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Enhanced Assumed Strain Elements and Locking in Membrane Problems

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Abstract

The assumptions which guarantee convergence of EAS elements first introduced by Simo and Rifai, can be reduced to two postulates. Specifically, the stability condition refers to a relaxed stability of an associated Hu–Washizu principle.

We investigate the problem of volume locking of membrane problems. Estimates of a certain type which for instance imply volume locking of the Q1-element, are fortunately not present with EAS elements.

Finally we verify an equivalence of the EAS concept with a Hellinger–Reissner method which was observed in numerical computations by Andelfinger and Ramm.

1 Introduction

The elements with Enhanced Assumed Strains which were introduced by Simo and Rifai [1990], are now popular in certain areas. Some computations with these *EAS elements* have turned out to show no locking in cases in which the corresponding non-enhanced elements suffer from locking. On the other hand, there are still some mysteries about the elements. Simo and Rifai have used the Hu–Washizu principle as a point of departure, but the connection can be only formal since the corresponding elements for the mixed method of Hu–Washizu may not be stable.

Recently, Reddy and Simo [1995] have established some rigorous arguments, but they changed to another variational principle when regarding the elements in the nearly incompressible case. They moved to a mixed method (saddle point problem) with penalty term already with the original (non discretized) problem.

We will provide a short analysis of EAS elements which is based on merely two conditions which are easily understood. There is a *stability condition* and a *consistency condition*. The first condition guarantees that the problem is coercive. If the elements are deduced from a Hu–Washizu principle like Simo and Rifai did, the condition is related to the stability of the corresponding elements. The stability condition can be formulated as a strengthened Cauchy inequality, and all the well-known tools for inequalities of this kind can be applied.

The consistency condition controls the consistency error in the second Strang lemma which is a famous tool for the analysis of nonconforming methods.

The enhancement of the finite element space leads to a softening of the stiffness matrix. It is bounded from below by a factor which is independent of h. Nevertheless, the softening decreases when $h \to 0$.

The study of the EAS elements and their behaviour in computations for nearly incompressible material will be a central point in this paper. It will be shown that locking phenomena can be substantially reduced, but the rest may induce irregular checkerboard modes. In contrast to similar situations in fluid flow problems, the checkerboard modes vanish for $h \to 0$. If they are annoying, e.g. in nonlinear problems, these modes can be completely suppressed. To this end we propose to extend the space of the enhanced assumed strains given by Simo and Rifai once more in difficult situations.

Andelfinger and Ramm [1993] observed already that the EAS concept in some model situation is equivalent to a Hellinger–Reissner formulation. We will provide a short proof of the equivalence in our mathematical setting referring to the weak equations. When we completed this paper, we noted that recently Yeo and Lee [1996] have given a formal proof which is based on the variational functionals. Since we show that the resulting Hellinger–Reissner formulation does not degenerate if the above mentioned stability condition holds, the proof is now complete. The two methods are only *nearly equivalent* if the elements are general quadrilaterals and not rectangles. Our analysis supports the preference of the Hellinger–Reissner formulation in the non-equivalent case, since it is closer to the concept of reduced integration.

2 The Hu–Washizu Principle as a Point of Departure

Following Simo and Rifai [1990] we will start from the Hu–Washizu principle and eliminate the stresses to obtain a variational formulation with displacements and strains. Fortunately, it will not matter whether the finite element spaces are stable for the Hu–Washizu principle. We emphasize this fact since here the construction of stable spaces is often very involved. It is just the aim of the concept of EAS elements to be more flexible. Therefore we establish stability only later after a reduction process has been done.

Let $\Omega \subset \mathbb{R}^d$ with $d = 2$ or $d = 3$. When the linear elasticity problem is to be solved with the setting of the Hu–Washizu principle, there are three finite element spaces $V_h \subset H^1(\Omega)^d$, $E_h, S_h \subset L_2(\Omega)^{d\times d}$. Specifically, E_h and S_h contain symmetric tensor fields. Given a load $\boldsymbol{f} \in H^{-1}(\Omega)^d$ and a functional $g \in L_2(\Omega)^{d\times d}$, we look for $\boldsymbol{\varepsilon}_h \in E_h$, $\boldsymbol{u}_h \in V_h$, and $\boldsymbol{\sigma}_h \in S_h$ such that

$$
\begin{aligned}
&(\boldsymbol{\varepsilon}_h, C\boldsymbol{\eta}) & & -(\boldsymbol{\eta}, \boldsymbol{\sigma}_h) &&= 0 && \text{for } \boldsymbol{\eta} \in E_h,\\
& & & (\nabla^s \boldsymbol{v}, \boldsymbol{\sigma}_h) &&= (\boldsymbol{f}, \boldsymbol{v}) && \text{for } \boldsymbol{v} \in V_h,\\
-&(\boldsymbol{\varepsilon}_h, \boldsymbol{\tau}) & +(\nabla^s \boldsymbol{u}_h, \boldsymbol{\tau}) & &&= (g, \boldsymbol{\tau}) && \text{for } \boldsymbol{\tau} \in S_h.
\end{aligned} \tag{2.1}
$$

Here C is the elasticity tensor. It depends on the Lamé constants λ and μ such that $\boldsymbol{\sigma} = C\boldsymbol{\varepsilon}$ is a short version of

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda(\operatorname{trace}\boldsymbol{\varepsilon})I. \tag{2.2}$$

Moreover $\boldsymbol{\varepsilon}(\boldsymbol{v}) := \nabla^s \boldsymbol{v} := \frac{1}{2}[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T]$ is the symmetric gradient

$$\varepsilon_{ik}(\boldsymbol{v}) = (\nabla^s \boldsymbol{v})_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right). \tag{2.3}$$

The inner product $(\cdot,\cdot)$ is the L_2 scalar product. In particular,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) := \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, dx := \int_\Omega \sum_{i,k} \sigma_{ik}\varepsilon_{ik}\, dx \tag{2.4}$$

and

$$\|\boldsymbol{\varepsilon}\|_0 := (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})^{1/2}.$$

The variational problem (2.1) is usually considered with $g = 0$. This will also be the case here with only one exception.

We recall that (2.1) refers to a saddle point problem with $\boldsymbol{\varepsilon}_h$ and $\boldsymbol{u}_h$ being direct variables. The stresses have the role of Lagrangian multipliers.

As usual the stored energy U can be given in terms of the strains by making use of (2.2) and $\operatorname{div}\boldsymbol{v} = \operatorname{trace}\boldsymbol{\varepsilon}(\boldsymbol{v})$

$$U(\boldsymbol{\varepsilon}(\boldsymbol{v})) := (\boldsymbol{\varepsilon}(\boldsymbol{v}), C\boldsymbol{\varepsilon}(\boldsymbol{v})) = 2\mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) dx + \lambda \int_\Omega (\operatorname{div}\boldsymbol{v})^2 dx. \tag{2.5}$$

The second term on the right-hand side will be called the *volume term of the stored energy.*

We recall that full stability of the Hu–Washizu principle means that displacements, strains, and stresses depend continuously on the loads and the other terms on the right hand side of the equations (2.1). In particular, zero energy modes are excluded, and in linear elasticity we have the bounds for the norms

$$\|\boldsymbol{\varepsilon}_h\|_0 + \|\nabla^s \boldsymbol{u}_h\|_0 + \frac{1}{\mu}\|\boldsymbol{\sigma}_h\|_0 \le c_1\,(\|\boldsymbol{f}\|_0 + \mu\|g\|_0).$$

(The factors μ have been inserted since strains and stresses are measured in different units.) In the following lemma we will require only a reduced stability. We need only the stability of the strains and only with respect to the functional g.

Lemma 1. *Assume that*

$$\|\nabla^s \boldsymbol{u}_h\|_0 \le c_1\mu\|g\|_0 \tag{2.6}$$

whenever $(\boldsymbol{\varepsilon}_h, \boldsymbol{u}_h, \boldsymbol{\sigma}_h)$ *is a solution of (2.1) and* $\boldsymbol{f} = 0$. *Then, given* $\boldsymbol{u}_h \in V_h$, *there exists a stress* $\boldsymbol{\sigma}_h \in S_h$ *such that* $\|\boldsymbol{\sigma}_h\|_0 = 1$ *and*

$$(\nabla^s \boldsymbol{u}_h, \boldsymbol{\sigma}_h) \ge \frac{1}{c_1\mu}\|\nabla^s \boldsymbol{u}_h\|_0. \tag{2.7}$$

Proof. Assume to the contrary that the lemma is wrong. Then there exists some $\boldsymbol{w}_h \in V_h$ such that

$$(\nabla^s \boldsymbol{w}_h, \boldsymbol{\tau}) \le \alpha\|\boldsymbol{\tau}\|_0 \quad \text{for all } \boldsymbol{\tau} \in S_h \tag{2.8}$$

and $\alpha < \frac{1}{c_1\mu}\|\nabla^s \boldsymbol{w}_h\|_0$. We will derive that this is impossible. If we put $\boldsymbol{f} := 0$, $g := \nabla^s \boldsymbol{w}_h$, then the solution of (2.1) is immediate: $\boldsymbol{u}_h = \boldsymbol{w}_h$, $\boldsymbol{\varepsilon}_h = \boldsymbol{0}$, and $\boldsymbol{\sigma}_h = \boldsymbol{0}$. From (2.1) and (2.8) we conclude that

$$(g, \boldsymbol{\tau}) = (\nabla^s \boldsymbol{w}_h, \boldsymbol{\tau}) = (\nabla^s \boldsymbol{u}_h, \boldsymbol{\tau}) \le \alpha\|\boldsymbol{\tau}\|_0 \quad \text{for } \boldsymbol{\tau} \in S_h.$$

