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Stability Estimates of the Mortar Finite Element Method for 3-Dimensional Problems

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Abstract

This paper is concerned with the mortar finite element method for three spatial variables. The two main issues are the proof of the LBB condition based on appropriate choices of Lagrange multipliers and optimal efficiency of corresponding multigrid schemes for the whole coupled systems of equations. The implementation of the smoothing procedure also differs from that one used in the 2-dimensional case.

Key words: Mortar method, domain decomposition, saddle point problems, L_2-stability of mortar projections, multigrid algorithms, error estimates, efficiency of smoothing procedures.

AMS subject classification: 65N55, 65N30, 65F10, 46E35.

1 Introduction

The mortar method is a domain decomposition method with non-overlapping subdomains, see e.g. [1, 2, 3, 6]. The matching of discretizations on adjacent subdomains is only enforced weakly which, in particular, facilitates employing *different types* of discretizations on different subdomains. Even in the case when only finite elements are used on each subdomain, and this is the setting to be treated here, this still offers, in principle, the advantage of freely choosing highly varying mesh sizes on different subdomains. This is a very promising feature with regard to abruptly changing diffusion coefficients or local anisotropies.

We view such types of *physical properties* of the underlying problem as a major motivation for employing a mortar technique rather than parallelization. Therefore we are primarily interested in developing highly efficient *multigrid schemes* for the whole *coupled system* of equations.

In a recent paper [6] such a multigrid algorithm for mortar elements was developed and analyzed with primary focus on the case of two spatial variables. The analysis heavily depended on an L_2-error estimate which, for convenience, was derived in a framework of mesh-dependent norms, see also [12]. By now there is a fairly complete theory for 2-dimensional domains.

However, when dealing with 3-dimensional problems, some extra considerations are necessary. This refers to the analysis as well as to computational aspects.

To give some indications of emerging complications we recall from [6] that our analysis of the discrete problem referred to mortar elements with continuity at cross points. The analogous constraint in 3D would impose continuity along edges. Of course, this is a severe constraint which would eliminate some advantages of the mortar element method. Fortunately, this kind of conformity can be abandoned. To do this we have to be once again careful with regard to the distinction of the spaces $H^{1/2}(\Gamma)$ and $H_{00}^{1/2}(\Gamma)$.

The hard part of extending the results from [6] to three dimensions is the L_2-stability of a mapping from the mortar functions to the traces of the finite elements functions on the interfaces. We will establish it in Section 3 via an *inf-sup condition*. Although the condition is multiplicative in its nature, it can be made additive via Young's inequality. In this way we can abandon the assumption in [2] that the grids induce triangulations on the interfaces which are uniform. This inf-sup condition will serve in Section 4 as the main ingredient of the proof of the LBB condition for the saddle point formulation of the mortar method. In contrast to [6] we do not require that the jumps across interfaces Γ_{kl} between adjacent subdomains belong to $H_{00}^{1/2}(\Gamma_{kl})$. Section 5 is devoted to verifying the ellipticity of the bilinear form induced by the domain partition and to establishing the main result of this section in terms of an L_2-error estimate. In addition to the results from Section 3 this requires collecting first some further prerequisites primarily concerning approximation properties of the Lagrange multipliers.

The results in Section 5, in turn, imply *h-independent convergence rates* of the W-cycle of a multigrid algorithm for the coupled system of equations in the same way as in [6]. However, the design of multigrid algorithms for mortar elements in higher spatial dimension is more involved. Each application of a smoother suggested by the treatment in [6] requires the solution of some auxilliary problems. While in 2D the natural choices lead to low-dimensional systems of equations that can be solved by a direct method, this is no longer true in the 3D case. To retain the efficiency of the multigrid method we have to split the auxilliary problems or solve them only approximately. We will therefore show in Section 6 that the auxiliary systems involve matrices with uniformly bounded spectral condition numbers so that conjugate gradient iterations converge very fast.

2 Problem Formulation and Notation

We follow the setting of [6] and consider boundary value problems of second order of the form

$$\begin{aligned} -\operatorname{div} a(x)\operatorname{grad} u(x) &= f(x) && \text{in } \Omega, \\ a(x)\frac{\partial u}{\partial n} &= g(x) && \text{on } \varGamma_N \subset \partial\Omega, \\ u &= 0 && \text{on } \varGamma_D := \partial\Omega \setminus \varGamma_N. \end{aligned} \tag{2.1}$$

We assume that Ω is partitioned into non-overlapping subdomains Ω_k,

$$\bar{\Omega} \;=\; \bigcup_{k=1}^{K} \bar{\Omega}_k, \quad \Omega_k \cap \Omega_l = \emptyset \;\text{ for } k \neq l. \tag{2.2}$$

This partition is supposed to be geometrically conforming, i.e., if p is a point in $\varGamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$, $(k \neq l)$, then either p is a vertex or the entire face respectively edge containing p lies in $\varGamma_{kl}$. Recall that the characterization

$$H^1_{0,D}(\Omega) = \{v \in X_\delta : \sum_{k=1}^{K} (\mathbf{q}\cdot\mathbf{n}_k, v)_{0,\partial\Omega_k} = 0 \;\; \forall \mathbf{q} \in H_{0,N}(\mathrm{div}, \Omega)\}, \tag{2.3}$$

(see Proposition III.1.1 in [8]) may serve as a starting point for domain decomposition methods and suggests the following setting. Each Ω_k is assumed to be further subdivided into tetrahedra *independently* of the neighboring subdomains. Such a partition is denoted as $\mathcal{T}_{k,h}$ and the corresponding spaces of piecewise linear finite elements on $\mathcal{T}_{k,h}$ are denoted by $S_h(\mathcal{T}_{k,h})$. We set

$$\begin{aligned} X_\delta &:= \{v \in L_2(\Omega) : v|_{\Omega_k} \in H^1(\Omega_k),\ k = 1, \ldots, K,\ v|_{\varGamma_D} = 0\}, \\ X_{00} &:= \{v \in X_\delta : [v]\,|_{\varGamma_{kl}} \in H^{1/2}_{00}(\varGamma_{kl}) \;\forall\, \varGamma_{kl} \subset \mathcal{S}\}, \end{aligned}$$

where the *skeleton* $\mathcal{S}$ is composed of the union of interfaces $\varGamma_{kl}$ and as usual $[v]\,|_{\varGamma_{kl}}$ denotes the *jump* of a function $v \in X_\delta$ accross $\varGamma_{kl}$. Here we will employ the *unconstrained* product spaces

$$X_h := \prod_{k=1}^{K} S_h(\mathcal{T}_{k,h}). \tag{2.4}$$

as trial spaces.

