{v} - \boldsymbol{v})\big) \\ &\geq \|q\|_0^2 - C_1h\|q\|_0\|\boldsymbol{\nabla}(\boldsymbol{v} - \boldsymbol{\Pi}_h\boldsymbol{v})\|_0\,. \end{aligned}$$

From (4.7) and (A.2) it follows that

$$\|\boldsymbol{\nabla}(\boldsymbol{v} - \boldsymbol{\Pi}_h\boldsymbol{v})\|_0 \leq \|\boldsymbol{\nabla v}\|_0 + \|\boldsymbol{\nabla\Pi}_h\boldsymbol{v}\|_0 \leq C\|\boldsymbol{\nabla v}\|_0 \leq C\|q\|_0.$$

A combination of the above two inequalities gives (A.1) for sufficiently small h. $\square$

Lemma A.2. *There exist constants C_1 and C_2 such that for sufficiently small h,*

$$\sup_{(\boldsymbol{w},r)\in \boldsymbol{X}_h\times M_h} \frac{\mathcal{A}(\boldsymbol{w},r;\boldsymbol{v},q)}{\|\boldsymbol{w},r\|} \geq C_1\|\boldsymbol{v},q\| - C_2\|\boldsymbol{v}\|_0, \qquad \forall(\boldsymbol{v},q)\in \boldsymbol{X}_h\times M_h. \tag{A.4}$$

Proof. In view of the *discrete LBB inequality* (A.1), there exists $C_b > 0$ such that

$$a_p(\boldsymbol{v}_1, q) = \|q\|_0^2 \quad \text{and} \quad \|\boldsymbol{v}_1\|_1 \leq C_b\|q\|_1.$$

Set $C_s := 1 + \|p\|_{C^0(\overline{\Omega})}$, and let C_p be the constant in the *Poincaré's inequality*, i.e., $\|\boldsymbol{v}\|_1 \leq C_p\|\boldsymbol{\nabla v}\|_0$. From Theorem 2.4 it follows that for any $\alpha > 0$

$$\begin{aligned}
\mathcal{A}(\boldsymbol{v}+\alpha\boldsymbol{v}_1, -q;\boldsymbol{v},q) &= \mathcal{A}(\boldsymbol{v},-q;\boldsymbol{v},q) + \alpha\mathcal{A}(\boldsymbol{v}_1,0;\boldsymbol{v},q)\\
&= a_u(\boldsymbol{v},\boldsymbol{v}) + \lambda^{-1}\|q\|_0^2 + \alpha a_{\boldsymbol{u}}(\boldsymbol{v}_1,\boldsymbol{v}) + \alpha a_p(\boldsymbol{v}_1,q)\\
&\geq 1/2\|\boldsymbol{\nabla v}\|_0^2 - 1/2\|p\|^2_{C^1(\overline{\Omega})}\|\boldsymbol{v}\|_0^2 + \lambda^{-1}\|q\|_0^2\\
&\quad + \alpha/2\|q\|_0^2 - \alpha C_b^2 C_s^2/2\|\boldsymbol{\nabla v}\|_0^2\\
&\geq \frac{1-\alpha C_b C_s^2}{2C_p}\|\boldsymbol{v}\|_1^2 + \frac{\alpha}{2}\|q\|_0^2 + \lambda^{-1}\|q\|_0^2 - \frac{1}{2}\|p\|^2_{C^1(\overline{\Omega})}\|\boldsymbol{v}\|_0^2.
\end{aligned}$$

Next we choose $\alpha := C_p/(1 + C_p C_b^2 C_s^2)$. Since $\frac{1}{2}(1-\alpha C_b C_s^2)C_p = \alpha/2$, it follows that

$$\begin{aligned}
\mathcal{A}(\boldsymbol{v}+\alpha\boldsymbol{v}_1,-q;\boldsymbol{v},q) &\geq \alpha/2\|\boldsymbol{v},q\|^2 - 1/2\|p\|^2_{C^1(\overline{\Omega})}\|\boldsymbol{v}\|_0^2\\
&\geq (\alpha/2\|\boldsymbol{v},q\| - 1/2\|p\|^2_{C^1(\overline{\Omega})}\|\boldsymbol{v}\|_0)\|\boldsymbol{v},q\|.
\end{aligned}$$

By construction,

$$\|\boldsymbol{v}+\alpha\boldsymbol{v}_1,-q\| \leq \max(\sqrt{2}, 1+\sqrt{2}\alpha C_b)\|\boldsymbol{v},q\| =: C_3\|\boldsymbol{v},q\|.$$

A combination of the last two inequalities yields (A.4) with $C_1 := \alpha/2C_3$ and $C_2 := \|p\|^2_{C^1(\overline{\Omega})}/2C_3$. □

Theorem A.3. *If h is sufficiently small, then there exists a constant C such that*

$$\sup_{(\boldsymbol{w},r)\in \boldsymbol{X}_h\times M_h} \frac{\mathcal{A}(\boldsymbol{w},r;\boldsymbol{v},q)}{\|\boldsymbol{w},r\|} \geq C\|\boldsymbol{v},q\| \qquad \forall(\boldsymbol{v},q)\in \boldsymbol{X}_h\times M_h. \tag{A.5}$$

Proof. We use a contradiction argument. Suppose that (A.5) does not hold. Then there exists a sequence $\{\boldsymbol{v}_n, q_n\} \in \boldsymbol{X}_{h_n}\times M_{h_n}$ such that $h_n < 1/n$, $\|\boldsymbol{v}_n, q_n\| = 1$, and

$$|\mathcal{A}(\boldsymbol{w},r;\boldsymbol{v}_n,q_n)| \leq \frac{1}{n}\|\boldsymbol{w},r\|, \quad \text{for all } (\boldsymbol{w},r)\in \boldsymbol{X}_{h_n}\times M_{h_n}. \tag{A.6}$$