By the Riesz representation theorem (see e.g. Oden(1979)) we can construct a representer $g_1 \in L_2(\Omega)^{d\times d}$ for the same functional, i.e.,

$$(g_1, \boldsymbol{\tau}) = (g, \boldsymbol{\tau}) \quad \text{for } \boldsymbol{\tau} \in S_h$$

and $\|g_1\|_0 \le \alpha$. From (2.6) we have $\|\nabla^s \boldsymbol{u}_h\|_0 \le c_1\mu\|g_1\|_0 = c_1\mu\alpha < \|\nabla^s \boldsymbol{u}_h\|_0$ which is a contradiction. Therefore, (2.8) cannot be true and (2.7) holds. □

3 The Decomposition of Simo and Rifai

Simo and Rifai assume that the finite element space E_h contains the strains which are compatible with the displacements and that E_h is obtained by an enhancement of the set of the compatible tensor fields. Thus, we have a decomposition

$$
\begin{aligned}
\boldsymbol{\varepsilon}_h &= \nabla^s \boldsymbol{v}_h + \tilde{\boldsymbol{\varepsilon}}_h, \qquad \boldsymbol{v}_h \in V_h, \quad \tilde{\boldsymbol{\varepsilon}}_h \in \Gamma_h, \\
E_h &= E_h^c \oplus \Gamma_h, \qquad E_h^c = \nabla^s V_h.
\end{aligned}
\tag{3.1}
$$

The superscript c is to indicate that the set E_h^c contains the *compatible* strains $\boldsymbol{\varepsilon}_h^c$ which correspond to displacements $\boldsymbol{v}_h$ from the finite element space V_h. The crucial assumption is that the enhanced strains $\tilde{\boldsymbol{\varepsilon}}_h$ do not contribute to the energy if $\boldsymbol{\sigma}_h \in S_h$, i.e.,

$$
(\boldsymbol{\sigma}_h, \tilde{\boldsymbol{\varepsilon}}_h) = 0 \quad \text{for } \boldsymbol{\sigma}_h \in S_h, \ \tilde{\boldsymbol{\varepsilon}}_h \in \Gamma_h. \tag{3.2}
$$

Now we return to the equations (2.1) and set $\boldsymbol{\varepsilon}_h = \nabla^s \boldsymbol{w}_h + \tilde{\boldsymbol{\varepsilon}}_h$ in the spirit of (3.1). Similarly, we decompose the test functions $\boldsymbol{\eta} = \nabla^s \boldsymbol{v} + \boldsymbol{\gamma}$. Moreover we make use of the orthogonality relations (3.2) and obtain

$$
\begin{array}{llllll}
(\nabla^s \boldsymbol{w}_h, C\,\nabla^s \boldsymbol{v}) & +(\tilde{\boldsymbol{\varepsilon}}_h, C\,\nabla^s \boldsymbol{v}) & & -(\nabla^s \boldsymbol{v}, \boldsymbol{\sigma}_h) & = 0 & \text{for } \boldsymbol{v} \in V_h, \\
(\nabla^s \boldsymbol{w}_h, C\boldsymbol{\gamma}) & +(\tilde{\boldsymbol{\varepsilon}}_h, C\boldsymbol{\gamma}) & & & = 0 & \text{for } \boldsymbol{\gamma} \in \Gamma_h, \\
 & & & (\nabla^s \boldsymbol{v}, \boldsymbol{\sigma}_h) & = (\boldsymbol{f}, \boldsymbol{v}) & \text{for } \boldsymbol{v} \in V_h, \\
-(\nabla^s \boldsymbol{w}_h, \boldsymbol{\tau}) & & +(\nabla^s \boldsymbol{u}_h, \boldsymbol{\tau}) & & = 0 & \text{for } \boldsymbol{\tau} \in S_h.
\end{array}
\tag{3.3}
$$

If the equations (2.1) for the Hu–Washizu principle are solvable, then given $\boldsymbol{v}_h \in V_h$, there is a $\boldsymbol{\tau} \in S_h$ such that $(\nabla^s \boldsymbol{v}_h, \boldsymbol{\tau}) \neq 0$. Note that this algebraic condition is weaker than the stability condition in Lemma 1. With this we conclude from the last equation in (3.3) that

$$
\nabla^s \boldsymbol{w}_h = \nabla^s \boldsymbol{u}_h. \tag{3.4}
$$

Moreover the third equation can be inserted into the first one. As a consequence, (3.3) is reduced to a variational formulation in which only the displacements and the enhanced strains are found

$$
\begin{array}{llll}
(\nabla^s \boldsymbol{u}_h, C\,\nabla^s \boldsymbol{v}) & + \ (\tilde{\boldsymbol{\varepsilon}}_h, C\,\nabla^s \boldsymbol{v}) & = (\boldsymbol{f}, \boldsymbol{v}) & \text{for } \boldsymbol{v} \in V_h, \\
(\nabla^s \boldsymbol{u}_h, C\boldsymbol{\gamma}) & + \ (\tilde{\boldsymbol{\varepsilon}}_h, C\boldsymbol{\gamma}) & = 0 & \text{for } \boldsymbol{\gamma} \in \Gamma_h.
\end{array}
\tag{3.5}
$$

This is the famous variational formulation with enhanced assumed strains of Simo and Rifai (or enhanced displacement formulation for short) for problems in linear elasticity. The equations would coincide with the classical displacement formulation if all terms with enhanced strains are ignored. Although we have started with the Hu–Washizu principle, the equations (3.5) do not refer to a mixed problem, but lead to a system with a symmetric positive definite matrix.

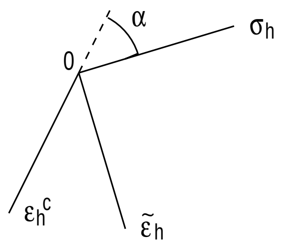

Fig. 1. If the finite element spaces V_h and S_h have been chosen to be appropriate for the Hu–Washizu prinziple, then by Lemma 1 the angle between $\boldsymbol{\varepsilon}_h^c$ and a suitable $\boldsymbol{\sigma}_h$ satisfies $\alpha \le \alpha_{\max}$. The orthogonality condition (3.2) implies that the angle between $\boldsymbol{\varepsilon}_h^c$ and $\tilde{\boldsymbol{\varepsilon}}_h$ is at least $\frac{\pi}{2} - \alpha_{\max}$.

We note that once we have reached the formulation (3.5), we need no longer take assumption (3.2) into account.

In order to facilitate the understanding of the following conditions, we recall that the symmetric gradient of a displacement field (2.3) refers to the corresponding strains and rewrite (3.5)

$$\begin{aligned} (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), C\boldsymbol{\varepsilon}(\boldsymbol{v})) + (\tilde{\boldsymbol{\varepsilon}}_h, C\boldsymbol{\varepsilon}(\boldsymbol{v})) &= (\boldsymbol{f}, \boldsymbol{v}) \quad \text{for } \boldsymbol{v} \in V_h, \\ (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), C\boldsymbol{\gamma}) \quad + (\tilde{\boldsymbol{\varepsilon}}_h, C\boldsymbol{\gamma}) \quad &= 0 \qquad\quad \text{for } \boldsymbol{\gamma} \in \Gamma_h. \end{aligned} \tag{3.5'}$$

In order to ensure convergence, Simo and Rifai [1990] and later Reddy and Simo [1995] have introduced some extra conditions. We will reduce them to two conditions, namely a stability condition and a consistency condition.

Postulates:

(P1) *Stability condition:* There is a constant $c_0 < 1$ which is independent of h such that a strengthened Cauchy inequality

$$(\boldsymbol{\varepsilon}_h^c, \boldsymbol{\gamma}_h) \le c_0 \, \|\boldsymbol{\varepsilon}_h^c\|_0 \, \|\boldsymbol{\gamma}_h\|_0 \quad \text{for } \boldsymbol{\varepsilon}_h^c \in \nabla^s V_h, \ \boldsymbol{\gamma}_h \in \Gamma_h \tag{3.6}$$

holds. (The symbol c_0 will be reserved for this constant.)

(P2) *Consistency condition:* The finite element space Γ_h is orthogonal (with respect to the inner product of the stored energy) to the piecewise constant strains, i.e.,

$$(\boldsymbol{\varepsilon}_h, C\tilde{\boldsymbol{\varepsilon}}_h) = 0 \quad \text{for } \tilde{\boldsymbol{\varepsilon}}_h \in \Gamma_h, \tag{3.7}$$

whenever $\boldsymbol{\varepsilon}_h$ is a symmetric tensor which is constant on each element of the mesh.

The postulate (P2) can already be found in the paper by Simo and Rifai. It implies that the enhanced strains have little influence if the strains are almost constant in each element. Therefore it prevents that consistency errors of order zero are present. Reddy and Simo discuss also (more restricitive) consistency conditions which require orthogonality to piecewise polynomials up to some degree k. In this way the enhanced strains produce only errors of higher order in h.

For understanding the EAS-concept the control of the stability is more important.

An inequality of the form (3.6) is called a *strengthened Cauchy inequality* and refers to geometric properties of linear subspaces in a Hilbert space. Here the inequality refers to the L_2 inner product, but the meaning is the same as for (Euclidean) n-space and is elucidated in figures 1 and 3. If there is an angle α between the two subspaces, then the inner product of two vectors from them is at most $\cos\alpha$ times the product of the lengths. Equivalent estimates for the projection onto a subspace and estimates for the sum of two vectors are listed in Lemma A in Appendix A. We note that Reddy and Simo [1995] refer to a property of the projections which corresponds to the alternative (4) in our Lemma A.

Lemma B in Appendix A shows that (P1) holds if the variational formulation (3.5) is derived from a Hu–Washizu principle thas has at least the reduced stability as stated in Lemma 1. Thus the stability condition (P1) for the EAS elements is less restrictive than the Babuška–Brezzi condition for the formulation (1.1) with the Hu–Washizu formulation.

