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BLOCK SSOR PRECONDITIONERS FOR 3-D GROUNDWATER FLOW PROBLEMS

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SUMMARY

In the simulation of 3-dimensional groundwater flow problems often meshes are generated for which the horizontal mesh size is much larger than the distances in the vertical direction. The matrices which arise from the discretization of differential equations are known to have large condition numbers. Here, the geometry leads to an additional increase of the condition number. While standard preconditioners as SSOR and ICC can cope with the differential operators, they are less suited for the geometrical difficulties. We show that here block-SSOR preconditioners are an appropriate tool.

INTRODUCTION

In groundwater flow simulations great systems of linear equations are to be solved. Stationary problems lead to the discretization of elliptic differential equations of second order. If Darcy's law

$$v = -K \operatorname{grad} h$$

is inserted in the continuity equation

$$\operatorname{div} v = q$$

then an elliptic equation of second order arises

$$-\operatorname{div}(K \operatorname{grad} h) = q\,.$$

When time-dependent problems are solved by implicit integration schemes, then equations with a similar structure are created.

In the modeling of 2-dimensional problems with several thousands of unknowns, preconditioned conjugate gradient methods (PCG-methods) have turned out to be efficient. In particular, the standard preconditioners which can be used as black-box codes are successful in most cases. There are no big differences between preconditioning by incomplete Cholesky decomposition (ICC) and symmetric successive overrelaxation (SSOR), when computing times are measured. There is a small preference for the ICC method [1, 4, 5].

The computing times are more sensitive to the question whether meshes with well-shaped triangles (and quadrilaterials) or with slim triangles have been created. Meshes are usually generated so that sources and wells become nodes in the mesh and (small) rivers become edges. Since there are these restrictions from a practical point of view, one is prepared to have a slightly larger number of iteration steps than in similar problems of other applied sciences. Indeed, this is the situation when 2-dimensional models are treated.

The situation turned out to be less favorable when 3-dimensional problems were first attacked. The iteration with ICC-preconditioner broke down in many cases and the iterations with SSOR-preconditioner suffered from a dramatic increase of the number of steps. This was our experience, and the same was reported by other groups.

It is our aim to understand this phenomenon and to find a remedy.

SOME CHARACTERISTICAL PROPERTIES OF 3-DIMENSIONAL MODELS

When 3-dimensional models for groundwater flow problems are performed, one has typically domains whose projection onto a horizontal plane covers an area of several km^2. On the other hand, the aquifer, i.e. the zone in which groundwater is transported, has a thickness of about 10 to 100 m. To achieve a good model of the geological situation, the domain is divided into 3 to 20 layers.

As a consequence, the 3-dimensional meshes have quite different meshsizes in horizontal and vertical direction. Typical are mean values of meshsizes of

$$\begin{array}{ll} 100\text{ m} - 500\text{ m} & \text{in the horizontal direction,} \\ 1\text{ m} - 10\text{ m} & \text{in the vertical direction.} \end{array}$$

Since the coefficients of the equations which arise from a differential equation of second order are proportional to h^{-2}, we get an anisotropy of

$$10^3 \text{ to } 10^4.$$

Here we have already taken into account that the permeability in the horizontal direction is greater than the vertical one.

The standard preconditioners SSOR and ICC are well suited to cope with the properties of the matrix which are due to the discretization of an elliptic equation. Also ICC can deal with problems with jumps in the dispersion coefficient since it is less point-oriented than SSOR. On the other hand it cannot compensate the increase of the condition number of the matrix induced by the anisotropy of the grid.

We will see that block preconditioners are suited [1, 2, 6].

The 3-dimensional meshes which are encountered in groundwater, usually have a special structure. They can be understood as copies of identical horizontal layers. The number of layers is between 3 and 20. Thus in the vertical direction there is a very regular structure while the meshes within every 2-dimensional layer may be a very irregular.