So there exists a subsequence of $\{\boldsymbol{v}_n, q_n\}_{n=1}^{\infty}$, that will be denoted by $\{\boldsymbol{v}_n, q_n\}$ again, and $(\overline{\boldsymbol{v}}, \overline{q}) \in \boldsymbol{X}\times M$ such that

$$\begin{aligned}
&\boldsymbol{v}_n \rightharpoonup \overline{\boldsymbol{v}} \quad \text{weakly in } \boldsymbol{H}_0^1(\Omega),\ \boldsymbol{v}_n \to \overline{\boldsymbol{v}} \text{ strongly in } \boldsymbol{L}^2(\Omega),\\
&q_n \rightharpoonup \overline{q} \quad \text{weakly in } L^2(\Omega).
\end{aligned}$$

From the discrete Gårding inequality (A.4) and the assumption (A.6), we obtain for large n the inequality $C_1\|\boldsymbol{v}_n, q_n\| - C_2\|\boldsymbol{v}_n\|_0 \leq \frac{1}{2}C_1\|\boldsymbol{v}_n, q_n\|$ which implies

$$\|\boldsymbol{v}_n\|_0 \geq C_1/2C_2 > 0.$$

Hence, $\|\overline{\boldsymbol{v}}\|_0 \geq C_1/2C_2$ and $(\overline{\boldsymbol{v}}, \overline{q}) \neq 0$. By Assumption **R**, there exists $(\boldsymbol{w}, r) \in \boldsymbol{X} \times M$ with $\|\boldsymbol{w}, r\| = 1$ such that

$$\mathcal{A}(\boldsymbol{w}, r; \overline{\boldsymbol{v}}, \overline{q}) \geq C_4 > 0. \tag{A.7}$$

The weak convergence of $\boldsymbol{v}_n$ and q_n together with (A.7) implies that for sufficiently large n, there holds

$$\mathcal{A}(\boldsymbol{w}, r; \boldsymbol{v}_n, q_n) \geq C_4/2. \tag{A.8}$$

The density of the finite element spaces guarantees that there exists some $(\boldsymbol{w}_n, r_n) \in \boldsymbol{X}_{h_n} \times M_{h_n}$ such that $\|\boldsymbol{w} - \boldsymbol{w}_n, r - r_n\| \leq C_4/4\|\mathcal{L}\|$, where $\|\mathcal{L}\|$ denotes the usual operator norm of $\mathcal{L}$. These estimates imply that for sufficiently large n, there holds

$$\begin{aligned}\mathcal{A}(\boldsymbol{w}_n, r_n; \boldsymbol{v}_n, q_n) &= \mathcal{A}(\boldsymbol{w}, r; \boldsymbol{v}_n, q_n) - \mathcal{A}(\boldsymbol{w} - \boldsymbol{w}_n, r - r_n; \boldsymbol{v}_n, q_n)\\ &\geq C_4/2 - \|\boldsymbol{w} - \boldsymbol{w}_n, r - r_n\|\,\|\mathcal{L}\|\\ &\geq C_4/4. \end{aligned}\tag{A.9}$$

This obviously contradicts (A.6), and the proof is complete. □

Now we turn to the Green's functions. By construction, $\|\delta_i\|_0 \leq Ch^{-1}$, $\|\boldsymbol{\nabla}\delta_i\|_0 \leq Ch^{-2}$, and as a simple corollary of Theorem 3.1, we have

Lemma A.4. *There exists a constant C which may depend on $\|\boldsymbol{u}\|_{\boldsymbol{\mathcal{C}}^2(\overline{\Omega})}$ and $\|p\|_{\mathcal{C}^1(\overline{\Omega})}$ such that*

$$\|\mathcal{D}^2\boldsymbol{G}_i\|_0 + \|\boldsymbol{\nabla}\zeta_i\|_0 \leq \begin{cases} Ch^{-1}, & \textit{for } \ i = 1,\\ Ch^{-2}, & \textit{for } \ i = 2, 3.\end{cases} \tag{A.10}$$

Having Theorem A.3, we obtain the following error bounds by a standard scaling argument (see [7, Lemma III.3.7]) and the duality technique of Aubin–Nitsche.

Lemma A.5. *If h is sufficiently small, then we have*

$$\|\boldsymbol{\nabla}(\boldsymbol{G} - \boldsymbol{G}^h)\|_0 + \|\zeta - \zeta^h\|_0 \leq Ch(\|\mathcal{D}^2\boldsymbol{G}\|_0 + \|\boldsymbol{\nabla}\zeta\|_0), \tag{A.11}$$

$$\|\boldsymbol{G} - \boldsymbol{G}^h\|_0 + \|\zeta - \zeta^h\|_{-1} \leq Ch^2(\|\mathcal{D}^2\boldsymbol{G}\|_0 + \|\boldsymbol{\nabla}\zeta\|_0), \tag{A.12}$$

where C may depend on $\|\boldsymbol{u}\|_{\boldsymbol{\mathcal{C}}^2(\overline{\Omega})}$ and $\|p\|_{\mathcal{C}^1(\overline{\Omega})}$.

Before estimating $\|\boldsymbol{\nabla}(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2}$ and $\|\zeta - \zeta^h\|_{\sigma^2}$ we cite an approximation result which will be frequently used in the subsequent analysis.

Lemma A.6. *Let K be as in* (3.6), *and let $\boldsymbol{\psi} := (\boldsymbol{G} - \boldsymbol{G}^h)\sigma^2$. If h is sufficiently small, then there exist constants $C_K > 0$ and $C > 0$ such that*

$$\begin{aligned}\|\boldsymbol{\nabla}(\boldsymbol{\psi} - \boldsymbol{\Pi}_h\boldsymbol{\psi})\|_{\sigma^{-2}} &\le C_K h^2(\|\mathcal{D}^2\boldsymbol{G}\|_0 + \|\boldsymbol{\nabla}\zeta\|_0) \\ &\quad + C_K h(\|\mathcal{D}^2\boldsymbol{G}\|_{\sigma^2} + \|\boldsymbol{\nabla}\zeta\|_{\sigma^2}) + C/K\|\boldsymbol{\nabla}(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2}.\end{aligned} \tag{A.13}$$

This assertion follows from [16, Lemma 4.1] due to Theorem 4.1.