For defining the weak matching conditions we associate with each interface $\varGamma_{kl}$ the *non-mortar-side* which by convention is Ω_k while Ω_l is the *mortar-side*. Our central objective is to construct suitable spaces $M_{kl,h}$ of Lagrange multipliers associated with each interface $\varGamma_{kl}$. As usual the Lagrange multipliers are chosen as traces of the finite element spaces on Γ_{kl} with respect to the triangulation on the non-mortar side, i.e., $M_{kl,h} \subset S_h(\mathcal{T}_{k,h})|_{\Gamma_{kl}}$. Specifically, our choice will be such that

$$\dim M_{kl,h} = \dim(S_h(\mathcal{T}_{k,h})|_{\Gamma_{kl}} \cap H_0^1(\Gamma_{kl})). \tag{2.5}$$

This is a natural condition in the 2D case when employing the constrained spaces $X_h \cap X_{00}$ as trial spaces so that the jumps $[v]\ |_{\Gamma_{kl}}$ belong to $H_0^1(\Gamma_{kl})$. The condition (2.5) is then often achieved by the constraint that the Lagrange multipliers are *constant functions* on the intervals next to the end points of Γ_{kl}. Although we will use the *full* spaces X_h in the 3D case and will not restrict the finite elements to X_{00}, we will show that (2.5) is still a reasonable condition. However, the corresponding constructions are more involved, cf [2]. We defer the precise definition of $M_{kl,h}$ to the next section since it will be essential for establishing an inf-sup condition. The space of Lagrange multipliers then has the form

$$M_h := \prod_{\Gamma_{kl} \subset \mathcal{S}} M_{kl,h}. \tag{2.6}$$

The mortar-element solution of (1.1) is the solution of the saddle point problem: *find* $(u_h, \lambda_h) \in X_h \times M_h$ *such that*

$$\begin{array}{lcll} a(u_h, v_h) + b(v_h, \lambda_h) & = & (f, v_h)_{0,\Omega} + (g, v_h)_{0,\Gamma_N}, & v_h \in X_h, \\ b(u_h, \mu_h) & = & 0, & \mu_h \in M_h, \end{array} \tag{2.7}$$

where

$$a(u, v) := \sum_k \int_{\Omega_k} (a(x)\nabla u(x)) \cdot \nabla v(x) dx, \tag{2.8}$$

and

$$b(v, \mu) := \sum_{\Gamma_{kl} \subset \mathcal{S}} \int_{\Gamma_{kl}} \mu[v] ds. \tag{2.9}$$

Of course, a central issue is the stability of this discretization. During the past three three years at least three different approaches have been pursued which are based on three different pairs of norms. To put things into proper perspective we will briefly comment on the distinctions and merits of each method. All of them are designed to make the bilinear form (2.9) continuous and refer to different dual pairings.

1) One may choose

$$[v]|_{\Gamma_{kl}} \in H^{1/2}(\Gamma_{kl}) \ \text{ and } \ \mu \in (H^{1/2}(\Gamma_{kl}))'. \tag{2.10}$$

The right hand side of (2.9) may then be estimated by $\|[v]\|_{H^{1/2}(\Gamma_{kl})} \|\mu\|_{(H^{1/2}(\Gamma_{kl}))'}$. We recall that the Sobolev spaces H^s with s being half integer have some extraordinary behavior, and it is better not to write $H^{-1/2}$ for $(H^{1/2})'$. (Sometimes the symbol $\tilde{H}^{-1/2}$ or $H_{00}^{-1/2}$ is used). This pairing is found in [2] and has the advantage that by the trace theorem

$$\|[v]\|_{H^{1/2}(\Gamma_{kl})} \le c(\|v\|_{1,\Omega_k} + \|v\|_{1,\Omega_l}). \tag{2.11}$$

On the other hand, in this framework the norm of the Lagrange multiplier is not estimated directly, see below.

2) If continuity at cross-points is assumed, one may choose

$$[v]|_{\Gamma_{kl}} \in H_{00}^{1/2}(\Gamma_{kl}) \ \text{ and } \ \mu \in (H_{00}^{1/2}(\Gamma_{kl}))'.$$

This pairing was introduced in [1]. Here it is easier to establish the inf-sup condition since the dual space may be identified with $H^{-1/2}(\Gamma_{kl})$. On the other hand, ellipticity is now uncertain since we cannot infer an estimate for $\|[v]\|_{H_{00}^{1/2}(\Gamma_{kl})}$ which is similar to (2.11).

3) The third variant, which is our preference and will be taken up here again, involves the mesh-dependent norms, c.f. [6, 12]

$$\begin{aligned}
\|w\|_{1/2,h,\Gamma_{kl}} &:= h^{-1/2}\|w\|_{0,\Gamma_{kl}}, \qquad \|\mu\|_{-1/2,h,\Gamma_{kl}} := h^{1/2}\|\mu\|_{0,h,\Gamma_{kl}} \\
\|v_h\|_{1,h}^2 &:= \|v_h\|_{1,\delta}^2 + \sum_{\Gamma_{kl}\subset\mathcal{S}} \|[v_h]\|_{1/2,h,\Gamma_{kl}}^2 = \|v_h\|_{1,\delta}^2 + \sum_{\Gamma_{kl}\subset\mathcal{S}} h^{-1}\|[v_h]\|_{0,\Gamma_{kl}}^2, \qquad (2.12) \\
\|\mu\|_{-1/2,h}^2 &:= \sum_{\Gamma_{kl}\subset\mathcal{S}} \|\mu\|_{-1/2,h,\Gamma_{kl}}^2 = \sum_{\Gamma_{kl}\subset\mathcal{S}} h\|\mu\|_{0,\Gamma_{kl}}^2. \qquad (2.13)
\end{aligned}$$

Here we have used the standard notation $\|v\|_{0,\Omega}^2 := \int_\Omega |v(x)|^2 dx$ and similarly $\|v\|_{m,\Omega}^2 := \sum_{|\alpha|\le m} \|D^\alpha v\|_{0,\Omega}^2$, $m \in \mathbb{N}$. For convenience, we always label the finite element spaces by a (global) mesh size parameter h. On the other hand, we do admit different individual discretization parameters on the subdomains. In order to avoid a severe cluttering of indices we will suppress additional indices attached to the mesh size parameter h whenever there is no risk of confusion. Moreover we will always adopt the convention that the mesh size h in the summands corresponding to Γ_{kl} in (2.12) and (2.13) refers to the mesh size h_k induced by the *non-mortar side* Ω_k.

Note that we also have as in 1)

$$\left| \int_{\Gamma_{kl}} \mu[v]ds \right| \le \|\mu\|_{-1/2,h,\Gamma_{kl}} \|[v]\|_{1/2,h,\Gamma_{kl}}.$$

Formally, this concept differs from that one with (2.10). On the other hand, the analysis in [2] resolves the terms with the norm $\|\cdot\|_{(H^{1/2}(\Gamma_{kl}))'}$ always by a duality argument going back to the duality pairing $(\cdot,\cdot)_{L_2(\Gamma_{kl})}$. Thus the distinction between 1) and 3) refers merely to different interpretations. What is essential in both cases is that L_2-duality avoids trouble with boundary conditions imposed by $H_{00}^{1/2}(\Gamma_{kl})$, a fact that will be essential in the subsequent development as well.