We will see in the next section that the stability condition induces a limitation of the reduction of the stored energy induced by the enhanced strains. First estimates into this direction follow from other implications of the strengthened Cauchy inequality given in Lemma A. The L_2-norm of the total strain can be bounded from below by its compatible part or by the sum of the two separate parts:

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h) + \boldsymbol{\gamma}_h\|_0 \geq \sqrt{1 - c_0^2}\, \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0, \tag{3.8}$$

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h) + \boldsymbol{\gamma}_h\|_0 \geq \sqrt{\frac{1}{2}(1 - c_0)}\, \Big(\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 + \|\boldsymbol{\gamma}_h\|_0\Big). \tag{3.9}$$

4 First Consequences of the Enhancement of the Assumed Strains

The implications of the additional degrees of freedom become apparent if we consider the equations (3.5') as a characterization of a variational problem

$$(\boldsymbol{\varepsilon}(\boldsymbol{v}_h) + \boldsymbol{\gamma}_h, C(\boldsymbol{\varepsilon}(\boldsymbol{v}_h) + \boldsymbol{\gamma}_h)) - 2(\boldsymbol{f}, \boldsymbol{v}_h) \longrightarrow \min! \tag{4.1}$$

Recalling (2.5) we have the equivalent formulation

$$U(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)+\boldsymbol{\gamma}_h)-2(\boldsymbol{f},\boldsymbol{v}_h)\longrightarrow \text{min!} \tag{4.2}$$

Here the minimum is taken for $\boldsymbol{v}_h \in V_h$ and $\boldsymbol{\gamma}_h \in \Gamma_h$.

Remark 2. Obviously, we have the classical displacement formulation if $\Gamma_h = \{0\}$. The enhanced assumed strains provide additional degrees of freedom for the stored energy, but they do not contribute to the load. Therefore, following Remark 3.3.1 of Simo and Rifai [1990], we can understand (4.2) as follows: *If $\boldsymbol{v}_h$ has been determined and the compatible strains $\boldsymbol{\varepsilon}_h^c = \nabla^s \boldsymbol{v}_h$ are known, the additional degrees of freedom will result in a tensor field $\tilde{\boldsymbol{\varepsilon}}_h$ such that the stored energy U for the total strain tensor*

$$\boldsymbol{\varepsilon}_h = \boldsymbol{\varepsilon}(\boldsymbol{v}_h) + \tilde{\boldsymbol{\varepsilon}}_h$$

will be minimal.

As a consequence, the stored energy is at most as large as in the case without the enhancement. We have the same effect as working with a *relaxed* stored energy. The crucial point is the question whether this relaxation supports or spoils convergence. There is an analogy to the situation when we move from a *conforming* to a *nonconforming* theory.

First we note that (2.5) implies that

$$U(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)+\boldsymbol{\gamma}_h) \geq 2\mu\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)+\boldsymbol{\gamma}_h\|_0^2. \tag{4.3}$$

Here μ is a Lamé constant. Combining this estimate with (3.8) and (3.9) we obtain

$$U(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)+\boldsymbol{\gamma}_h) \geq 2\mu(1-c_0^2)\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2, \tag{4.4}$$

$$U(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)+\boldsymbol{\gamma}_h) \geq \mu(1-c_0)\left(\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2+\|\boldsymbol{\gamma}_h\|_0^2\right). \tag{4.5}$$

Therefore, the stability condition (P1) guarantees that the problem remains coercive with a constant which is independent of the meshsize h.

So at first glance, the softening seems to be bounded from below. Actually, this is true if we have a compressible material, i.e., if λ/μ is of moderate size. On the other hand, (4.4) and (4.5) provide less information if $\lambda \gg \mu$. Indeed, we will see in the next section that volume locking may be reduced substantially by EAS elements.

An error estimate is established by the well-known theory for nonconforming elements. In particular, Strang's second lemma (see e.g. Braess [1997] p. 102) yields

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|\tilde{\varepsilon}_h\|_0 \le c[\inf_{\boldsymbol{v}_h\in V_h} \|\boldsymbol{u}-\boldsymbol{v}_h\|_1 + \sup_{\gamma_h\in\Gamma_h} \frac{(\boldsymbol{\varepsilon}(u), C\boldsymbol{\gamma}_h)}{\|\boldsymbol{\gamma}_h\|_0}]. \tag{4.6}$$

The first term contains the error of the best approximation in the finite element space and is common to all theories. The second term contains the consistency error and is typical for nonconforming elements. Here it refers to the solution $\boldsymbol{u}$ in the Sobolev space and to the enhanced strains in the finite element space Γ_h. The compatible strains are not involed.

If the problem is H^2 regular, then the hypothesis (P2) together with the Bramble–Hilbert lemma yields $|(\boldsymbol{\varepsilon}(u), C\boldsymbol{\gamma}_h)| \le c_1 h\|\boldsymbol{\varepsilon}(u)\|_1\,\|\boldsymbol{\gamma}_h\|_0$, and one has

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|\tilde{\varepsilon}_h\|_0 \le c[\inf_{\boldsymbol{v}_h\in V_h} \|\boldsymbol{u}-\boldsymbol{v}_h\|_1 + h\|\boldsymbol{\varepsilon}(u)\|_1]. \tag{4.7}$$

For details, see Reddy and Simo [1995]. We only want to be more specific concerning constants

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|\tilde{\varepsilon}_h\|_0 \le (1+\frac{\lambda}{\mu})\frac{2}{\alpha(1-c_0)}[\inf_{\boldsymbol{v}_h\in V_h} \|\boldsymbol{u}-\boldsymbol{v}_h\|_1 + c_1 h\|\boldsymbol{\varepsilon}(u)\|_1]. \tag{4.8}$$

Here, λ and μ are the Lamé constants, α is the constant in Korn's inequality, and c_1 is a constant which arises from the application of the Bramble–Hilbert lemma. We note that (4.8) is not sharp for nearly incompressible material, i.e. when $\lambda \gg \mu$.

5 The EAS4 Element and Volume Locking

The most numerical investigations within the framework of EAS elements have been done with the EAS4 element, the EAS7 element, and the PS5 element for the 2-dimensional elasticity problem. Moreover they are attractive because of the resemblance with Wilson's nonconforming element, see Taylor, Beresford, and Wilson [1976] and with the mixed method of Pian and Sumihara [1984]. The three models coincide, if we assume that the domain Ω is a rectangle and that it can be decomposed into rectangular or quadratic elements of meshsize h. Moreover, the restriction to this model situation enables us to keep the technical effort at a minimum while studying the problem of volume locking in a rigorous mathematical way and we avoid the discussion of the appropriate version of the EAS concept for isoparametric elements, see Arunakirinathar and Reddy [1995].

We note that Reddy and Simo [1995] have circumvented such an analysis by changing the model when going to the limit of an incompressible material.

They left the concept of EAS-elements and moved to a mixed method with penalty term which is often suitable for a problem with a small parameter, cf. Braess [1997], p. 274 or Babuška and Suri [1992].

We consider a uniform mesh of square elements generated by affine maps from the reference square

$$T_{\text{ref}} = [-1,+1]^2 = \{(\xi,\eta) \in \mathbb{R}^2; -1 \le \xi, \eta \le +1\}. \tag{5.1}$$

The space V_h contains Q_1 elements, i.e., continuous bilinear elements for each component u_i of the displacements $\boldsymbol{u} = (u_1, u_2)$, and we have with local coordinates (ξ,η)

$$u_i = a_i + b_i\xi + c_i\eta + d_i\xi\eta, \quad i = 1,2. \tag{5.2}$$

In plane elasticity the strains are commonly written in the vector form $\boldsymbol{\varepsilon} = (\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12})$. Hence, the space $E_h^c = \nabla^s V_h$ for the ansatz (5.2) is locally spanned by the columns of the matrix (which coincides with BN in the setting of Andelfinger and Ramm)

$$\begin{pmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & \xi & \eta \end{pmatrix}. \tag{5.3}$$

The corresponding basis functions will be denoted as $\boldsymbol{\varepsilon}_{h,i}^c$, $i = 1,2,\ldots,5$. The space Γ_h for the EAS4 element that is also called Q1/E4 element, is spanned by the columns of the matrix

$$M := \begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \xi & \eta \end{pmatrix}. \tag{5.4}$$

The corresponding basis functions will be denoted by $\boldsymbol{\gamma}_{h,i}$, $i = 1,2,\ldots,4$. Simo and Rifai [1990], Reddy and Simo [1995], Andelfinger and Ramm [1993] use variants with different quadratic polynomials, but this will not matter for our considerations.

Obviously, we have for square elements

$$\begin{aligned} (\boldsymbol{\varepsilon}_{h,4}^c, \boldsymbol{\gamma}_{h,3}) &= \tfrac{1}{\sqrt{2}} \|\boldsymbol{\varepsilon}_{h,4}^c\|_0 \, \|\boldsymbol{\gamma}_{h,3}\|_0 \,, \\ (\boldsymbol{\varepsilon}_{h,5}^c, \boldsymbol{\gamma}_{h,4}) &= \tfrac{1}{\sqrt{2}} \|\boldsymbol{\varepsilon}_{h,5}^c\|_0 \, \|\boldsymbol{\gamma}_{h,4}\|_0 \,, \end{aligned} \tag{5.5}$$

and it is well known that $c_0 = \sqrt{1/2}$ is the best constant in the strengthened Cauchy inequality for the pair E_h^c, Γ_h.

First we observe that the Q_1 elements without enhancement suffer from volume locking. Actually, $H^1(\Omega)^2$ contains an infinite dimensional subspace of vector fields with

$$\operatorname{div} \boldsymbol{u} = 0.$$

On the other hand, the intersection of the kernel with the finite space V_h of Q_1 elements contains only functions of the form

$$\begin{aligned} v_1^{\text{kern}}(x,y) &= v_{1,0}(y) + bx, \\ v_2^{\text{kern}}(x,y) &= v_{2,0}(x) - by. \end{aligned} \tag{5.6}$$

Here $v_{1,0}$ and $v_{2,0}$ are (arbitrary) piecewise linear functions of one variable and b is a real number. The two axes of the coordinate system are distinguished. A pure shear orthogonal to the x-axis with an x-dependent amount belongs to displacements of the form (5.6). The same is true for the y-axis and y-dependent shears. On the other hand, this is no longer true for shears orthogonal to the diagonal in 2-space which vary along the diagonal. Similarly, a rotation around a point does not belong the set if the angle depends on the distance from the center.