A MODEL PROBLEM

In order to describe the phenomena more clearly and to simplify the representation, we turn to a model problem with the same structure. We consider the Poisson equation on a rectangle

$$\begin{aligned} -\Delta u &= f \ \text{ on } [0, n_x \Delta x] \times [0, n_z \Delta z] \\ u &= 0 \ \text{ on the lateral boundary,} \\ \frac{\partial u}{\partial n} &= 0 \ \text{ on the top and on the bottom.} \end{aligned}$$

The typical stencil for the discretization, i.e. the typical coefficients in the matrix associated to an interior point is given by

$$\begin{pmatrix} & -1 & \\ -\varepsilon & 2+\varepsilon & -\varepsilon \\ & -1 & \end{pmatrix}.$$

Here $\varepsilon = (\Delta z/\Delta x)^2$. In order to have a comparable model, we consider $\varepsilon \approx 10^{-3}$, $n_x = 40$ and $n_z = 10$. The coupling in the vertical direction is much higher than in the horizontal one.

Now it is clear that a point relaxation scheme cannot be successful. The point Gauss-Seidel scheme can be understood as follows. Each update of a component of the vector is performed such that the induced quadratic form is minimized. A difference to the vertical neighbours, however, increases the quadratic term by a large amount and is penalized by a large factor. The updates are damped by this large factor. A slow-down of the iteration is the consequence.

The ICC method has a different shortcoming. The matrix has a block structure and the diagonal blocks

$$\begin{pmatrix} 1+\varepsilon & -1 & & & \\ -1 & 2+\varepsilon & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+\varepsilon & -1 \\ & & & -1 & 1+\varepsilon \end{pmatrix}$$

are almost singular. The truncation errors which are caused by the incompleteness of the decomposition meet with a nearly singular matrix.

Table 1. Number of iteration steps for model problem and three preconditioners. ($n_x = 40$, $n_z = 10$, accuracy$= 10^{-5}$)

anisotropy ε	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}
preconditioner					
ICC	26	35	42	74	94
SSOR	26	41	105	161	> 200
block-SSOR	19	18	17	17	16

Candidates for overcoming the difficulties are block-SSOR and block-ICC methods. Specifically the first method turns out to be successful. When the updates of all the points on a vertical line are done simultaneously, the strong coupling is no longer a disadvantage.

The results in Table 1 support the considerations above.

In principle, the same holds for block ICC provided that the inversion of the diagonal block matrices is performed exactly. In this case, all the blocks of the matrix which contain only one nonzero entry are completely filled. To avoid this drawback, block-ICC is usually combined with some approximations of the inverses. Inverses of tridiagonal matrices are approximated by tridiagonal ones. These approximations are here not appropriate for the same reason as mentioned for point ICC.

Table 2. Three examples of matrices in groundwater simulations. Number of iteration steps k and computing time t for an accuracy of 10^{-5}. (nc = no convergence)

region	Düsseldorf		Heidelberg		Ruhr-Lippe	
variables	9246		16464		45666	
non-zero entries	85499		161645		546000	
horizontal layers	6		12		6	
area in km^2	4.5		0.71		152.8	
	k	t	k	t	k	t
preconditioner						
ICC	49	7.8	*nc*		*nc*	
point SSOR	183	24.1	620	150	750	280
block SSOR	30	4.6	278	83	150	60
AMG with block SSOR	7	5.4	128	105	30	81

NUMERICAL RESULTS

We have examined several preconditioners for 7 real-life groundwater flow problems. Table 2 contains the results for three of them. Here the number of iteration steps is listed together with the computer time (in seconds) on a workstation with 7 MFlops. Table 3 contains only the iteration count. It is obvious that the block-SSOR preconditioner leads to the fastest iteration and is the most reliable. The reduction of the number of iteration steps is always significant. In some cases it is more than a factor of 5. Specifically, the difference bewtween the preconditioners are very great in those cases in which a region with a great area is modelled.

We emphasize that the codes are (black-box) general purpose codes. No individual programming was performed in connection with these investigations.

In Table 2 also results for preconditioners with algebraic multigrid methods (AMG) is shown [3]. The costs for one cycle are now as high as for about 4 steps with the other preconditioner. Therefore, we do not save computer time although the number of iteration steps was reduced once more.

Table 3. Four examples of matrices in groundwater simulations. Number of iteration steps for an accuracy of 10^{-5}.

region	*Brandenb.*	*Anhalt*	*CokePlant*	*Dorsten*
variables	24000	24260	75816	34074
non-zero entries	290000	292000	987000	412000
horizontal layers	6	17	13	6
area in km^2	23.5	98.9	0.49	113.0
preconditioner				
ICC	1411	859	128	173
point SSOR	2369	1149	184	282
block SSOR	155	112	86	81

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