Most of the analysis in [6] is independent of the spatial dimension except the definition and properties of the Lagrange multiplier spaces. Therefore our main goal is to provide a suitable version of this ingredient in the 3D case.

3 Projection of the Mortar Functions

The understanding of the discrete Lagrange multipliers in [6] rests on Lemma 3.1 there which is stated only for the case of two spatial variables. We wish to derive next an analog for the $3D$ case. As before, Ω is assumed to be a bounded domain in $\mathbb{R}^3$ which is partitioned into subdomains Ω_k. We will consider a typical interface $\Gamma = \Gamma_{kl}$ between two adjacent subdomains Ω_k and Ω_l. Without loss of generality we assume that Γ is a bounded polygonal domain in $\mathbb{R}^2$. Let $\mathcal{T}$ denote the quasiuniform triangulation of Γ that is induced by the partitioning on the non-mortar side. Consider the finite elements in (2.4). The traces of those functions on the non-mortar side span a space of functions which also consists of piecewise linear finite elements but now of two spatial variables. Specifically, let $S(\mathcal{T})$ denote the space of all piecewise linear functions with respect to $\mathcal{T}$ which are continuous on Γ and vanish on its boundary, i.e.,

$$S(\mathcal{T}) := S_h(\mathcal{T}_{k,h})|_{\Gamma_{kl}} \cap H_0^1(\Gamma_{kl}) \subset H_0^1(\Gamma) \subset H_{00}^{1/2}(\Gamma).$$

Motivated by the analysis in [6] we wish to define next a space $M(\mathcal{T}) := M_{kl,h}$ which will host the Lagrange multipliers in a typical factor of (2.6) with the following properties.

(i) $M(\mathcal{T})$ is contained in the trace of the finite element spaces on Ω.

(ii) $\dim M(\mathcal{T}) = \dim S(\mathcal{T})$.

(iii) Approximations from $M(\mathcal{T})$ to functions which are *not* constrained by boundary conditions should be at least of order one.

Since (iii) requires that $M(\mathcal{T})$ contains constants on all of Γ, we cannot simply take $M(\mathcal{T}) = S(\mathcal{T})$. Requirement (ii) can be satisfied in the univariate case by lowering the degree of the finite elements on the intervals next to the end points of the interface. This does not work in higher dimensions, cf. also [2].

We now describe a candidate for $M(\mathcal{T})$ satisfying the above requirements (i)–(iii). Following [2] we first fix some notation. Let $\mathcal{V}$, $\mathcal{V}_0$, $\partial\mathcal{V}$ denote the set of all vertices of $\mathcal{T}$, the vertices in the interior of Γ, and the vertices on the boundary of Γ, respectively. The finite element basis functions will be denoted by ϕ_a, $a \in \mathcal{V}$. Thus

$$S(\mathcal{T}) = \operatorname{span}\{\phi_a : a \in \mathcal{V}_0\}.$$

Moreover,

$$\sigma_a := \bigcup \{T \in \mathcal{T} : a \in T\} = \operatorname{supp} \phi_a$$

denotes the support of ϕ_a. Since $\mathcal{T}$ is assumed to be quasiuniform, it makes sense to speak about the mesh size h. Normalizing the basis functions such that

$$\phi_a(b) = h^{-1}\delta_{a,b}, \tag{3.1}$$

a standard scaling argument shows that

$$\| \sum_{a\in\mathcal{V}_0} c_a\phi_a \|_{0,\Gamma} \sim \|\{c_a\}\|_{\ell_2}. \tag{3.2}$$

Obviously, $\dim S(\mathcal{T}) = \#\mathcal{V}_0$. Let

$$\mathcal{N}_a := \{b \in \mathcal{V}_0 : b \in \sigma_a\}$$

the set of neighboring vertices in $\mathcal{V}_0$ of a. Accordingly

$$\mathcal{N} := \bigcup_{a\in\partial\mathcal{V}} \mathcal{N}_a$$

is the set of those interior vertices which have a neighbor on the boundary of Γ. Now set

$$M(\mathcal{T}) := \operatorname{span}\{\hat{\phi}_a : a \in \mathcal{V}_0\}, \tag{3.3}$$

where the basis functions $\hat{\phi}_a$ are defined as follows:

$$\hat{\phi}_a := \begin{cases} \phi_a, & a \in \mathcal{V}_0 \setminus \mathcal{N}, \\ \phi_a + \sum\limits_{b\in\partial\mathcal{V}\cap\sigma_a} A_{b,a}\phi_b, & a \in \mathcal{N}. \end{cases} \tag{3.4}$$

Thus for each node in $\mathcal{N}$ the corresponding basis function consists of a *fixed* linear combination of the original basis function and its neighbors rooted on the boundary. Since one still has $\hat{\phi}_a(b) = 0$, $a, b \in \mathcal{V}_0$, $a \neq b$, the $\hat{\phi}_a$ are still linearly independent, and we have

$$\dim M(\mathcal{T}) = \dim S(\mathcal{T}), \tag{3.5}$$

confirming (ii) above. — Usually, we will have coefficients $A_{b,a,} \geq 0$.

Actually some care is required about the following situation. If some triangle $T \in \mathcal{T}$ has *all* its vertices on the boundary of Γ, then there exists one (corner) vertex which has *no* neighbor in $\mathcal{V}_0$. If one wants to reproduce constants, one would have to include also the original basis function of the corner node in some new basis function. To avoid technical complication we will exclude this situation for the moment and will point out how to cover this case after the following observation.