The divergence of the functions in the finite element space which do not have the form (5.6), can be bounded from below.

Theorem 3. *Let Ω be a rectangle with maximal sidelength d and V_h be the space of bilinear elements on meshes with squares with sides of length h. Then*

$$\|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega} \geq \frac{h}{12d} \inf_{\boldsymbol{v}^{\text{kern}}} |\boldsymbol{v}_h - \boldsymbol{v}^{\text{kern}}|_{1,\Omega} \quad \textit{for all } \boldsymbol{v}_h \in V_h. \tag{5.7}$$

Here the infimum is taken over all displacements $\boldsymbol{v}^{\text{kern}} \in V_h$ with the form (5.6).

The proof of (5.7) will given in Appendix B.

It follows now from the general theory of locking phenomena, see Braess [1997], p. 272 or Babuška and Suri [1992] that an error estimate of the form

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 = O(h)$$

cannot hold *uniformly* when the parameter λ^{-1} tends to zero.

In order to understand this, consider the variational equation for the displacement formulation:

$$2\mu(\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v})) + \lambda(\operatorname{div} \boldsymbol{u}_h, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \text{for } \boldsymbol{v} \in V_h.$$

By setting $\boldsymbol{v} = \boldsymbol{u}_h$ we obtain

$$2\mu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_0^2 + \lambda\|\operatorname{div} \boldsymbol{u}_h\|_0^2 \leq \|\boldsymbol{f}\|_{-1}\, \|\boldsymbol{u}_h\|_1. \tag{5.8}$$

If we drop the second term on the left-hand side and use Korn's inequality $\|\boldsymbol{u}_h\|_1 \le c_2 \|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_0$, we obtain

$$\|\boldsymbol{u}_h\|_1 \le \frac{c_2^2}{2\mu}\|\boldsymbol{f}\|_{-1}. \tag{5.9}$$

When we insert this in the right-hand side of (5.8) and abandon the first term on the left-hand side, it follows that

$$\|\operatorname{div} \boldsymbol{u}_h\|_0^2 \le \frac{1}{\lambda}\|\boldsymbol{u}_h\|_1 \|\boldsymbol{f}\|_{-1} \le \frac{c_2^2}{2\mu\lambda}\|\boldsymbol{f}\|_{-1}^2.$$

Now Theorem 3 yields that there is a displacement $\boldsymbol{v}^{\text{kern}}$ of the form (5.6) such that

$$|\boldsymbol{u}_h - \boldsymbol{v}^{\text{kern}}|_{1,\Omega}^2 \le \frac{1}{\lambda h^2}\frac{c_2^2 d^2}{2\mu c}\|\boldsymbol{f}\|_{-1}^2. \tag{5.10}$$

In the incompressible limit $\lambda \to \infty$ finite element computations in V_h will yield only numerical solutions of $\boldsymbol{u}_h$ which are very close to functions of the special form (5.6).

From (5.7) and (5.10) we conclude that the volume locking of the Q1 element stems from the fact that the volume term $\lambda \int_\Omega (\operatorname{div} \boldsymbol{v}_h)^2 dx$ is too stiff. The stiffness is reduced by the enhanced assumed strains, and the following questions are natural:

In which way do the enhanced assumed strains reduce the stiffness?

Is the reduction sufficient?

The first question is easily answered for the rectangular EAS4 element. First we will decompose the integral $\int_T (\operatorname{div} \boldsymbol{v}_h)^2 dx$ on each element T. To this end, observe that the square of a linear function is easily computed on a rectangle with the center at the origin (as e.g. the reference square in (5.1))

$$\int (a + b\xi + c\eta)^2 d\xi d\eta = \int a^2 d\xi d\eta + \int (b\xi)^2 d\xi d\eta + \int (c\eta)^2 d\xi d\eta. \tag{5.11}$$

The first term on the right hand side refers to the mean value of the given linear function.

In the following, let $\bar{f}$ denote the mean value of a function on an element T, and $\operatorname{osc} f := f - \bar{f}$ its oscillation or deviation from its mean value. By definition

$$\operatorname{div} \boldsymbol{v}_h = \overline{\operatorname{div} \boldsymbol{v}_h} + \operatorname{osc} \frac{\partial}{\partial x} v_{h,1} + \operatorname{osc} \frac{\partial}{\partial y} v_{h,2}.$$

Since $\boldsymbol{v}_h$ is bilinear on each element, $\frac{\partial}{\partial x} u_{h,1}$ is a linear function of y and $\frac{\partial}{\partial y} v_{h,2}$

is linear in x. Therefore we can apply the integral formula (5.11) and obtain

$$\int_T (\operatorname{div} \boldsymbol{v}_h)^2 dx = \int_T (\overline{\operatorname{div} \boldsymbol{v}_h})^2 dx + \int_T (\operatorname{osc} \frac{\partial}{\partial x} v_{h,1})^2 dx + \int_T (\operatorname{osc} \frac{\partial}{\partial y} v_{h,2})^2 dx \quad \text{for } \boldsymbol{v}_h \in V_h. \tag{5.12}$$

Next, referring to the basis functions defined by (5.3) and (5.4) we have

$$\operatorname{trace} \boldsymbol{\gamma}_{h,1} = \xi, \quad \operatorname{trace} \boldsymbol{\gamma}_{h,2} = \eta.$$

Hence, given $\boldsymbol{v}_h \in V_h$, there is a $\boldsymbol{\gamma}_h \in \Gamma_h$ such that

$$\int_T (\operatorname{div} \boldsymbol{v}_h + \operatorname{trace} \boldsymbol{\gamma}_h)^2 dx = \int_T (\overline{\operatorname{div} \boldsymbol{v}_h})^2 dx. \tag{5.13}$$

On the other hand, this is the greatest reduction which the enhancement with EAS4 elements can induce since the integrand on the right-hand side of (5.13) is constant and there is no $\boldsymbol{\gamma}_h \in \Gamma_h$ with a nonzero mean value in an element. [1] We may interprete (5.13) as a reduced integration.

Now we want to investigate the problem whether the reduction of the stiffness is sufficient. Obviously we have to prevent that an estimate of the type (5.10) holds which in turn excludes that an inequality of the type (5.7) holds. Recalling (5.13) this leaves us with the following question:

Question 4. Assume that $\boldsymbol{u} \in H^1(\Omega)^2$ and $\operatorname{div} \boldsymbol{u} = 0$ in Ω. Is there a function $\boldsymbol{w}_h \in V_h$ such that

$$\overline{\operatorname{div} \boldsymbol{w}_h} = 0 \text{ in each element } T \tag{5.14}$$

and $\|\boldsymbol{u}-\boldsymbol{w}_h\|_1 \leq c \inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u}-\boldsymbol{v}_h\|_1$ with some constant c which is *independent of h and $\boldsymbol{u}$*? We note that the infimum runs over all finite elements without any restriction on their divergence.

In order to *guarantee* a positive answer to Question 4, we have to replace condition (5.14) by a weaker one. We will first discuss a solution which is mo-

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[1] More precisely, the *total* stored energy is minimal if $\boldsymbol{\gamma}_h$ is chosen such that

$$\operatorname{div} \boldsymbol{v}_h + \operatorname{trace} \boldsymbol{\gamma}_h = \overline{\operatorname{div} \boldsymbol{v}_h} + \frac{2\mu}{\lambda + 2\mu} (\operatorname{div} \boldsymbol{v}_h - \overline{\operatorname{div} \boldsymbol{v}_h}) \quad \text{in each element } T.$$

Obviously, the second term on the right-hand side is small for $\lambda \gg \mu$, and it may be ignored here in our discussion.

tivated by the Taylor–Hood element which is well-known from computational fluid mechanics, see e.g. Girault and Raviart [1986].

Assume that Ω can be decomposed into macro elements with each macro element consisting of 2×2 elements of the original mesh, see Fig. 2. The meshes with the macro elements will be denoted as $\mathcal{T}_{2h}$. Moreover, for simplicity we assume that $\boldsymbol{u}$ is piecewise linear on the boundary of the domain Ω.