The following lemma is crucial for the estimate of $\|\boldsymbol{\nabla}(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2}$. It is similar to [16, Lemma 4.2], the proof is, however, quite different.

Lemma A.7. *Let $\xi := (\mathcal{Q}_h\zeta - \zeta^h)\sigma^2$ and $\psi := (\boldsymbol{G} - \boldsymbol{G}^h)\sigma^2$, then for any $\epsilon > 0$, there exists a constant $C(\epsilon)$ which is independent of h such that*

$$\begin{aligned}&|a_p(\boldsymbol{\psi}, \zeta - \zeta^h) - \lambda^{-1}\big(\mathcal{Q}_h\zeta - \zeta^h, \xi\big)| \\ &\quad \le \epsilon\|\boldsymbol{\nabla}(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2}^2 + C(\epsilon)h^4\big(\|\mathcal{D}^2\boldsymbol{G}\|_0 + \|\boldsymbol{\nabla}\zeta\|_0\big)^2 + C(\epsilon)h^2\|\boldsymbol{\nabla}\zeta\|_{\sigma^2}^2.\end{aligned} \tag{A.14}$$

Proof. Integration by parts yields

$$\begin{aligned}a_p(\boldsymbol{\psi}, \zeta - \zeta^h) &= (\zeta - \zeta^h, \mathrm{Cof}(\mathrm{Id} + \boldsymbol{\nabla}\boldsymbol{u}) : (\boldsymbol{G} - \boldsymbol{G}^h) \otimes \boldsymbol{\nabla}\sigma^2) \\ &\quad + a_p(\boldsymbol{G} - \boldsymbol{G}^h, (\zeta - \zeta^h)\sigma^2) \\ &= (\zeta - \zeta^h, \mathrm{Cof}(\mathrm{Id} + \boldsymbol{\nabla}\boldsymbol{u}) : (\boldsymbol{G} - \boldsymbol{G}^h) \otimes \boldsymbol{\nabla}\sigma^2) \\ &\quad + a_p(\boldsymbol{G} - \boldsymbol{G}^h, (\zeta - \mathcal{Q}_h\zeta)\sigma^2) + a_p(\boldsymbol{G} - \boldsymbol{G}^h, \xi).\end{aligned}$$

The last term on the right-hand side is rewritten by using the weak equations of $\boldsymbol{G}$ and $\boldsymbol{G}^h$

$$\begin{aligned}a_p(\boldsymbol{G} - \boldsymbol{G}^h, \xi) &= a_p(\boldsymbol{G} - \boldsymbol{G}^h, \xi - \mathcal{Q}_h\xi) + a_p(\boldsymbol{G} - \boldsymbol{G}^h, \mathcal{Q}_h\xi) \\ &= a_p(\boldsymbol{G} - \boldsymbol{G}^h, \xi - \mathcal{Q}_h\xi) + \lambda^{-1}(\mathcal{Q}_h\xi, \zeta - \zeta^h) \\ &= a_p(\boldsymbol{G} - \boldsymbol{G}^h, \xi - \mathcal{Q}_h\xi) + \lambda^{-1}(\mathcal{Q}_h\xi, \mathcal{Q}_h\zeta - \zeta^h).\end{aligned}$$

The last two equations yield

$$\begin{aligned}&a_p(\boldsymbol{\psi}, \zeta - \zeta^h) - \lambda^{-1}\big(\mathcal{Q}_h\zeta - \zeta^h, \xi\big) \\ &= a_p(\boldsymbol{G} - \boldsymbol{G}^h, (\zeta - \mathcal{Q}_h\zeta)\sigma^2) + a_p(\boldsymbol{G} - \boldsymbol{G}^h, \xi - \mathcal{Q}_h\xi) \\ &\quad + (\zeta - \zeta^h, \mathrm{Cof}(\mathrm{Id} + \boldsymbol{\nabla}\boldsymbol{u}) : (\boldsymbol{G} - \boldsymbol{G}^h) \otimes \boldsymbol{\nabla}\sigma^2).\end{aligned} \tag{A.15}$$

The first two terms on the right-hand side of (A.15) are bounded by

$$C\|\boldsymbol{\nabla}(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2}(\|\zeta - \mathcal{Q}_h\zeta\|_{\sigma^2} + \|\xi - \mathcal{Q}_h\xi\|_{\sigma^{-2}}). \tag{A.16}$$

The superapproximation property (4.15) together with (A.11) yields

$$\|\xi - \mathcal{Q}_h\xi\|_{\sigma^{-2}} \le Ch\|\mathcal{Q}_h\zeta - \zeta^h\|_0 \le Ch^2(\|\mathcal{D}^2\boldsymbol{G}\|_0 + \|\boldsymbol{\nabla}\zeta\|_0).$$

Inserting this into (A.16) and recalling the approximation property (4.14), we obtain a bound of (A.16) that is rewritten by using Young's inequality,

$$\begin{aligned}
&C\|\boldsymbol{\nabla}(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2}\big(h\|\boldsymbol{\nabla}\zeta\|_{\sigma^2}+h^2(\|\mathcal{D}^2\boldsymbol{G}\|_0+\|\boldsymbol{\nabla}\zeta\|_0)\big)\\
&\le \epsilon\|\boldsymbol{\nabla}(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2}^2+\frac{C}{4\epsilon}\Big(h^2\|\boldsymbol{\nabla}\zeta\|_{\sigma^2}^2+h^4\|\mathcal{D}^2\boldsymbol{G}\|_0+h^4\|\boldsymbol{\nabla}\zeta\|_0)^2\Big).
\end{aligned}\tag{A.17}$$