Remark 3.1 *$M(\mathcal{T})$ contains the constants if*

$$\sum_{b\in\mathcal{N}_a} A_{a,b} = 1, \quad a \in \mathcal{N}. \tag{3.6}$$

Proof: By definition (3.4), one has for the sum of the basis functions

$$\begin{aligned}
\sum_{a\in\mathcal{V}_0} \hat{\phi}_a &= \sum_{a\in\mathcal{V}_0\setminus\mathcal{N}} \phi_a + \sum_{a\in\mathcal{N}} \left(\phi_a + \sum_{b\in\partial\mathcal{V}, b\in\sigma_a} A_{b,a}\phi_b\right) \\
&= \sum_{a\in\mathcal{V}_0} \phi_a + \sum_{b\in\partial\mathcal{V}} \left(\sum_{a\in\mathcal{N}, a\in\sigma_b} A_{b,a}\right)\phi_b \\
&= \sum_{a\in\mathcal{V}_0} \phi_a + \sum_{a\in\partial\mathcal{V}} \phi_a = h^{-1},
\end{aligned}$$

where we have used (3.6) in the last step. □

Let us return briefly to the case of a corner vertex described above, i.e., $T = [c, a, a']$ is a triangle whose vertices all belong to $\partial\mathcal{V}$ and c is the corner vertex, see the right part of Figure 2. Then there must be some $b \in \mathcal{V}_0$ such that $[a, a', b]$ is a triangle in $\mathcal{T}$. Since by Remark 3.1 the functions $\hat{\phi}_e$, $e \in \mathcal{V}_0$, form a partition of unity on $\Gamma \setminus T$, one only has to replace the basis function $\hat{\phi}_b$ defined in (3.4) by $\phi_b + \sum_{e\in\partial\mathcal{V}\cap\sigma_b} A_{e,b}\phi_e + \phi_c$ to ensure that one now reproduces constants on all of Γ. Therefore, such a corner vertex obviously does not cause any essential difficulty. However, in order to avoid related technical difficulties, we assume that the triangulations near the boundaries are covered by the situations shown in Figures 1 and 2.

The spaces $M(\mathcal{T})$ obviously satisfy (i), (ii). We will see later that Remark 3.1 implies (iii). Before we turn to the following central lemma, we refer to an assumption which is weaker than shape regularity of the triangulation of Γ.

Hypothesis. There is a constant $\kappa \geq 1$ such that

$$\frac{|T_1|}{|T_2|} \leq \kappa \tag{3.7}$$

whenever T_1 and T_2 are adjacent triangles in Γ that intersect its boundary.

Here and in the sequel $|T|$ denotes the area of T.

The following lemma provides the hard part of the LBB condition.

Lemma 3.2 *The weights $A_{a,b}$ in (3.4) subject to condition (3.6) can be chosen in such a way that*

$$\inf_{v\in M(\mathcal{T})} \sup_{u\in S(\mathcal{T})} \frac{(u,v)_{0,\Gamma}}{\|u\|_{0,\Gamma}\|v\|_{0,\Gamma}} \geq \frac{1}{4(1+\kappa)}. \tag{3.8}$$

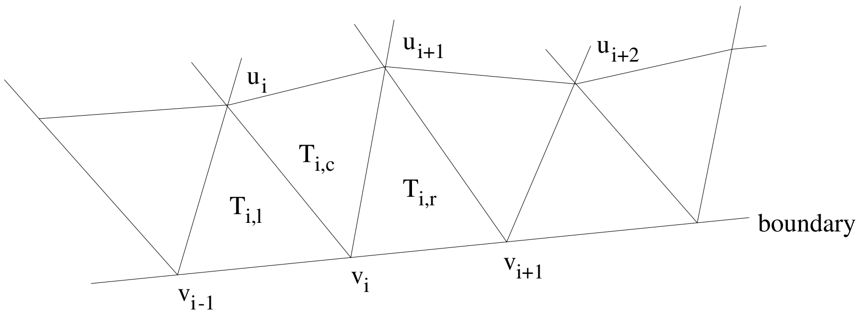

Figure 1: Significant values of u and v at the nodes of the triangles next to the boundary. (v coincides with u where u is specified, and $u = 0$ where v is specified.)

In fact, if $T_c = [a, b, c]$ is a triangle sharing only the vertex c with the boundary of Γ and if $T_{c,l}$, $T_{c,r}$ denote the left and right neigboring triangle both having an edge on $\partial\Gamma$, then the weights may be chosen such that

$$A_{c,a} + A_{c,b} = 1, \quad |T_{c,l}|A_{c,b} = |T_{c,r}|A_{c,a}, \tag{3.9}$$

and this choice yields (3.8).

Proof: We need some preliminaries. For any triangle $T = [a, b, c]$ we denote its barycentric coordinates by λ_e, $e \in \{a, b, c\}$, i.e., any $x \in T$ has the unique representation $x = \lambda_a a + \lambda_b b + \lambda_c c$, $\lambda_a + \lambda_b + \lambda_c = 1$. Elementary calculations show that (cf. (II.8.4) in [5]):

$$\int_T \lambda_e \lambda_{e'} dx = \begin{cases} \frac{|T|}{6}, & e = e' \in \{a, b, c\}, \\ \frac{|T|}{12}, & e, e' \in \{a, b, c\},\ e \neq e'. \end{cases} \tag{3.10}$$

Given a linear function f, we may represent it in the form $f = f(a)\lambda_a + f(b)\lambda_b + f(c)\lambda_c$. As a consequence, if f and g are linear functions on the triangle T and f_i and g_i denote their values at the vertices, we have

$$\int_T f^2(x)dx = \frac{|T|}{12}[f_1^2 + f_2^2 + f_3^2 + (f_1 + f_2 + f_3)^2] \tag{3.11}$$

and by polarisation

$$\int_T f(x)g(x)dx = \frac{|T|}{12}[f_1 g_1 + f_2 g_2 + f_3 g_3 + (f_1 + f_2 + f_3)(g_1 + g_2 + g_3)]. \tag{3.12}$$

Given v, we will choose $u \in M(\mathcal{T})$ as close as possible, i.e., we will fix u at the interior vertices by

$$u(a) := v(a) \quad \forall a \in \mathcal{V}_0, \tag{3.13}$$

which means

$$u = h \sum_{a \in \mathcal{V}_0} v(a)\phi_a.$$

As a consequence of (3.4), we have

$$v(b) = \sum_{a \in \sigma_b \cap \mathcal{N}} A_{b,a} u(a), \quad b \in \partial\mathcal{V}.$$

When we estimate $\int uvdx$ from below, we need only consider the integrals over triangles next to the boundary. For convenience, we adopt the notation from Fig. 1. Specifically, we have from (3.9)

$$v_i = A_i u_i + B_i u_{i+1} \tag{3.14}$$

with

$$A_i, B_i \geq 0, \quad A_i + B_i = 1, \quad A_i |T_{i,r}| = B_i |T_{i,l}|. \tag{3.15}$$

By using the quadrature rule (3.12) we obtain

$$\begin{aligned} I_{i,c} &:= \int_{T_{i,c}} uvdx = \frac{|T_{i,c}|}{12} \left[u_i^2 + u_{i+1}^2 + (u_i + u_{i+1})\ (u_i + u_{i+1} + v_i)\right] \\ &= \frac{|T_{i,c}|}{12} \left[u_i^2 + u_{i+1}^2 + (u_i + u_{i+1})^2 + D_i\right], \end{aligned} \tag{3.16}$$

where the term D_i is induced by the difference between u and v

$$\begin{aligned} D_i &:= (u_i + u_{i+1})v_i \\ &= (u_i + u_{i+1})\ (A_i u_i + B_i u_{i+1}) \\ &= \frac{1}{2}(u_i + u_{i+1})^2 + (u_i + u_{i+1})(A_i - \frac{1}{2})\ (u_i - u_{i+1}) \\ &\geq \frac{1}{2}(u_i + u_{i+1})^2 - \frac{1}{2}|u_i^2 - u_{i+1}^2| \\ &\geq -\frac{1}{4}(u_i^2 + u_{i+1}^2). \end{aligned} \tag{3.17}$$