Theorem 5. *Let $\boldsymbol{u} \in H^1(\Omega)^2$ and $\operatorname{div} \boldsymbol{u} = 0$ in Ω. Then there is a function $\boldsymbol{w}_h \in V_h$ such that*

$$\overline{\operatorname{div} \boldsymbol{w}_h} = 0 \text{ in each macro element } T_2 \in \mathcal{T}_{2h} \tag{5.15}$$

and

$$\|\boldsymbol{u} - \boldsymbol{w}_h\|_1 \le c \inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 . \tag{5.16}$$

Proof. Let M_{2h} be the space of functions which are constant on the macro elements, i.e.,

$$M_{2h} := \{q \in L_2(\Omega);\ \int_\Omega q dx = 0,\ q|_{T_2} = \text{ const for each } T_2 \in \mathcal{T}_{2h}\}.$$

We consider the Stokes problem: Given $\boldsymbol{u} \in H^1(\Omega)^2$, find $\boldsymbol{w}_h \in V_h$ and $p_h \in M_{2h}$ such that

$$\begin{aligned}
(\nabla \boldsymbol{w}_h, \nabla \boldsymbol{v}) + (\operatorname{div} \boldsymbol{v}, p_h) &= (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) \quad && \text{for } \boldsymbol{v} \in V_h, \\
(\operatorname{div} \boldsymbol{w}_h, q) &= 0 && \text{for } q \in M_{2h}.
\end{aligned} \tag{5.17}$$

Let $\boldsymbol{v}_h \in V_h$, then we have

$$\begin{aligned}
(\nabla(\boldsymbol{w}_h - \boldsymbol{v}_h), \nabla \boldsymbol{v}) + (\operatorname{div} \boldsymbol{v}, p_h) &= (\nabla(\boldsymbol{u} - \boldsymbol{v}_h), \nabla \boldsymbol{v}) \quad && \text{for } \boldsymbol{v} \in V_h, \\
(\operatorname{div}(\boldsymbol{w}_h - \boldsymbol{v}_h), q) &= (\operatorname{div}(\boldsymbol{u} - \boldsymbol{v}_h), q) && \text{for } q \in M_{2h}.
\end{aligned} \tag{5.18}$$

The Stokes problem (5.17) with this pair of spaces is known to be stable. The proof proceeds as for the Taylor–Hood element, see Girault and Raviart [1986]. Hence, we obtain fom (5.18) that

$$\|\boldsymbol{w}_h - \boldsymbol{v}_h\|_1 \le c(\|\nabla(\boldsymbol{u} - \boldsymbol{v}_h)\|_0 + \|\operatorname{div}(\boldsymbol{u} - \boldsymbol{v}_h)\|_0) \le c\|\boldsymbol{u} - \boldsymbol{v}_h\|_1 .$$

Next, $\|\boldsymbol{u} - \boldsymbol{w}_h\|_1 \le \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \|\boldsymbol{v}_h - \boldsymbol{w}_h\|_1 \le (c+1)\|\boldsymbol{u} - \boldsymbol{v}_h\|_1$. Since this is true for each $\boldsymbol{v}_h \in V_h$, the proof of (5.16) is complete. ∎

Theorem 5 ensures that there is no volume locking if the volume term of the stored energy in (2.5) is replaced by

$$\lambda \sum_{T_2 \in \mathcal{T}_{2h}} \int_{T_2} (\overline{\operatorname{div} \boldsymbol{u}_h})^2 dx. \tag{5.19}$$

In this way we have gone one step further than Simo and Rifai [1990]. From (5.12) we conclude that they relaxed the volume term only to

$$\lambda \sum_{T \in \mathcal{T}_h} \int_{T} (\overline{\operatorname{div} \boldsymbol{u}_h})^2 dx. \tag{5.20}$$

Unfortunately an approximation property which covers (5.20), does not hold. An appriximation $\boldsymbol{w}_h$ which is good in the sense of (5.14) would be obtained by solving (5.17) with M_h instead of M_{2h}. The corresponding Stokes problem is known to be unstable and to suffer from spurious checkerboard modes. In computational fluid mechanics these elements with bilinear functions for the velocities and piecewise constant pressures are called Q_1-P_0 element, see e.g. the discussion by Girault and Raviart [1986] p. 160. Indeed, checkerboard modes are mentioned in several papers with numerical results on EAS elements.

On the other hand the reader may ask whether the difference between (5.19) and (5.20) is essential and whether there are substantial checkerboard modes and whether they can spoil the finite element computations. The answer to this question which is obtained from well-known experience in fluid mechanics, is more sophisticated than the reader may expect.

$+1$	-1
-1	$+1$

Fig. 2. Checkerboard mode on a macro element consisting of 4 squares.

When the Stokes problem is solved by this (unstable) Q_1-P_0 element, the checkerboard modes in the pressure can be eliminated by a simple linear filter, and the L_2-errors of the velocities are quite satisfactory. Consequently, *we expect that at most very small checkerboard terms will appear in the solution of linear elasticity problems if the displacements are considered (and measured with the L_2-norm). Moreover, here the checkerboard modes become smaller with $h \to 0$. The situation changes if we consider the strains and here specifically* $\operatorname{div} \boldsymbol{u}$. *Therefore, computations with nonlinear constitutive laws or the determination of plastic regions can be spoiled.* As far as the author can see, this is consistent with the observations by Simo and Rifai [1990], Wriggers and Hueck [1996], and Betsch [1996].

6 Note on Convergence for Nearly Incompressible Materials

The discussion in the preceding section was concentrated on the fact that lower estimates as we have with (5.7) for the standard bilinear elements, are not present with appropriate EAS elements. Although we expect that in this way volume locking is eliminated, the arguments do not provide a rigorous proof. Specifically, it is not proven that the convergence is uniform in λ if λ^{-1} is small.

For treating the nearly incompressible case we proceed along well-established lines and transform the problem into a mixed problem with penalty term. To be more precise, we will perform the transformation on the finite element level, since we intend to analyse a given EAS element.

We start from (4.1) and recall that $C\boldsymbol{\varepsilon} = 2\mu\boldsymbol{\varepsilon} + \lambda(\operatorname{trace}\boldsymbol{\varepsilon})I$. Hence, for all $\boldsymbol{v} \in V_h$ and $\boldsymbol{\gamma} \in \Gamma_h$:

$$2\mu(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \tilde{\boldsymbol{\varepsilon}}_h, \boldsymbol{\varepsilon}(\boldsymbol{v}) + \boldsymbol{\gamma}) + \lambda(\operatorname{div}\boldsymbol{u}_h + \operatorname{trace}\tilde{\boldsymbol{\varepsilon}}_h, \operatorname{div}\boldsymbol{v} + \operatorname{trace}\gamma) = (\boldsymbol{f}, \boldsymbol{v}). \quad (6.1)$$

Now we set $p_h := \lambda(\operatorname{div}\boldsymbol{u}_h + \operatorname{trace}\tilde{\boldsymbol{\varepsilon}}_h)$, which means $M_h := \operatorname{div} V_h \oplus \operatorname{trace}\Gamma_h$ for the associated finite element spaces. The following problem is equivalent to (6.1): *Find $\boldsymbol{u}_h \in V_h$, $\tilde{\boldsymbol{\varepsilon}}_h \in \Gamma_h$, and $p_h \in M_h$ such that*

$$\begin{aligned}
2\mu(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \tilde{\varepsilon}_h, \boldsymbol{\varepsilon}(\boldsymbol{v}) + \gamma) + (\operatorname{div}\boldsymbol{v} + \operatorname{trace}\gamma, p_h) &= (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V_h, \boldsymbol{\gamma}_h \in \Gamma_h, \\
(\operatorname{div}\boldsymbol{u}_h + \operatorname{trace}\tilde{\boldsymbol{\varepsilon}}_h, q) \quad -\tfrac{1}{\lambda}(p_h, q) &= 0 \qquad \forall q \in M_h.
\end{aligned} \quad (6.2)$$

In the incompressible limit, i.e. in the case $1/\lambda = 0$, we know that p_h is piecewise constant, and we can redefine M_h as

$$M_h := \{q \in L_{2,0}(\Omega); q|_T = \text{const for each } T \in \mathcal{T}_h\}. \quad (6.3)$$

The well-known results for the Stokes problem are now applicable. On the other hand, if the Lamé constant λ is finite, then we have

$$\frac{\lambda}{\lambda + 2\mu}\operatorname{div}\boldsymbol{u}_h + \operatorname{trace}\tilde{\boldsymbol{\varepsilon}}_h \quad \text{is piecewise constant.}$$

If we set M_h as given in (6.3), we have only an approximation of the actual situation.

We note that Reddy and Simo [1995] circumvented this problem by restarting from the continuous equations.

7 The EAS4 Element and Shear Locking

The EAS concept can also be a remedy for shear locking. On the other hand, a successful elimination of shear locking is obtained only with additional assumptions. The reason is the fact that here the small parameter refers to the geometry of the domain.

We will start with a negative result. Assume that we have a material with Poisson number $\nu \leq 1/3$, i.e. a compressible material. Then $\lambda \leq 2\mu$, and from (2.5) we know that

$$U(\boldsymbol{\varepsilon}(\boldsymbol{v})) \leq 3\mu\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0^2. \tag{7.1}$$

Combining this with (3.8) and (4.3) we have

$$\begin{aligned} U(\boldsymbol{\varepsilon}(\boldsymbol{v})+\boldsymbol{\gamma}_h) &\geq 2\mu\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)+\boldsymbol{\gamma}_h\|_0^2 \\ &\geq 2\mu(1-c_0^2)\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2 \geq \tfrac{2}{3}(1-c_0^2)\cdot U(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)). \end{aligned} \tag{7.2}$$

There is a direct consequence. In the often studied model problem with the constant $c_0 = \sqrt{1/2}$ in the strengthened Cauchy inequality, the relaxation by the enhanced assumed strains can yield a softening at most by a factor of 3. Since much higher stiffening factors are reported with shear locking, the remedy looks to be very restrictive and less attractive.

We cannot do with a fixed constant c_0 in the strengthened Cauchy inequality.

On the other hand, the constant depends heavily on the shape of the elements, and the value $c_0 = \sqrt{\frac{1}{2}}$ is only obtained for the square.

Consider the matrix elements of the system matrix for the rectangle $[-a,+a]\times[-b,+b]$. Its area is $F = 4ab$. Specifically, let $\boldsymbol{\varepsilon}_h^c$ refer to the fourth column of (5.3) and $\tilde{\boldsymbol{\varepsilon}}_h$ to the third one of (5.4). A simple computation yields

$$\begin{aligned} \|\boldsymbol{\varepsilon}_h^c\|_0^2 &= \int (\xi^2+\eta^2)\, d\xi d\eta = \tfrac{F}{3}(a^2+b^2), \\ \|\tilde{\boldsymbol{\varepsilon}}_h\|_0^2 = (\boldsymbol{\varepsilon}_h^c, \tilde{\boldsymbol{\varepsilon}}_h) &= \int \xi^2 d\xi d\eta \qquad = \tfrac{F}{3}\, a^2. \end{aligned} \tag{7.3}$$

Hence, we have

$$(\boldsymbol{\varepsilon}_h^c, \tilde{\boldsymbol{\varepsilon}}_h) = \sqrt{\frac{a^2}{a^2+b^2}}\, \|\boldsymbol{\varepsilon}_h^c\|_0\, \|\tilde{\boldsymbol{\varepsilon}}_h\|_0, \tag{7.4}$$

and the constant in the strengthened Cauchy inequality is

$$\sqrt{\frac{a^2}{a^2+b^2}} \quad \text{if } a \geq b.$$

We see that the value gets close to 1 if the rectangle becomes a slim one. Therefore, slim rectangles may imply large relaxations of the stored energy.