Here ϵ can be any positive number. Notice that $\mathrm{Cof}(\mathrm{Id}+\boldsymbol{\nabla u}):(\boldsymbol{G}-\boldsymbol{G}^h)\otimes\boldsymbol{\nabla}\sigma^2\in\boldsymbol{X}$. Using Young's inequality once again we estimate the fourth term of (A.15),

$$\begin{aligned}
&\|\zeta-\zeta^h\|_{-1}\|\boldsymbol{\nabla}\big(\mathrm{Cof}(\mathrm{Id}+\boldsymbol{\nabla u}):(\boldsymbol{G}-\boldsymbol{G}^h)\otimes\boldsymbol{\nabla}\sigma^2\big)\|_0\\
&\le C\|\zeta-\zeta^h\|_{-1}(\|\boldsymbol{G}-\boldsymbol{G}^h\|_0+\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\sigma^2}+\|\boldsymbol{\nabla}(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2})\\
&\le \epsilon\|\boldsymbol{\nabla}(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2}^2+\frac{C}{4\epsilon}h^4(\|\mathcal{D}^2\boldsymbol{G}\|_0+\|\boldsymbol{\nabla}\zeta\|_0)^2.
\end{aligned}$$

This and (A.17) yield the required estimate (A.14). □

We start the discussion of weighted norms with a direct corollary of [15, Lemma 3.3].

Lemma A.8. *Let $p\in\mathcal{C}^1(\overline{\Omega})$, then for any $\rho\in W^{1,\infty}(\Omega)$ with $\rho\ge 0$, we have*

$$a_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{v}\rho^2)+C\int_\Omega \boldsymbol{v}^2(\rho^2+|\boldsymbol{\nabla}\rho|^2)\,d\boldsymbol{x}\ge C\int_\Omega|\boldsymbol{\nabla v}|^2\rho^2\,d\boldsymbol{x}.\tag{A.18}$$

Lemma A.9. *If h is sufficiently small, then*

$$\begin{aligned}
&\|\boldsymbol{\nabla}(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2}+\lambda^{-1/2}\|\zeta-\zeta^h\|_{\sigma^2}\le Ch^2(\|\mathcal{D}^2\boldsymbol{G}\|_0+\|\boldsymbol{\nabla}\zeta\|_0)\\
&\qquad+Ch(\|\mathcal{D}^2\boldsymbol{G}\|_{\sigma^2}+\|\boldsymbol{\nabla}\zeta\|_{\sigma^2})+C(h\|\zeta-\zeta^h\|_{\sigma^2}+\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\sigma^2}).
\end{aligned}\tag{A.19}$$

Proof. Setting again $\boldsymbol{\psi}:=(\boldsymbol{G}-\boldsymbol{G}^h)\sigma^2$ and $\xi:=(\mathcal{Q}_h\zeta-\zeta^h)\sigma^2$, we have

$$\begin{aligned}
&a_{\boldsymbol{u}}\big(\boldsymbol{G}-\boldsymbol{G}^h,(\boldsymbol{G}-\boldsymbol{G}^h)\sigma^2\big)+\lambda^{-1}(\mathcal{Q}_h\zeta-\zeta^h,(\mathcal{Q}_h\zeta-\zeta^h)\sigma^2)\\
&=a_{\boldsymbol{u}}(\boldsymbol{G}-\boldsymbol{G}^h,\boldsymbol{\psi}-\boldsymbol{\Pi}_h\boldsymbol{\psi})-a_p(\boldsymbol{\Pi}_h\boldsymbol{\psi},\zeta-\zeta^h)+\lambda^{-1}(\mathcal{Q}_h\zeta-\zeta^h,\xi),\\
&=a_{\boldsymbol{u}}(\boldsymbol{G}-\boldsymbol{G}^h,\boldsymbol{\psi}-\boldsymbol{\Pi}_h\boldsymbol{\psi})+a_p(\boldsymbol{\psi}-\boldsymbol{\Pi}_h\boldsymbol{\psi},\zeta-\zeta^h)\\
&\quad+\lambda^{-1}(\mathcal{Q}_h\zeta-\zeta^h,\xi)-a_p(\boldsymbol{\psi},\zeta-\zeta^h).
\end{aligned}\tag{A.20}$$

By recalling (4.3) we decompose the second term on the right-hand side of the above identity

$$\begin{aligned}
|a_p(\boldsymbol{\psi}-\boldsymbol{\Pi}_h\boldsymbol{\psi},\zeta-\zeta^h)|&=\big|\big(\zeta-\zeta^h,\mathrm{div}((\mathrm{adj}-\mathrm{adj}_h)(\mathrm{Id}+\boldsymbol{\nabla u})(\boldsymbol{\psi}-\boldsymbol{\Pi}_h\boldsymbol{\psi}))\big)\\
&\quad+\big(\zeta-\mathcal{Q}_h\zeta,\mathrm{div}(\mathrm{adj}_h(\mathrm{Id}+\boldsymbol{\nabla u})(\boldsymbol{\psi}-\boldsymbol{\Pi}_h\boldsymbol{\psi}))\big)\big|\\
&\le C(h\|\zeta-\zeta^h\|_{\sigma^2}+\|\zeta-\mathcal{Q}_h\zeta\|_{\sigma^2})\|\boldsymbol{\nabla}(\boldsymbol{\psi}-\boldsymbol{\Pi}_h\boldsymbol{\psi})\|_{\sigma^{-2}}.
\end{aligned}\tag{A.21}$$