Here we have made use of the formula $(x + y)^2 + x^2 - y^2 = (\frac{3}{2}x + \frac{2}{3}y)^2 - \frac{1}{4}x^2 - \frac{4}{9}y^2 \geq -\frac{1}{2}(x^2 + y^2)$. Combining (3.16) and (3.17) yields

$$I_{i,c} \geq \frac{|T_{i,c}|}{12} \left[\frac{3}{4}(u_i^2 + u_{i+1}^2) + (u_i + u_{i+1})^2\right] \geq \frac{3}{4} \int_{T_{i,c}} u^2 dx. \tag{3.18}$$

Next we determine the integral over the triangles which have two vertices on the boundary

$$\begin{aligned} I_{i,r} &:= \int_{T_{i,r}} uvdx = \frac{|T_{i,r}|}{12} \left[u_{i+1}^2 + u_{i+1}(u_{i+1} + v_i + v_{i+1})\right] \\ &= \frac{|T_{i,r}|}{12} \left[2u_{i+1}^2 + E_i + F_i\right] = \int_{T_{i,r}} u^2 dx + \frac{|T_{i,r}|}{12} \left[E_i + F_i\right] \end{aligned} \tag{3.19}$$

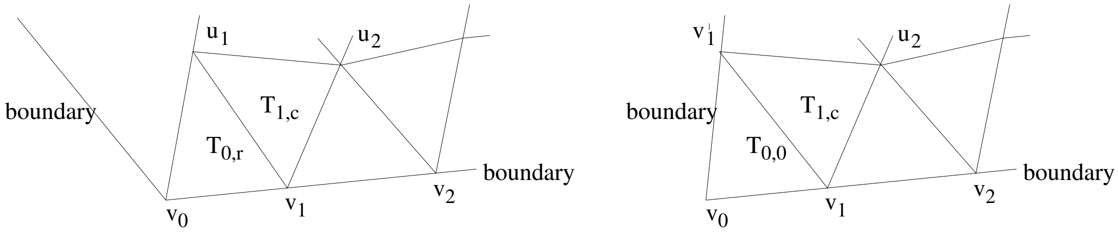

Figure 2: Two different situations of the boundary near cross points.

with $E_i := u_{i+1}v_i$, $F_i := u_{i+1}v_{i+1}$. Note that we can rearrange the extra terms from (3.19) which allows us to make use of condition (3.15)

$$\begin{aligned}
|T_{i,r}|E_i + |T_{i-1,r}|F_{i-1} &= |T_{i,r}|u_{i+1}v_i + |T_{i,l}|u_i v_i \\
&= (A_i + B_i)\left(|T_{i,r}|u_{i+1} + |T_{i,l}|u_i\right) v_i \\
&= (|T_{i,r}| + |T_{i,l}|)\,(B_i u_{i+1} + A_i u_i)\, v_i \\
&= (|T_{i,r}| + |T_{i,l}|)v_i^2 \geq 0.
\end{aligned}$$

Now we sum the terms over the elements in the strip of Γ next to the boundary. First we consider the case that the triangulations in the vicinity of the cross points have the form as shown in the left part of Figure 2:

$$\Omega_b = T_{0,r} \cup T_{1,c} \cup T_{1,r} \cup \ldots \cup T_{j,c} \cup T_{j,r}\,.$$

We obtain

$$\int\limits_{\Omega_b} uv dx \geq \frac{3}{4} \int\limits_{\Omega_b} u^2 dx. \tag{3.20}$$

Finally we estimate $\|v\|_{0,\Gamma}$. From (3.11) and Young's inequality we conclude that

$$\begin{aligned}
\int\limits_{T_{i,c}} v^2 dx &= \frac{|T_{i,c}|}{12}\,[u_i^2 + u_{i+1}^2 + v_i^2 + (u_i + u_{i+1} + v_i)^2] \\
&\leq \frac{|T_{i,c}|}{12}\,[u_i^2 + u_{i+1}^2 + 2(u_i + u_{i+1})^2 + 3v_i^2] \\
&\leq 2\int\limits_{T_{i,c}} u^2 dx + \frac{|T_{i,c}|}{12}\,3v_i^2.
\end{aligned}$$

The extra term is estimated by elementary calculations

$$v_i^2 = (A_i u_i + B_i v_i)^2 \leq \max\{u_i^2, u_{i+1}^2\} \leq \frac{12}{T_{i,c}}\frac{2}{3}\int\limits_{T_{i,c}} u^2 dx. \tag{3.21}$$

Combining the last two inequalities we obtain for $T_{i,c}$ the estimate $\int v^2 dx \le 4 \int u^2 dx$. For estimating the integrals on the other kind of triangles we need the hypothesis on the area of neighboring triangles

$$|T_{i,r}| \le \kappa |T_{i,c}|, \quad |T_{i,r}| \le \kappa |T_{i+1,c}|.$$

Recalling (3.21) we obtain

$$\begin{aligned}
\int\limits_{T_{i,r}} v^2 dx &= \frac{|T_{i,r}|}{12} [u_{i+1}^2 + v_i^2 + v_{i+1}^2 + (u_i + v_i + v_{i+1})^2] \\
&\le \frac{|T_{i,r}|}{12} 4(u_{i+1}^2 + v_i^2 + v_{i+1}^2) \\
&\le 4\frac{|T_{i,r}|}{12} u_{i+1}^2 + 4\kappa \frac{|T_{i,c}|}{12} v_i^2 + 4\kappa \frac{|T_{i+1,c}|}{12} v_{i+1}^2) \\
&\le 2 \int\limits_{T_{i,r}} u^2 dx + \frac{8}{3}\kappa \int\limits_{T_{i,c}} u^2 dx + \frac{8}{3}\kappa \int\limits_{T_{i+1,c}} u^2 dx.
\end{aligned}$$

By summing over all triangles of Ω_b we obtain

$$\int\limits_{\Omega_b} v^2 dx \le (4 + \frac{16}{3}\kappa) \int\limits_{\Omega_b} u^2 dx.$$

Hence,

$$\int\limits_{\Omega_b} uv dx \ge \frac{1}{8(1+\kappa)} \int\limits_{\Omega_b} (u^2 + v^2) dx.$$

Since even a stronger inequality holds for the triangles in the interior, we conclude with Young's inequality that

$$(u, v)_{0,\Gamma} \ge \frac{1}{8(1+\kappa)} (\|u\|_{0,\Gamma}^2 + \|v\|_{0,\Gamma}^2) \ge \frac{1}{4(1+\kappa)} \|u\|_{0,\Gamma} \|v\|_{0,\Gamma}.$$

The proof of (3.8) is now complete if we have the situation shown in the left figure of Figure 2. The other situation requires some extra consideration which only leads to a different constant. □

In view of the hypothesis (3.7) on adjacent triangles we could have weakened our assumption of quasi-uniformity. The simple scaling from (3.1) has to be replaced, for instance, by $\phi'_a(b) = \delta_{a,b}$, $\phi_a := \phi'_a / \|\phi'_a\|_{0,\Gamma}$. Obviously the relation (3.9) has then to be rescaled properly.