The general convergence theory is no longer applicable since we find the term $1 - c_0$ in the denominator on the right-hand side of (4.8).

At this moment we look back to our point of departure. We can have c_0 close to 1 only if the value of c_1 in (2.6) is large, which by definition is not compatible with the stability of the Hu–Washizu discretization.

On the other hand, we may regard a situation which is sensitive to shear locking. Consider a cantilever beam of lenght l and thickness t with the displacement

$$\begin{aligned} u_1 &= +3\xi^2\eta, \quad & \boldsymbol{\varepsilon}_{11} &= 3\xi\eta, \\ u_2 &= -\xi^3, \quad & \boldsymbol{\varepsilon}_{12} &= \boldsymbol{\varepsilon}_{22} = 0. \end{aligned} \tag{7.5}$$

The stored energy is const $(l \cdot t)^3$. In order to model the functions by bilinear elements, the higher order polynomials in ξ have to be approximated by piecewise linear functions. The stored energy turns out to be

$$\text{const}\; l^3 t(t^2 + \Delta x^2)$$

if Δx is the horizontal lenght of the element. The dependence of the finite element functions on the ξ-variable is easily calculated: $\frac{\partial}{\partial y} u_{h,1}$ is a linear and continuous function of ξ while $\frac{\partial}{\partial x} u_{h,2}$ is piecewise constant. There is a mismatch which is the larger, the longer the elements are. The situation (7.5) cannot be modeled by these finite element functions such that the shear energy is small.

On the other hand, there is no mismatch at the centers of the elements, and we can have

$$\int_T \overline{\Big(\frac{\partial}{\partial x_2} u_{h,1} + \frac{\partial}{\partial x_1} u_{h,2}\Big)}\, dx = 0\,.$$

Now we see that we can yield

$$\int_\Omega (\boldsymbol{\varepsilon}_{12}(\boldsymbol{u}_h) + \gamma_{h,12})^2 dx = 0$$

by choosing an appropriate enhancement γ_h. The arguments are the same as in the discussion which has led to (5.13).

Therefore, shear locking can be avoided for small thickness, but as we have seen before, only at the expense of a nearly degenerate strengthened Cauchy inequality.

8 An Equivalent Hellinger–Reissner Concept

In 1993 Andelfinger and Ramm observed that the EAS4 elements give the same numerical results as some elements for the Hellinger–Reissner concept. We will show that elements with enhanced assumed strains are in general equivalent to mixed methods for the Hellinger–Reissner formulation. Indeed, the analysis of the EAS4 element was facilitated since the optimal enhanced strain $\tilde{\boldsymbol{\varepsilon}}_h$ which is associated to $\nabla^s \boldsymbol{u}_h$ could be explicitly given, see e.g. (5.13), and belong to very simple finite element spaces.

In view of the considerations in Section 6 we emphasize that the equivalence will be established for the finite element spaces and not for the variational problems in the Sobolev spaces.

Our concept differs slightly from the concept of Bischoff, Ramm, and the author [1997]. We start with considerations within the general framework of Simo and Rifai, and we assume that we have already left the formulation with the Hu–Washizu functional and have arrived at (3.5'). So we are free to define (possibly) new stresses $\boldsymbol{\sigma}$ in the spirit of (3.1) by

$$\boldsymbol{\sigma}_h = C\boldsymbol{\varepsilon}_h, \quad \boldsymbol{\varepsilon}_h = \boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \tilde{\boldsymbol{\varepsilon}}_h. \tag{8.1}$$

This equation may be rewritten as

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_h) = C^{-1}\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\varepsilon}}_h. \tag{8.2}$$

This decomposition of the compatible strains is only possible for each $\boldsymbol{u}_h \in V_h$ if

$$\nabla^s V_h \subset C^{-1} S_h \oplus \Gamma_h, \tag{8.3}$$

or $E_h^c \subset E_h \oplus \Gamma_h$ with the notation of (3.1). Here the space S_h may differ from the finite element space for the stresses in (2.1). We note that an appropriate space S_h can be defined in a constructive way. An example is

$$S_h := \{\boldsymbol{\sigma}_h \in C\nabla^s V_h \text{ which satisfy } (\boldsymbol{\sigma}_h, \boldsymbol{\gamma}) = 0 \text{ for all } \boldsymbol{\gamma} \in \Gamma_h\}. \tag{8.4}$$

Since we have $\boldsymbol{\sigma}_h = C(\boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \tilde{\boldsymbol{\varepsilon}}_h)$ and C is symmetric, the second equation in (3.5') may be rewritten as $(\boldsymbol{\sigma}_h, \boldsymbol{\gamma}_h) = 0$ for all $\boldsymbol{\gamma} \in \Gamma_h$. Therefore, we regain the orthogonality relation (3.2). This means that (8.3) is an orthogonal decomposition for the inner product

$$(\boldsymbol{\varepsilon}_h, \boldsymbol{\gamma}_h)_C := (C\boldsymbol{\varepsilon}_h, \boldsymbol{\gamma}_h) \tag{8.5}$$

and the decomposition (8.2) is always unique.

Now we will follow the ideas of Yeo and Lee [1996] to characterize equivalent EAS elements and Hellinger–Reissner formulations. Moreover we will assume the stability condition (P1) in order to guarantee that pathological cases cannot occur.

We recall that the orthogonality relation (3.2) for S_h and Γ_h is always postulated in a framework which follows Simo and Rifai.

Theorem 6. *Assume that the stability condition (P1) holds. The variational formulation (3.5') for the EAS method is equivalent to the following mixed method for the Hellinger–Reissner concept if the splitting conditions (8.3) holds. Find $\boldsymbol{\sigma}_h \in S_h, \boldsymbol{u}_h \in V_h$ such that*

$$\begin{aligned} (\boldsymbol{\sigma}_h, C^{-1}\boldsymbol{\tau}) \; -(\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{u}_h)) &= 0 && \textit{for } \boldsymbol{\tau} \in S_h, \\ -(\boldsymbol{\sigma}_h, \boldsymbol{\varepsilon}(\boldsymbol{v})) \qquad\qquad &= -(\boldsymbol{f}, \boldsymbol{v}) && \textit{for } \boldsymbol{v} \in V_h. \end{aligned} \tag{8.6}$$

Proof. Let $(\boldsymbol{u}_h, \tilde{\boldsymbol{\varepsilon}}_h)$ be a solution of (3.5') and let $\boldsymbol{\varepsilon}_h, \boldsymbol{\sigma}_h$ be defined by (8.1). The second equation of (3.5') reads $(\boldsymbol{\sigma}_h, \gamma) = 0$ for all $\boldsymbol{\gamma} \subset \Gamma_h$. So by definition of $\boldsymbol{\varepsilon}_h$, we have a decomposition $\boldsymbol{\varepsilon}(\boldsymbol{u}_h) = \boldsymbol{\varepsilon}_h - \tilde{\boldsymbol{\varepsilon}}_h$ with $\tilde{\boldsymbol{\varepsilon}}_h \in \Gamma_h$ and $C\boldsymbol{\varepsilon}_h$ orthogonal to Γ_h. Thus $C\boldsymbol{\varepsilon}_h \in S_h$, i.e. $\boldsymbol{\sigma}_h \in S_h$. Since $\tilde{\boldsymbol{\varepsilon}}_h \in \Gamma_h$, the orthogonality relation (3.2) implies that $(\tilde{\boldsymbol{\varepsilon}}_h, \boldsymbol{\tau}) = 0$ for all $\boldsymbol{\tau} \in S_h$. Hence, $(C^{-1}\boldsymbol{\sigma}_h - \boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\tau}) = (\tilde{\boldsymbol{\varepsilon}}_h, \boldsymbol{\tau}) = 0$, and the first equation in (8.6) has been proved.

The first equation of (3.5') yields

$$(\boldsymbol{f}, \boldsymbol{v}) = (\boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \tilde{\boldsymbol{\varepsilon}}_h, C\boldsymbol{\varepsilon}(\boldsymbol{v})) = (\boldsymbol{\varepsilon}_h, C\boldsymbol{\varepsilon}(\boldsymbol{v})) = (\boldsymbol{\sigma}_h, \boldsymbol{\varepsilon}(\boldsymbol{v})),$$

and also the second equation in (8.6) is true.

To complete the proof we have to show the uniqueness of the solution of (8.6). Since the quadratic form $(\boldsymbol{\sigma}_h, C^{-1}\boldsymbol{\sigma}_h)$ is positive definite, it is sufficient to verify a sup-condition, i.e. given $\boldsymbol{v}_h \in V_h$ with $\boldsymbol{\varepsilon}(\boldsymbol{v}_h) \neq 0$, there is a $\boldsymbol{\tau}_h \in S_h$ such that $(\boldsymbol{\tau}_h, \boldsymbol{\varepsilon}(\boldsymbol{v})) \neq 0$; see e.g. Braess [1997], p.130. Referring to (8.3) we decompose $\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = C^{-1}\boldsymbol{\tau}_h + \boldsymbol{\gamma}_h$. From (3.8) we conclude that $\|C^{-1}\boldsymbol{\tau}_h\|_0 \geq (1 - c_0^2)^{1/2}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|^2 \neq 0$. The orthogonality implies $(\boldsymbol{\tau}_h, \boldsymbol{\varepsilon}(\boldsymbol{v}_h)) = (\boldsymbol{\tau}_h, \boldsymbol{\varepsilon}(\boldsymbol{v}_h) - \boldsymbol{\gamma}_h) = (\boldsymbol{\tau}_h, C^{-1}\boldsymbol{\tau}_h) \neq 0$. Now the uniqueness of the solution follows from the general theory of saddle point problems, and the proof of the equivalence is complete. □

Yeo and Lee [1996] emphasize that they need a different proof if there are general quadrilaterals in the mesh and the Jacobians of the isoparametric mappings are not constant on all elements. In this case the finite element spaces contain also rational functions, and equivalence is only obtained if the restrictions of the functions to the Gauss points are considered.