From Lemma A.7 and (A.21) it follows that for any $\epsilon > 0$,

$$\begin{aligned} &a_{\boldsymbol{u}}\big(\boldsymbol{G}-\boldsymbol{G}^h,(\boldsymbol{G}-\boldsymbol{G}^h)\sigma^2\big)+\lambda^{-1}\|\xi\|^2_{\sigma^2} \\ &\leq C(\|\boldsymbol{\nabla}(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2}+h\|\zeta-\zeta^h\|_{\sigma^2}+h\|\boldsymbol{\nabla}\zeta\|_{\sigma^2})\|\boldsymbol{\nabla}(\boldsymbol{\psi}-\boldsymbol{\Pi}_h\boldsymbol{\psi})\|_{\sigma^{-2}} \\ &+\epsilon\|\boldsymbol{\nabla}(\boldsymbol{G}-\boldsymbol{G}^h)\|^2_{\sigma^2}+C(\epsilon)h^4(\|\mathcal{D}^2\boldsymbol{G}\|_0+\|\boldsymbol{\nabla}\zeta\|_0)^2+C(\epsilon)h^2\|\boldsymbol{\nabla}\zeta\|_{\sigma^2}. \end{aligned} \tag{A.22}$$

The weighted Gårding inequality (A.18) yields

$$\begin{aligned} C\|\boldsymbol{\nabla}(\boldsymbol{G}-\boldsymbol{G}^h)\|^2_{\sigma^2} &\leq a_{\boldsymbol{u}}(\boldsymbol{G}-\boldsymbol{G}^h,(\boldsymbol{G}-\boldsymbol{G}^h)\sigma^2) \\ &+C(\|\boldsymbol{G}-\boldsymbol{G}^h\|^2_{\sigma^2}+\|\boldsymbol{G}-\boldsymbol{G}^h\|^2_0). \end{aligned} \tag{A.23}$$

We substitute (A.23) into (A.22). Now Lemma A.6 as well as Lemma A.5 result in

$$\begin{aligned} &\|\boldsymbol{\nabla}(\boldsymbol{G}-\boldsymbol{G}^h)\|^2_{\sigma^2}+\lambda^{-1}\|\zeta-\zeta^h\|^2_{\sigma^2}\leq(\frac{C}{K}+\epsilon)\|\boldsymbol{\nabla}(\boldsymbol{G}-\boldsymbol{G}^h)\|^2_{\sigma^2} \\ &+C(\epsilon)h^4(\|\mathcal{D}^2\boldsymbol{G}\|_0+\|\boldsymbol{\nabla}\zeta\|_0)^2+(C_K+C(\epsilon))h^2(\|\mathcal{D}^2\boldsymbol{G}\|_{\sigma^2}+\|\boldsymbol{\nabla}\zeta\|_{\sigma^2})^2 \\ &+Ch^2\|\zeta-\zeta^h\|^2_{\sigma^2}+C\|\boldsymbol{G}-\boldsymbol{G}^h\|^2_{\sigma^2}. \end{aligned} \tag{A.24}$$

Let K be big enough, e.g. $K := 4C$, and choose $\epsilon := 1/4$. Now the first term on the right-hand side of (A.24) can be absorbed by spending a factor of 2. This completes the proof of (A.19). ☐

Next, in order to estimate the last two terms on the right-hand side of (A.19), we apply a duality argument. First, for any $\boldsymbol{w},\boldsymbol{v}\in\boldsymbol{X}$ and $r\in M$, we define

$$\begin{aligned} [\boldsymbol{w},\boldsymbol{v}]_{\boldsymbol{u}} &:= a_{\boldsymbol{u}}(\boldsymbol{w},\sigma^2\boldsymbol{v})-a_{\boldsymbol{u}}(\sigma^2\boldsymbol{w},\boldsymbol{v}), \\ [r,\boldsymbol{v}]_p &:= a_p(r,\sigma^2\boldsymbol{v})-a_p(\sigma^2 r,\boldsymbol{v}). \end{aligned}$$

A direct manipulation yields the following two identities:

$$\begin{aligned} [\boldsymbol{w},\boldsymbol{v}]_{\boldsymbol{u}} &:= (\boldsymbol{\nabla}\boldsymbol{w},\boldsymbol{v}\otimes\boldsymbol{\nabla}\sigma^2)-(\boldsymbol{w}\otimes\boldsymbol{\nabla}\sigma^2,\boldsymbol{v}) \\ &-(\boldsymbol{\nabla}p,(\boldsymbol{\nabla}\sigma^2)^{\perp}\otimes\boldsymbol{w}^{\perp}\boldsymbol{v}), \\ [r,\boldsymbol{v}]_p &:= -(r,\mathrm{Cof}(\mathrm{Id}+\boldsymbol{\nabla}\boldsymbol{u}):\boldsymbol{\nabla}\sigma^2\otimes\boldsymbol{v}). \end{aligned} \tag{A.25}$$

Lemma A.10.

$$\begin{aligned} \|\zeta-\zeta^h\|_{\sigma^2} &\leq C(\|\boldsymbol{\nabla}(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2}+\lambda^{-1/2}\|\zeta-\zeta^h\|_{\sigma^2}) \\ &+Ch^2(\|\mathcal{D}^2\boldsymbol{G}\|_0+\|\boldsymbol{\nabla}\zeta\|_0)+Ch\|\boldsymbol{\nabla}\zeta\|_{\sigma^2}. \end{aligned} \tag{A.26}$$

Proof. Consider the following auxiliary problem: *Find* $(\boldsymbol{w},r)\in\boldsymbol{X}\times M$ *such that*

$$\mathcal{A}(\boldsymbol{w},r;\boldsymbol{v},q)=(\zeta-\zeta^h,q),\quad\forall(\boldsymbol{v},q)\in\boldsymbol{X}\times M. \tag{A.27}$$

Lemma 3.2 and Lemma 3.4 provide the following a priori estimates:

$$\begin{aligned} \|\boldsymbol{w}\|_0+\|r\|_{-1} &\leq C\|\zeta-\zeta^h\|_{-1}, \\ \|\boldsymbol{\nabla}\boldsymbol{w}\|_0+\|r\|_0 &\leq C\|\zeta-\zeta^h\|_0, \\ \|\boldsymbol{\nabla}\boldsymbol{w}\|_{\sigma^2}+\|r\|_{\sigma^2} &\leq C(\|\zeta-\zeta^h\|_{-1}+h\|\zeta-\zeta^h\|_0+\|\zeta-\zeta^h\|_{\sigma^2}). \end{aligned} \tag{A.28}$$