4 LBB-Condition

For each $\Gamma = \Gamma_{kl}$ the spaces $M_{kl,h}$ of Lagrange multipliers in (2.6) will henceforth be assumed to have the form $M(\mathcal{T})$ defined by (3.3) where $\mathcal{T}$ is the triangulation of Γ_{kl}

induced by the mesh for the non-mortar side Ω_k. We are now in a position to establish the LBB-condition for the discretization (2.7).

Theorem 4.1 *Assume that the triangulation in each subdomain Ω_k is quasi-uniform. The discretizations (2.7) based on the spaces X_h, M_h defined by (2.4) and (2.6), respectively, satisfy the LBB-condition, i.e., there exists some $\beta > 0$ such that*

$$\inf_{\mu_h \in M_h} \sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_{1,h} \|\mu_h\|_{-1/2,h}} \geq \beta, \tag{4.1}$$

holds uniformly in h.

Proof: The proof follows exactly the lines of the proof in [6]. In fact, recall that the proof there relied on three ingredients, namely (i) the inverse estimate $\|v_h\|_{H_{00}^{1/2}(\Gamma)} \leq c\,h^{-1/2}\|v_h\|_{0,\Gamma}$ for all $v_h \in S(\mathcal{T})$, (see (3.14) in [6]), (ii) the extension result from Lemma 5.1 in [4], and (iii) given $\mu \in M(\mathcal{T})$, there is a $v \in S(\mathcal{T})$ such that an estimate of the type

$$\int_{\Gamma_{kl}} \mu v ds \geq c\|\mu\|_{0,\Gamma} \|v\|_{0,\Gamma}$$

holds. (i) and (ii) persist to hold in the 3D case. The essential point is (iii) whose validity has been ensured by Lemma 3.2. Therefore the theorem follows from the same arguments as in the proof of Theorem 3.2 in [6]. □

B. Wohlmuth has pointed out to us that the discrete harmonic extension from Lemma 5.1 in [4] can actually be avoided. The reason is that the extension is encountered together with an inverse inequality. An alternative argument with an alternative extension can be sketched as follows. Given $\mu \in M_{kl,h}$ define $d_{kl} \in S(\Gamma_{kl})$ as above. Next define $v_{kl} \in S_h(\mathcal{T}_{kl})$ at each nodal point a of the triangulation $\mathcal{T}_{k,h}$ in Ω_k and then on the rest of Ω by

$$v_{kl}(a) := \begin{cases} d_{kl}(a) & \text{if } a \in \Gamma_{kl} \cap \bar{\Omega}_k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

The support of v_{kl} is a neighborhood of Γ_{kl} with a width of order h, and by construction $v_{kl}|_{\partial\Omega_k \setminus \Gamma_{kl}} = 0$. Now we obtain by a standard scaling argument

$$\|v_{kl}\|_{0,\Omega}^2 = \|v_{kl}\|_{0,\Omega_k}^2 \leq ch^3 \sum_{a \in \mathcal{T}_{k,h}} |v_{kl}(a)|^2 = ch\,h^2 \sum_{a \in \mathcal{V}_0} |d_{kl}(a)|^2 \leq ch\|d_{kl}\|_{0,\Gamma_{kl}}^2.$$

Combining this with an inverse inequality we finally have $\|v_{kl}\|_{1,\Omega_k} \leq ch^{-1}\|v_{kl}\|_{0,\Omega_k} \leq ch^{-1/2}\|d_{kl}\|_{0,\Gamma_{kl}}$, and we can again continue with the proof of Theorem 3.2 in [6] to conclude (4.1).

5 An L_2-Error Estimate

In order to confirm stability of the discretization (2.7) we yet have to verify ellipticity. Moreover, the analysis of the multigrid scheme for the mortar method hinges on an L_2-error estimate [6]. Both issues require some preliminaries which we will collect next.

The first key fact is the following approximation property of the spaces $M_{kl,h}$:

$$\inf_{w \in M_{kl,h}} \|w - v\|_{0,\Gamma_{kl}} \le c\, h^{1/2} |v|_{1/2,\Gamma_{kl}} \quad \text{for all} \;\; v \in H^{1/2}(\Gamma_{kl}). \tag{5.1}$$

In principle, this estimate is straight forward. However, the spaces $M_{kl,h}$ are not completely standard due to the reduced number of degrees of freedom near the boundary of an interface, and we briefly comment on the derivation of (5.1). We adhere to the notation in Section 3 and consider a typical case $\Gamma = \Gamma_{kl}$, $M(\mathcal{T}) = M_{kl,h}$. On account of the low regularity, we cannot employ standard interpolation procedures but have to resort to averaging processes like Clément's interpolation process [9]. Specifically, following [11], see also [10], one can construct a *dual basis* $\{\eta_a : a \in \mathcal{V}\}$ with the following properties

$$(\phi_a, \eta_b)_{0,\Gamma} = \delta_{a,b}, \quad a, b \in \mathcal{V}, \quad \|\eta_a\|_{0,\Gamma} \sim 1, \quad \operatorname{supp} \eta_a \subseteq \operatorname{supp} \phi_a, \;\; a \in \mathcal{V}, \tag{5.2}$$

where as in Section 3, the ϕ_a are the standard finite element hat functions. Since therefore $(\phi_e, \eta_a)_{0,\Gamma} = 0$ for $e \in \partial\mathcal{V}$ and $a \in \mathcal{V}_0$, it immediately follows from (3.4) and (5.2) that one also has

$$(\hat{\phi}_a, \eta_b)_{0,\Gamma} = \delta_{a,b}, \quad a, b \in \mathcal{V}_0, \tag{5.3}$$

where again the $\hat{\phi}_a$ form the basis for $M(\mathcal{T})$. Hence the mapping $P : L_2(\Gamma) \to M(\mathcal{T})$ defined by

$$Pu := \sum_{a \in \mathcal{V}_0} (u, \eta_a)_{0,\Gamma} \hat{\phi}_a \tag{5.4}$$

is a *linear projector*. Moreover, the local support property of the $\hat{\phi}_a$ and (5.2) imply that

$$\|Pu\|_{0,\Gamma} \le c \|u\|_{0,\Gamma}, \quad u \in L_2(\Gamma). \tag{5.5}$$

From Remark 3.1 we conclude that P reproduces constants, and the Bramble–Hilbert lemma with standard arguments provides

$$\|Pv - v\|_{0,\Gamma} \le c\, h |v|_{1,\Gamma}, \quad v \in H^1(\Gamma). \tag{5.6}$$

The estimate (5.1) follows now from (5.5) and (5.6) by interpolation.