Fortunately, Theorem 6 covers also this case since our proof does not refer to special properties of the L_2-spaces. We may replace the inner product (2.4) by

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) := \sum_m w_m \sum_{i,k} \boldsymbol{\sigma}_{ik}(x_m)\boldsymbol{\varepsilon}_{ik}(x_m) \tag{8.7}$$

where m runs over all Gauss points in Ω and the numbers w_m are the weights in the quadrature formulas. If relations as (8.1) or (8.2) are understood to be true at the Gauss points and if the inner products in (8.4) and (8.6) are understood to be given by (8.7), then Theorem 6 remains true.

From Theorem 6 it is also clear that each hybrid stress element is equivalent to an EAS-method. If a mixed method (8.6) with finite element spaces S_h and V_h is given, we can define

$$\Gamma_h := \{\boldsymbol{\gamma}_h \in \nabla^s V_h \text{ which satisfy } (\boldsymbol{\gamma}_h, \boldsymbol{\tau}) = 0 \text{ for all } \boldsymbol{\tau} \in S_h\}.$$

Then (8.3) holds and we have even an equality here. Now Theorem 6 yields the equivalence when applied in the backwards direction. Another formulation is the following. *The stored internal energy is not computed directly from the strains $\nabla^s \boldsymbol{u}_h$ but from their projection into $C^{-1}S_h$, where the projection is performed with respect to the inner product (8.5).*

When we regard the EAS4 element on rectangles then the decomposition and projection is easily performed. Here the resulting stresses in (8.1) have in each element the following form

$$\begin{aligned} \sigma_{11} &= a_{11} + b_{11}\eta, \\ \sigma_{22} &= a_{22} + b_{22}\xi, \\ \sigma_{12} &= a_{12} \end{aligned} \tag{8.8}$$

with $a_{ij}, b_{ij} \in \mathbb{R}$. Specifically, the orthogonality to Γ_h implies that the expression for σ_{11} does not contain a term which is linear in ξ. Similarly, there is no term with η in the expression for σ_{22}. These functions define a space S_h associated to the EAS4 element, when the equivalent Hellinger–Reissner concept is considered, and the observation by Andelfinger and Ramm is proved.

At this moment we note that we have restricted ourselves to rectangular elements, and so have Reddy and Simo [1995]. There is a simple reason. The strains and stresses are related by C, and C is considered fixed once we have chosen a material. On the other hand, if general quadrilaterals are used, the stresses and strains are first fixed with respect to the local coordinates and then transformed as covariant and contravariant tensors, respectively. So the tensor fields in the spaces in (8.3) show different behaviour under the transformations, and it has not yet been analyzed whether the choice of coordinates has as essential effect on the finite element computations in this framework. If one wants for example that only mean values of the divergence on each element enter into the calculation of the internal stored energy, then this is more easily performed with hybrid stress elements.

For numerical examples the reader is referred to Andelfinger and Ramm [1993], Betsch [1996], Bischoff, Ramm, and Braess [1997], Simo and Rifai [1990], Wriggers and Hueck [1996].

9 Conclusion

Although Simo and Rifai start from the mixed method of Hu–Washizu when they considered the concept of EAS elements, one does not need full stability of the Hu–Washizu method. A restricted stability condition which is equivalent to a strengthened Cauchy inequality, is sufficient for the stabily of the EAS concept.

It is shown that the enhanced assumed strains can eliminate volume locking. To this end we have established a lower estimate for the divergence of bilinear elements and have shown that such an inequality cannot hold for the EAS4 element. If the enhancement of Simo and Rifai is used in its original form, then one may get checkerboard modes. They are small for linear problems, but it may be better to go one step further if nonlinear problems are treated or if plasticity is to be included.

On the other hand, a reduction of shear locking is only possible if the factor in a strengthened Cauchy inequality comes close to 1. This is not recommended for stability reasons.

The EAS elements are also equivalent to a mixed method for the Hellinger–Reissner principle. These are the elements of Pian and Sumihara [1984] if we have a membrane in 2-D with a rectangular mesh. These in turn are equivalent to a reduced integration of the volume term of the stored energy. Both the EAS concept and the mixed method with the Hellinger–Reissner principle have the effect that the (compatible) strains which are caused by the displacements, are projected onto a subspace. When the hybrid stress elements are used, the target space for the projection is explicitly defined while it is the orthogonal complement which is given in the EAS concept. Equivalent complementary spaces are easily constructed for rectangular spaces. If on the other hand, general quadrilaterials are used, then the construction of the complements is more involved, and this holds in both directions. On the other hand, each method will profit from the knowledge of the other one.

A Appendix. The Strengthened Cauchy Inequality

Let X be a Hilbert space with inner product (x, y) and norm $\|x\| = (x, x)^{1/2}$. Moreover, let V and W be closed linear subspaces. Therefore the orthogonal projectors

$$P_V : X \to V, \quad P_W : X \to W$$

are well defined.

Definition. The subspaces V and W satisfy a *strengthened Cauchy inequality* if there is a constant $c_0 < 1$ such that

$$(v, w) \le c_0 \|v\| \, \|w\| \quad \text{for all } v \in V, w \in W. \tag{A.1}$$

The inequality means that there is an angle α between the two subspaces with $\cos \alpha = c_0$, see Fig. 3.

The following well-known properties can be obtained by using two-dimensional arguments and are illustrated in the figure.

Lemma A: *Let V, W be subspaces of X. The following properties are equivalent.*

(1) A strengthened Cauchy inequality (A.1) holds with $c_0 < 1$.

(2) $\|P_W v\| \le c_0 \|v\|$ holds for all $v \in W$.

(3) $\|P_V w\| \le c_0 \|w\|$ holds for all $w \in W$.

(4) $\|v + w\| \ge \sqrt{1 - c_0^2}\, \|v\|$ holds for all $v \in V, w \in W$.

(5) $\|v + w\| \ge \sqrt{\tfrac{1}{2}(1 - c_0)}\, (\|v\| + \|w\|)$ holds for all $v \in V, w \in W$.

Proof. (1) $\Rightarrow$ (2). Let $w = P_W v$. Hence $w \perp v - w$ and

$$\|w\|^2 = (w, w) = (w, w) + (v - w, w) = (v, w) \le c_0 \|v\| \cdot \|w\|.$$

After dividing by $\|w\|$ we have property (2).

(2) $\Rightarrow$ (4). $P_W v$ is the nearest point to v in W. The orthogonality implies $\|v\|^2 = \|P_W v\|^2 + \|v - P_W v\|^2$, and for any $w \in W$ we have

$$\|v + w\|^2 \ge \|v - P_W v\|^2 = \|v\|^2 - \|P_W v\|^2 \ge \|v\|^2 (1 - c_0^2).$$

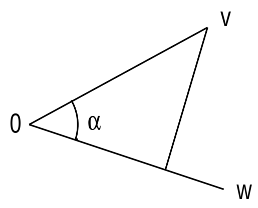

Fig. 3. If there is an angle α between the vectors v and w, then $(v, w) = \cos\alpha \|v\| \cdot \|w\|$. The projection has the length $\|P_W v\| = \sin\alpha\,\|v\|$, and the distance to $W = \mathrm{span}(w)$ is $\cos\alpha\,\|v\|$. Similarly, we see that $\|v + w\| \geq \sin\frac{\alpha}{2}(\|v\| + \|w\|)$, and we note that $\sin\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1 - \cos\alpha)}$.

(4) $\Rightarrow$ (1). From $\|v - w\|^2 \geq (1 - c_0^2)\|v\|^2$ we conclude that

$$2(v, w) = \|v\|^2 + \|w\|^2 - \|v - w\|^2 \leq c_0^2\|v\|^2 + \|w\|^2.$$

In particular, if $\|w\| = c_0\|v\|$, it follows that $2(v, w) \leq 2c_0\|v\|\|w\|$. A homogeneity argument yields that the inequality is true for all $v \in V$ and for all $w \in W$.

(1) $\Leftrightarrow$ (3). From the preceding steps we know that (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1). Hence, we have (1) $\Leftrightarrow$ (2). Since V and W come into (A.1) in a symmetrical way, we conclude that (1) $\Leftrightarrow$ (3) is also true.

(1) $\Rightarrow$ (5). We apply Young's inequality $2ab \leq a^2 + b^2$ twice

$$\begin{aligned}
\|v + w\|^2 &= \|v\|^2 + \|w\|^2 + 2(v, w) \geq \|v\|^2 + \|w\|^2 - 2c_0\|v\| \cdot \|w\| \\
&\geq \|v\|^2 + \|w\|^2 - c_0(\|v\|^2 + \|w\|^2) = (1 - c_0)(\|v\|^2 + \|w\|^2) \\
&\geq \frac{1}{2}(1 - c_0)(\|v\| + \|w\|)^2.
\end{aligned}$$

(5) $\Rightarrow$ (1). From $\|v + w\|^2 \geq \frac{1}{2}(1 - c_0)(\|v\| + \|w\|)^2$ and the Binomial formula we conclude $2(v, w) \leq \|v\|^2 + \|w\|^2 - \frac{1}{2}(1 - c_0)(\|v\| + \|w\|)^2$. In particular, if $\|v\| = \|w\| = 1$, then

$$2(v, w) \leq (2 - \frac{1}{2}(1 - c_0) \cdot 4)\|v\|\,\|w\| = 2c_0\|v\|\,\|w\|.$$

A homogeneity argument yields again that the inequality holds for all $v \in V$ and for all $w \in W$. $\square$

A strengthened Cauchy inequality for the EAS elements can be derived from the (relaxed) stability property (2.6) of the Hu–Washizu formulation. For this purpose we need the following lemma. In essence, the angle between two spaces is expressed in terms of an orthogonal space, cf. Fig. 1.