Set $q := (\zeta - \zeta^h)\sigma^2$ and $\boldsymbol{v} := (\boldsymbol{G} - \boldsymbol{G}^h)\sigma^2$ in (A.27).

$$
\begin{aligned}
\|\zeta - \zeta^h\|_{\sigma^2}^2 &= \mathcal{A}(\boldsymbol{w}, r; (\boldsymbol{G} - \boldsymbol{G}^h)\sigma^2, (\zeta - \zeta^h)\sigma^2) = \mathcal{A}(\sigma^2\boldsymbol{w}, \sigma^2 r; \boldsymbol{G} - \boldsymbol{G}^h, \zeta - \zeta^h) \\
&\quad + [\boldsymbol{w}, \boldsymbol{G} - \boldsymbol{G}^h]_{\boldsymbol{u}} + [r, \boldsymbol{G} - \boldsymbol{G}^h]_p + [\zeta - \zeta^h, \boldsymbol{w}]_p.
\end{aligned} \tag{A.29}
$$

We decompose the first term on the right-hand side of (A.29) as

$$
\begin{aligned}
&\mathcal{A}(\sigma^2\boldsymbol{w}, \sigma^2 r; \boldsymbol{G} - \boldsymbol{G}^h, \zeta - \zeta^h) \\
&\qquad = \mathcal{A}(\sigma^2\boldsymbol{w} - \boldsymbol{\Pi}_h(\sigma^2\boldsymbol{w}), \sigma^2 r - \mathcal{Q}_h(\sigma^2 r); \boldsymbol{G} - \boldsymbol{G}^h, \zeta - \zeta^h).
\end{aligned}
$$

Similar as in (A.21), the expression $a_p(\sigma^2\boldsymbol{w} - \boldsymbol{\Pi}_h(\sigma^2\boldsymbol{w}), \zeta - \zeta^h)$ can be bounded by

$$
C(h\|\zeta - \zeta^h\|_{\sigma^2} + \|\zeta - \mathcal{Q}_h\zeta\|_{\sigma^2})\|\boldsymbol{\nabla}(\sigma^2\boldsymbol{w} - \boldsymbol{\Pi}_h(\sigma^2\boldsymbol{w}))\|_{\sigma^{-2}}.
$$

From (A.25) and (A.29) it follows that

$$
\begin{aligned}
\|\zeta - \zeta^h\|_{\sigma^2}^2 \leq{}& C(\|\boldsymbol{\nabla}(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2} + \lambda^{-1/2}\|\zeta - \zeta^h\|_{\sigma^2}) \\
&\times (\|\boldsymbol{\nabla}(\sigma^2\boldsymbol{w} - \boldsymbol{\Pi}_h(\sigma^2\boldsymbol{w}))\|_{\sigma^{-2}} + \|\sigma^2 r - \mathcal{Q}_h(\sigma^2 r)\|_{\sigma^{-2}}) \\
&+ C(h\|\zeta - \zeta^h\|_{\sigma^2} + \|\zeta - \mathcal{Q}_h\zeta\|_{\sigma^2})\|\boldsymbol{\nabla}(\sigma^2\boldsymbol{w} - \boldsymbol{\Pi}_h(\sigma^2\boldsymbol{w}))\|_{\sigma^{-2}} \\
&+ C\|\boldsymbol{\nabla}(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2}\|\boldsymbol{w}\|_0 + C\|\boldsymbol{G} - \boldsymbol{G}^h\|_0(\|\boldsymbol{\nabla}\boldsymbol{w}\|_{\sigma^2} + \|r\|_{\sigma^2}).
\end{aligned} \tag{A.30}
$$

A simple calculation yields

$$
\|\boldsymbol{\nabla}(\sigma^2\boldsymbol{w} - \boldsymbol{\Pi}_h(\sigma^2\boldsymbol{w}))\|_{\sigma^{-2}} + \|\sigma^2 r - \mathcal{Q}_h(\sigma^2 r)\|_{\sigma^{-2}} \leq C(\|\boldsymbol{w}\|_0 + \|\boldsymbol{\nabla}\boldsymbol{w}\|_{\sigma^2} + \|r\|_{\sigma^2}).
$$

Inserting this into the right-hand side of (A.30) and using Lemma A.5 as well as (A.28), we obtain (A.26). □

To facilitate the subsequent analysis, we introduce another norm

$$
|||\boldsymbol{G} - \boldsymbol{G}^h, \zeta - \zeta^h||| := \|\boldsymbol{\nabla}(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2} + \|\zeta - \zeta^h\|_{\sigma^2}.
$$

Lemma A.11. *For arbitrary $\epsilon > 0$, there exists a constant $C(\epsilon)$ which is independent of h such that for sufficiently small h, there holds*

$$
\|\boldsymbol{G} - \boldsymbol{G}^h\|_{\sigma^2} \leq C(h + \epsilon)|||\boldsymbol{G} - \boldsymbol{G}^h, \zeta - \zeta^h||| + C(\epsilon)\|\boldsymbol{G} - \boldsymbol{G}^h\|_0. \tag{A.31}
$$

Proof. We consider the following auxiliary problem: *Find $(\boldsymbol{w}, r) \in \boldsymbol{X} \times M$ such that*

$$
\mathcal{A}(\boldsymbol{w}, r; \boldsymbol{v}, q) = \langle \boldsymbol{G} - \boldsymbol{G}^h, \boldsymbol{v}\rangle, \quad \forall(\boldsymbol{v}, q) \in \boldsymbol{X} \times M. \tag{A.32}
$$