The second ingredient needed below concerns approximation from the spaces X_h. Let I_h denote the Lagrange interpolation operator onto the space X_h. The same argument as used in [6, Remark 3.3] yields

$$\|v - I_h v|_{\Omega_m}\|_{0,\Gamma_{kl}} \le c h_m^{3/2} \|v\|_{2,\Omega_m} \quad \text{for } m = k \text{ or } m = l. \tag{5.7}$$

With the above prerequisites at hand we can now follow the lines of [6] and recall that a key step for proving ellipticity of the bilinear form $a(\cdot,\cdot)$ on the kernel

$$V_h := \{v_h \in X_h : b(v_h,\mu_h) = 0 \ \text{ for } \ \mu_h \in M_h\}, \tag{5.8}$$

is to establish the estimate

$$\|[v]\|_{1/2,h,\Gamma_{kl}} \le c\,(\|v\|_{1,\Omega_k} + \|v\|_{1,\Omega_l}) \quad \text{for } v \in V_h + H^1(\Omega), \tag{5.9}$$

if the triangulations in each subdomain Ω_k are shape regular. Following the proof of Lemma 3.5 in [6], let $P_{kl} : L_2(\Gamma_{kl}) \to M_{kl,h}$ be the L_2-projector. If $v_h \in V_h$, then by the matching condition, $\int\limits_{\Gamma_{kl}} \mu[v_h]ds = 0$ for $\mu \in M_{kl,h}$ and $P_{kl}([v_h]|_{\Gamma_{kl}}) = 0$. Since also $b(v,\mu_h) = 0$ for $\mu_h \in M_h$ and $v \in H^1(\Omega)$, we have $P_{kl}(v) = 0$ for $v \in V_h + H^1(\Omega)$. Combining this with the approximation property (5.1) and the trace theorem we obtain

$$\begin{aligned}\|[v]\|_{0,\Gamma_{kl}} &= \|(id - P_{kl})[v]\|_{0,\Gamma_{kl}} \le ch_k^{1/2}\|[v]\|_{1/2,\Gamma_{kl}} \\ &\le ch_k^{1/2}(\|v\|_{1,\Omega_k} + \|v\|_{1,\Omega_l}).\end{aligned}$$

After dividing by $h_k^{1/2}$ the proof of (5.9) is complete. Note that we have slightly deviated here from the argument in [6] in that *only* the mesh size corresponding to the non-mortar side enters the estimate. Therefore any preference for the mortar side depending on the respective mesh size is not necessary in this context.

The remaining reasoning from [6] applies without any change and yields the desired ellipticity estimate

$$a(v_h, v_h) \ge c\|v_h\|_{1,h}^2 \quad \text{for } \ v_h \in V_h. \tag{5.10}$$

Finally, we turn to an L_2-error estimate which is the key to the approximation property of the coarse-grid correction in the multigrid algorithm.

Theorem 5.1 *Assume that the triangulations $\mathcal{T}_{h,k}$ are shape regular and that the elliptic problem (2.1) is H^2-regular. Then the finite element solution $u_h \in V_h$ satisfies*

$$\|u - u_h\|_{0,\Omega} + h\|u - u_h\|_{1,h} \le c\,h^2\|u\|_{2,\Omega}, \tag{5.11}$$

where h is the maximum of the mesh sizes of the triangulations.

This result has been proved in [6] for the 2D case and for trial spaces with continuity at cross points. It is interesting that the latter constraint is irrelevant. In fact, following [6], employing Strang's second lemma in order to obtain a bound of $\|u - u_h\|_{1,h}$ the *consistency error* $L_u(v_h) := a(u, v_h) - (f, v_h)_{0,\Omega} - (g, v_h)_{0,\partial\Omega}$ can be rewritten as

$$\begin{aligned}L_u(v_h) &= \sum_{\Gamma_{kl}\subset\mathcal{S}} (a\frac{\partial u}{\partial n} - \mu_h, [v_h])_{0,\Gamma_{kl}} \\ &\le \sum_{\Gamma_{kl}\subset\mathcal{S}} \|a\frac{\partial u}{\partial n} - \mu_h\|_{-1/2,h,\Gamma_{kl}}\|[v_h]\|_{1/2,h,\Gamma_{kl}}\end{aligned} \tag{5.12}$$

where we have used that $(\mu_h, [v_h])_{0,\Gamma_{kl}} = 0$ for any $\mu_h \in M_h$ and $v_h \in V_h$. Note that the jumps need not belong to $H_{00}^{1/2}(\Gamma_{kl})$ since they are essentially measured in the L_2-norm. What matters is only the approximation property (5.1) of the Lagrange multipliers so that, in view of (5.9), the same reasoning as in [6] shows that

$$|L_u(v_h)| \le ch\, \|u\|_{2,\Omega}\, \|v_h\|_{1,h}\, .$$

The remainder of the proof of Theorem 4.1 in [6] applies without change since (5.7) holds and all other arguments are independent of the spatial dimension. □

6 The Condition Numbers of the Schur Complement in the Smoother

The variational equations (2.7) read in matrix form

$$\begin{pmatrix} A & B^T \\ B & \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{6.1}$$

The smoothing procedure in the multigrid algorithm for the mortar method proposed in [6] is based on an iteration of the form

$$\begin{aligned} \begin{pmatrix} u_h^{j+1} \\ \lambda_h^{j+1} \end{pmatrix} &:= \begin{pmatrix} u_h^{j} \\ \lambda_h^{j} \end{pmatrix} - \begin{pmatrix} C & B^T \\ B & \end{pmatrix}^{-1} \left\{ \begin{pmatrix} A & B^T \\ B & \end{pmatrix} \begin{pmatrix} u_h^{j} \\ \lambda_h^{j} \end{pmatrix} - \begin{pmatrix} f \\ 0 \end{pmatrix} \right\} \\ &= \begin{pmatrix} u_h^{j} \\ 0 \end{pmatrix} - \begin{pmatrix} C & B^T \\ B & \end{pmatrix}^{-1} \begin{pmatrix} Au_h^j - f \\ Bu_h^j \end{pmatrix}. \end{aligned} \tag{6.2}$$

Here superscripts denote iteration indices and C is a preconditioner for A satisfying

$$v^T A v \le v^T C v, \quad v \in X_h. \tag{6.3}$$

We note that u_h^{j+1} always satisfies the constraint, i.e.,

$$Bu_h^{j+1} = 0, \tag{6.4}$$

see [7]. On account of the L_2-error estimate in Theorem 5.1 above the same arguments as in [6] can be employed to establish *h-independent convergence rates* of W-cycles as in the 2D-case.