From Theorem 3.1 we have the following regularity estimates

$$
\|\boldsymbol{w}\|_2 + \|r\|_1 \leq C\|\boldsymbol{G} - \boldsymbol{G}^h\|_0, \tag{A.33}
$$

$$
\|\mathcal{D}^2\boldsymbol{w}\|_{\sigma^2} + \|\boldsymbol{\nabla} r\|_{\sigma^2} \leq C(\|\boldsymbol{G} - \boldsymbol{G}^h\|_0 + \|\boldsymbol{G} - \boldsymbol{G}^h\|_{\sigma^2}). \tag{A.34}
$$

Inserting $(\boldsymbol{G}-\boldsymbol{G}^h)\sigma^2$ into the right-hand side of (A.32), and proceeding along the same lines as in Lemma A.10 we get

$$\begin{aligned}\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\sigma^2}^2 &= \mathcal{A}(\sigma^2\boldsymbol{w}-\boldsymbol{\Pi}_h(\sigma^2\boldsymbol{w}),\sigma^2 s-\mathcal{Q}_h(\sigma^2 s);\boldsymbol{G}-\boldsymbol{G}^h,\zeta-\zeta^h)\\ &\quad+[\boldsymbol{w},\boldsymbol{G}-\boldsymbol{G}^h]_{\boldsymbol{u}}+[s,\boldsymbol{G}-\boldsymbol{G}^h]_p+[\zeta-\zeta^h,\boldsymbol{w}]_p.\end{aligned}\tag{A.35}$$

From (3.10) and Hölder's inequality it follows that

$$\|\boldsymbol{w}\|_{\sigma^{-2}}\le\Big(\int_\Omega|\boldsymbol{w}(\boldsymbol{x})|^{2s}\,d\boldsymbol{x}\Big)^{1/2s}\Big(\int_\Omega\sigma(\boldsymbol{x})^{-2s/(s-1)}\,d\boldsymbol{x}\Big)^{(s-1)/2s}\le Cs^{1/2}\theta^{-1/s}\|\boldsymbol{w}\|_1,$$

where we have used $\|\boldsymbol{w}\|_{L^{2s}}\le Cs^{1/2}\|\boldsymbol{w}\|_1$. Setting $s=|\ln\theta|$ in the above inequality, recalling that K has been fixed in (A.24), and using *Poincaré's inequality*, we obtain

$$\|\boldsymbol{w}\|_{\sigma^{-2}}\le C|\ln h|^{1/2}\|\boldsymbol{\nabla w}\|_0.\tag{A.36}$$

Applying the weighted approximation property (4.5) with $j=2$ and $k=1$, and recalling (A.36) we have the following approximation estimate

$$\begin{aligned}\|\boldsymbol{\nabla}(\sigma^2\boldsymbol{w}-\boldsymbol{\Pi}_h(\sigma^2\boldsymbol{w}))\|_{\sigma^{-2}}&\le Ch\|\mathcal{D}^2(\sigma^2\boldsymbol{w})\|_{\sigma^{-2}}\\ &\le Ch(|\ln h|^{1/2}\|\boldsymbol{\nabla w}\|_0+\|\mathcal{D}^2\boldsymbol{w}\|_{\sigma^2}),\end{aligned}$$

and invoking the weighted approximation property (4.14), we obtain

$$\|\sigma^2 r-\mathcal{Q}_h(\sigma^2 r)\|_{\sigma^{-2}}\le Ch(\|r\|_0+\|\boldsymbol{\nabla}r\|_{\sigma^2}).$$

The last two inequalities provide bounds of the first four terms on the right-hand side of (A.35)

$$Ch(\|r\|_0+\|\boldsymbol{\nabla}r\|_{\sigma^2}+|\ln h|^{1/2}\|\boldsymbol{\nabla w}\|_0+\|\mathcal{D}^2\boldsymbol{w}\|_{\sigma^2})|||\boldsymbol{G}-\boldsymbol{G}^h,\zeta-\zeta^h|||,\tag{A.37}$$

and the remaining three terms on the right-hand side of (A.35) are bounded by

$$C\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\sigma^2}(\|\boldsymbol{\nabla w}\|_0+\|r\|_0)+C|||\boldsymbol{G}-\boldsymbol{G}^h,\zeta-\zeta^h|||\,\|\boldsymbol{w}\|_0.\tag{A.38}$$

Inserting (A.37) and (A.38) into (A.35), taking into account (A.33) and (A.34), we have

$$\begin{aligned}\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\sigma^2}^2&\le C|||\boldsymbol{G}-\boldsymbol{G}^h,\zeta-\zeta^h|||(h|\ln h|^{1/2}\|\boldsymbol{G}-\boldsymbol{G}^h\|_0+h\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\sigma^2})\\ &\quad+C\|\boldsymbol{G}-\boldsymbol{G}^h\|_0^2+C\|\boldsymbol{G}-\boldsymbol{G}^h\|_0\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\sigma^2}\\ &\quad+C\|\zeta-\zeta^h\|_{\sigma^2}\|\boldsymbol{G}-\boldsymbol{G}^h\|_0.\end{aligned}$$

Since $h|\ln h|^{1/2}|<1$, recalling the definition of $|||\cdot,\cdot|||$ we obtain

$$\begin{aligned}\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\sigma^2}^2&\le C|||\boldsymbol{G}-\boldsymbol{G}^h,\zeta-\zeta^h|||(\|\boldsymbol{G}-\boldsymbol{G}^h\|_0+h\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\sigma^2})\\ &\quad+C\|\boldsymbol{G}-\boldsymbol{G}^h\|_0^2+C\|\boldsymbol{G}-\boldsymbol{G}^h\|_0\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\sigma^2}\end{aligned}$$

Young's inequality is applied twice. In particular, the product of the first two factors on the right-hand side is dealt with the ϵ-technique to obtain (A.31). $\square$

Now we are in a position to prove Lemma 5.4 and Theorem 5.5.

Proof of Lemma 5.4. By Theorem 3.3, we only need to estimate $\|\boldsymbol{\delta}_1\|_{\sigma_1^2}$ and $\|\boldsymbol{\delta}_1\|_{-1}$ to prove (5.14) for $i = 1$. The construction of $\boldsymbol{\delta}_1$ implies that $\|\boldsymbol{\delta}_1\|_{\sigma_1^2} = \mathcal{O}(1)$. Next, set $\boldsymbol{\pi}_h \boldsymbol{v} := (I_h v_1, I_h v_2)$, where I_h is the Clément interpolation operator [13]. Let $\mathcal{B}$ be the unit ball of $\boldsymbol{H}_0^1(\Omega)$. By the Sobolev imbedding inequality in 2-D, we have

$$\|v\|_{L^s} \le C s^{1/2} \|v\|_1, \quad \forall\, v \in H^1(\Omega), \quad 2 < s < \infty. \tag{A.39}$$

The inverse inequality for finite element spaces [11, p. 140] together with (A.39) yields

$$\|v\|_{L^\infty} \le C h^{-2/s} \|v\|_{L^s} \le C h^{-2/s} s^{1/2} \|v\|_1.$$

Setting $s = |\ln h|$ in the above inequality, we get

$$\|v\|_{L^\infty} \le C |\ln h|^{1/2} \|v\|_1.$$

We conclude that

$$\begin{aligned}\|\boldsymbol{\delta}_1\|_{-1} = \sup_{\boldsymbol{v}\in\mathcal{B}} (\boldsymbol{\delta}_1, \boldsymbol{v}) &\le \sup_{\boldsymbol{v}\in\mathcal{B}} (\boldsymbol{\delta}_1, \boldsymbol{v} - \boldsymbol{\pi}_h \boldsymbol{v}) + \sup_{\boldsymbol{v}\in\mathcal{B}} (\boldsymbol{\delta}_1, \boldsymbol{\pi}_h \boldsymbol{v}) \\ &\le \sup_{\boldsymbol{v}\in\mathcal{B}} (C + \|\boldsymbol{\pi}_h \boldsymbol{v}\|_{L^\infty}) \le C + C |\ln h|^{1/2}.\end{aligned}$$

This proves (5.14). Proceeding along the same lines we get (5.14) for $i = 2, 3$.

To prove (5.15), we first consider the case $i = 2$. By Lemma 3.4, we have

$$\|\boldsymbol{\nabla} \boldsymbol{G}_2\|_{\sigma_2^2} + \|\zeta_2\|_{\sigma_2^2} \le C(\theta \|\mathcal{D}\boldsymbol{\delta}_2\|_{-1} + \sum_{j=1}^{2} \|\mu_j \mathcal{D}\boldsymbol{\delta}_2\|_{-1} + \|\mathcal{D}\boldsymbol{\delta}_2\|_{-2}).$$

An integration by parts yields

$$\|\mu_j \mathcal{D}\boldsymbol{\delta}_2\|_{-1} \le \|\boldsymbol{\delta}_2\|_{-1} + \|\mu_j \boldsymbol{\delta}_2\|_0.$$

A combination of the above two inequalities gives

$$\|\boldsymbol{\nabla} \boldsymbol{G}_2\|_{\sigma_2^2} + \|\zeta_2\|_{\sigma_2^2} \le C(\theta \|\boldsymbol{\delta}_2\|_0 + \sum_{j=1}^{2} \|\mu_j \boldsymbol{\delta}_2\|_0 + \|\boldsymbol{\delta}_2\|_{-1}) \le C |\ln h|^{1/2}.$$

In the same way, we get (5.15) for $i = 1, 3$. □

Proof of Theorem 5.5. As a first step, we insert (A.26) into (A.19), and obtain for sufficiently small h, we have

$$\begin{aligned}\|\boldsymbol{\nabla}(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2} + \lambda^{-1/2} \|\zeta - \zeta^h\|_{\sigma^2} &\le C h^2 (\|\mathcal{D}^2 \boldsymbol{G}\|_0 + \|\boldsymbol{\nabla}\zeta\|_0) \\ &\quad + C h (\|\mathcal{D}^2 \boldsymbol{G}\|_{\sigma^2} + \|\boldsymbol{\nabla}\zeta\|_{\sigma^2}) + C \|\boldsymbol{G} - \boldsymbol{G}^h\|_{\sigma^2}.\end{aligned} \tag{A.40}$$

Secondly, inserting (A.40) into (A.26), we have for sufficiently small h,

$$\begin{aligned}\|\zeta - \zeta^h\|_{\sigma^2} \le C h^2 (\|\mathcal{D}^2 \boldsymbol{G}\|_0 + \|\boldsymbol{\nabla}\zeta\|_0) &+ C h (\|\mathcal{D}^2 \boldsymbol{G}\|_{\sigma^2} + \|\boldsymbol{\nabla}\zeta\|_{\sigma^2}) \\ &+ C \|\boldsymbol{G} - \boldsymbol{G}^h\|_{\sigma^2}.\end{aligned} \tag{A.41}$$

Inserting (A.31) into (A.40) and (A.41), we get

$$\begin{aligned}|||\boldsymbol{G}-\boldsymbol{G}^h,\zeta-\zeta^h||| &\leq C(h+\epsilon)|||\boldsymbol{G}-\boldsymbol{G}^h,\zeta-\zeta^h||| \\ &\quad + C(\epsilon)h^2(\|\mathcal{D}^2\boldsymbol{G}\|_0+\|\boldsymbol{\nabla}\zeta\|_0)+Ch(\|\mathcal{D}^2\boldsymbol{G}\|_{\sigma^2}+\|\boldsymbol{\nabla}\zeta\|_{\sigma^2}).\end{aligned}$$

Finally we choose $\epsilon := 1/4C$. If also $Ch < 1/4$, the first term on the right-hand side can be absorbed by spending a factor of 2. Using Lemma A.4 as well as Lemma 5.4, we complete the proof of Theorem 5.5. □

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