In the 3D-case, however, an essential difference arises with regard to the computational realization of the above smoother. To explain this in more detail note that each application of the smoothing procedure requires solving *auxiliary systems* of the form

$$\begin{pmatrix} C & B^T \\ B & \end{pmatrix} \begin{pmatrix} v \\ \mu \end{pmatrix} = \begin{pmatrix} d \\ e \end{pmatrix}. \tag{6.5}$$

In actual computations the solution of (6.5) boils down to implementing

$$S\mu = BC^{-1}d - e, \tag{6.6}$$
$$v = C^{-1}(d - B^T\mu), \tag{6.7}$$

where

$$S := BC^{-1}B^T \tag{6.8}$$

is the *Schur complement* of (6.5).

In particular, if we order the vector v by collecting all nodal values in the interior of the subdomains in one block and those on the skeleton in a second block and if we choose a non-diagonal preconditioner only for the first block, then the linear system (6.5) becomes

$$\begin{pmatrix} C_1 & 0 & 0 \\ 0 & \alpha I & B_2^T \\ 0 & B_2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \mu \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ e \end{pmatrix}. \tag{6.9}$$

Hence, in this case the Schur complement matrix

$$S_2 = \frac{1}{\alpha} B_2 B_2^T$$

has a simple structure.

In the $3D$ case the auxiliary linear system (6.6) corresponds to a *two-dimensional* problem, and direct solvers will become expensive. To retain efficiency of the smoothing procedure the application of iterative solvers to (6.6) has many advantages. Obviously S_2 is positive definite. Therefore to evaluate the efficiency of conjugate gradient iterations we estimate the spectral condition numbers of $B_2B_2^T$.

To this end, note that the Rayleigh quotients $\mu^T(B_2B_2^T)\mu/(\mu^T\mu)$ can be written as $\left(\|B_2^T\mu\|_{\ell_2}/\|\mu\|_{\ell_2}\right)^2$. Their estimation will be based on the interrelation of the matrix B_2 and the bilinear form

$$b_2 : L_2(\mathcal{S} + \mathcal{S}) \times L_2(\mathcal{S}) \to \mathbb{R}$$
$$b_2(v, \mu) = \sum_{\Gamma_{kl} \in \mathcal{S}} (\mu_{kl}, [v])_{0,\Gamma_{kl}}$$

living on the trace spaces. Here the elements of $L_2(\mathcal{S} + \mathcal{S})$ are understood to be defined separately on both sides of $\mathcal{S}$, i.e.,

$$L_2(\mathcal{S} + \mathcal{S}) = \{v = \{(v_{kl}, v_{lk}) : v_{k,l}, v_{lk} \in L_2(\Gamma_{kl}),\ \Gamma_{kl} \subset \mathcal{S}\}\},$$

and $\|v\|^2_{0,\mathcal{S}+\mathcal{S}} = \sum_{\Gamma_{kl}\subset\mathcal{S}}\{\|v_{kl}\|^2_{0,\Gamma_{kl}} + \|v_{lk}\|^2_{0,\Gamma_{kl}}\}$. Note that as a set $L_2(\mathcal{S} + \mathcal{S})$ can be identified with $X_\delta/H^1_{0,D}(\Omega)$ so that for $v \in L_2(\mathcal{S} + \mathcal{S})$ the notation $[v]\ |_{\Gamma_{kl}} = v_{kl} - v_{lk}$ is consistent. In order to emphasize the domain $L_2(\mathcal{S} + \mathcal{S}) \times L_2(\mathcal{S})$ of b_2 and hence the topology of the range of B_2^T, we distinguish $b_2(\cdot,\cdot)$ from its counterpart $b(\cdot,\cdot)$ in (2.9). In fact, identifying as usually finite element functions and their vector representations, it is clear that $b_2(v,\mu) = v^T B_2^T \mu$. Since the condition number of a matrix is independent of scaling factors, we may assume that the basis functions in M_h are normalized as in (3.2) so that $\|\mu\|_{0,\mathcal{S}} \sim \|\mu\|_{\ell_2}$ and likewise $\|v\|_{0,\mathcal{S}+\mathcal{S}} \sim \|v\|_{l_2}$. This implies

$$\begin{aligned} b_2(v,\mu) &= \sum_{\Gamma_{kl}\in\mathcal{S}} ([v],\mu_{kl})_{0,\Gamma_{kl}} \le \sum_{\Gamma_{kl}\in\mathcal{S}} \|[v]\|_{0,\Gamma_{kl}} \|\mu_{kl}\|_{0,\Gamma_{kl}} \\ &\le c\,\|v\|_{\ell_2}\|\mu\|_{\ell_2}. \end{aligned} \tag{6.10}$$

Moreover, making use of $\|B_2^T\mu\|_{l_2} = \sup_v\{vB_2^T\mu/\|v\|_{l_2}\} = \sup_v\{b_2(v,\mu)/\|v\|_{l_2}\}$, continuity (6.10) of b_2 now implies

$$\|B_2^T\mu\|_{\ell_2} = \sup_v \frac{b_2(v,\mu)}{\|v\|_{\ell_2}} \le c \sup_v \frac{\|[v]\|_{0,\mathcal{S}}\|\mu\|_{0,\mathcal{S}}}{\|v\|_{\ell_2}} \le c\|\mu\|_{\ell_2}. \tag{6.11}$$

Hence the largest eigenvalues of $B_2 B_2^T$ remain uniformly bounded.

On the other hand, given μ, the proof of the inf-sup condition (cf. Theorem 3.2 in [6] or Theorem 4.1 here) yields a function $d \in L_2(\mathcal{S}+\mathcal{S})$ such that

$$b_2(d,\mu) \ge c\|d\|_{0,\mathcal{S}+\mathcal{S}}\|\mu\|_{0,\mathcal{S}}\,. \tag{6.12}$$

Therefore,

$$\|B_2^T\mu\|_{\ell_2} \ge c\frac{b_2(d,\mu)}{\|d\|_{0,\mathcal{S}+\mathcal{S}}} \ge c\|\mu\|_{0,\mathcal{S}} \ge c\|\mu\|_{\ell_2}\,. \tag{6.13}$$

From the two estimates (6.11) and (6.13) it follows that

$$\kappa(B_2 B_2^T) \le c, \tag{6.14}$$

i.e., the spectral condition numbers of the Schur complement S_2 remain uniformly bounded.

Therefore the auxiliary linear systems that are encountered in the smoothing procedure can be efficiently solved by a cg-iteration.

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