Lemma B. *Assume that there is a linear subspace $Z \subset X$ which is orthogonal* *to W. Moreover, given $v \in V$, let there be a $z \in Z$ such that $\|z\| = 1$ and*

$$(v, z) \geq c'\|v\|. \tag{A.2}$$

Then a strengthened Cauchy inequality holds for the pair V, W with

$$c_0 = \sqrt{1 - c'^2}.$$

Proof. Given $v \in V$, let $z \in Z$ be as stated in the assumption of the lemma. Then from (A.2) and $\|z\| = 1$ it follows that for all $w \in W$.

$$\|v + w\| \geq (z, v + w) = (z, v) \geq c'\|v\|.$$

Now the assertion follows from the equivalence of the properties (1) and (4) stated in Lemma A. $\square$

B Appendix. Proof of Theorem 3

We start with a discrete version of Friedrich's inequality.

Lemma C. *Assume that $N \geq 1$ and the $(N+1)^2$ numbers $(p_{ik})_{i,k=0}^{N}$ satisfy*

$$\sum_{i,k} p_{ik} = 0. \tag{B.1}$$

Then we have

$$\sum_{i,k} p_{ik}^2 \leq N^2 \left[\sum_{i,k} (p_{i,k} - p_{i,k-1})^2 + \sum_{i,k} (p_{i,k} - p_{i-1,k})^2 \right]. \tag{B.2}$$

Here, the summations run over all indices for which the expressions are defined.

Proof. Consider the optimization problem

$$\sum_{i,k} p_{ik}^2 \to \max!$$

subject to the constraints (B.1) and

$$\sum_{i,k} (p_{i,k} - p_{i,k-1})^2 + \sum_{i,k} (p_{i,k} - p_{i-1,k})^2 = 1.$$

The associated Lagrange function is

$$L(p,\lambda,\mu) := \sum_{i,k} p_{ik}^2 - 2\lambda\left[\sum_{i,k}(p_{i,k}-p_{i,k-1})^2 + \sum_{i,k}(p_{i,k}-p_{i-1,k})^2 - 1\right] - \mu\sum_{ik} p_{ik},$$

where λ and μ are Lagrange multipliers. Thus the maximum is characterized by the equations

$$2p_{ik} - 2\lambda(\Delta_h p)_{ik} - \mu = 0, \quad i,k = 0,1,\dots,N. \tag{B.3}$$

Here Δ_h is the 5-point-stencil for the Laplace operator

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

which is to be modified at the boundary points due to Neumann boundary conditions. The sum of coefficients equals zero and $\sum_{ik}(\Delta_h p)_{ik} = 0$. Thus, by summing (B.3) over all i,k we obtain

$$2\sum_{i,k} p_{ik} - \mu(N+1)^2 = 0.$$

Hence, (B.1) implies $\mu = 0$, and from (B.3) we obtain

$$(\Delta_h p)_{ik} = \frac{1}{\lambda} p_{ik}, \quad i,k = 0,1,\dots,N. \tag{B.4}$$

By multiplying (B.3) with p_{ik} and summing, over i and k we see that we are interested in the smallest eigenvalue of Δ_h. It is associated to the eigenvectors

$$p_{ik} = \cos\frac{k}{N}\pi \quad \text{or} \quad p_{ik} = \cos\frac{i}{N}\pi.$$

The maximum is given by $\lambda^{-1} = 1 - \cos\frac{\pi}{N} = \sin^2\frac{\pi}{2N} \geq N^{-2}$. This completes the proof of (B.2). $\square$

Obviously, the following one-dimensional analogon can be proved in the same way.

Lemma D. *Assume that $N \geq 1$ and the $N+1$ numbers p_i, $i = 0,1,2,\dots,N$ satisfy $\sum_i p_i = 0$. Then we have*

$$\sum_{i=0}^{N} p_i^2 \leq N^2 \sum_{i=1}^{N}(p_i - p_{i-1})^2. \tag{B.5}$$

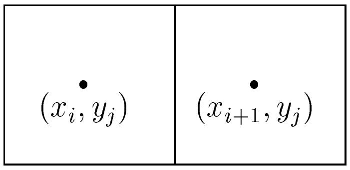

Fig. 4. The squares T_{ij} and $T_{i+1,j}$ from the partition of Ω.

Now we are prepared to prove Theorem 3. Let Ω be partitioned into $N \times M$ squares with sides of length h. Let (x_i, y_j) be the center of the square T_{ij}, see Fig. 4. The set of bilinear finite elements for this mesh is denoted as V_h.

Given $\boldsymbol{v} = (u, v) \in V_h$, we note that the derivatives

u_x *and* v_x *are piecewise linear in* y *and continuous at the upper and lower edges of the elements,*
u_y *and* v_y *are piecewise linear in* x *and continuous at the left and right edges of the elements.*

Consequently, when we consider one of the four derivatives and pass from the center of a cell to its boundary in the direction of a coordinate, we will find a nontrivial increment only in one direction

$$\begin{aligned} \Delta_{ij} u_x &:= u_x(x_i, \frac{y_j + y_{j+1}}{2}) - u_x(x_i, y_j), \\ \Delta_{ij} u_y &:= u_y(\frac{x_i + x_{i+1}}{2}, y_j) - u_y(x_i, y_j). \end{aligned} \tag{B.6}$$

Specifically, the difference operator Δ need not show the direction and it can carry the indices for the specification of the element. The expressions $\Delta_{ij} v_x$ and $\Delta_{ij} v_y$ are analogously defined.

Obviously, we have in each square T_{ij}

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= \operatorname{div} \boldsymbol{v}(x_i, y_j) + [u_x(x, y) - u_x(x_i, y_j)] + [v_y(x, y) - v_y(x_i, y_j)] \\ &= \overline{\operatorname{div} \boldsymbol{v}} + (u_x - \bar{u}_x) + (v_y - \bar{v}_y). \end{aligned} \tag{B.7}$$

The terms on the right hand side of (B.7) are constant, proportional to $y - y_j$ and proportional to $x - x_i$, respectively. Hence, we can apply formula (5.11). If we set the area of the element $|T_{ij}| = |T|$, we have

$$\begin{aligned} D_{ij} &:= \frac{1}{|T|} \int_{T_{ij}} (\operatorname{div} \boldsymbol{v})^2 dx dy \\ &= \frac{1}{|T|} \left(\int_{T_{ij}} (\overline{\operatorname{div} \boldsymbol{v}})^2 dx dy + \int_{T_{ij}} (u_x - \bar{u}_x)^2 dx dy + \int_{T_{ij}} (v_y - \bar{v}_y)^2 dx dy \right) \end{aligned}$$

$$= (\overline{\operatorname{div} \boldsymbol{v}})^2 + \frac{1}{3}(\Delta_{ij} u_x)^2 + \frac{1}{3}(\Delta_{ij} v_y)^2. \tag{B.8}$$

Since u_x is continuous on the horizontal edges, the difference between vertical neighbours can be expressed in terms of increments from the cell centers to the edge in between

$$u_x(x_i, y_{j+1}) - u_x(x_i, y_j) = \Delta_{ij} u_x + \Delta_{i,j+1} u_x.$$

From Young's inequality we know that $(a+b)^2 \le 2(a^2+b^2)$ and

$$\begin{aligned} |u_x(x_i, y_{j+1}) - u_x(x_i, y_j)|^2 &\le 2(\Delta_{ij} u_x)^2 + 2(\Delta_{i,j+1} u_x)^2 \\ &\le 6(D_{ij} + D_{i,j+1}). \end{aligned} \tag{B.9}$$

The estimate of differences in the horizontal direction is more indirect and makes use of $u_x = \operatorname{div} \boldsymbol{v} - v_y$ and the continuity of the derivatives v_y

$$\begin{aligned} &u_x(x_{i+1}, y_j) - u_x(x_i, y_j) \\ &\qquad = \operatorname{div} \boldsymbol{v}(x_{i+1}, y_j) - v_y(x_{i+1}, y_j) - \operatorname{div} \boldsymbol{v}(x_i, y_j) + v_y(x_i, y_j) \\ &\qquad = \operatorname{div} \boldsymbol{v}(x_{i+1}, y_j) - \operatorname{div} \boldsymbol{v}(x_i, y_j) - \Delta_{ij} v_y - \Delta_{i+1,j} v_y. \end{aligned}$$

Having sums with four terms we refer to $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$ and obtain

$$\begin{aligned} &|u_x(x_{i+1}, y_j) - u_x(x_i, y_j)|^2 \\ &\qquad \le 4[(\operatorname{div} \boldsymbol{v}(x_{i+1}, y_j))^2 + (\operatorname{div} \boldsymbol{v}(x_i, y_j))^2 + (\Delta_{ij} v_y)^2 + (\Delta_{i+1,j} v_y)^2] \\ &\qquad \le 12[D_{ij} + D_{i+1,j}]. \end{aligned} \tag{B.10}$$

Since each term D_{ij} may occur in two horizontal and two vertical estimates, we eventually have

$$\sum_{i,j} |u_x(x_i, y_{j+1}) - u_x(x_i, y_j)|^2 + \sum_{i,j} |u_x(x_{i+1}, y_j) - u_x(x_i, y_j)|^2 \le 36 \sum_{i,j} D_{ij}.$$

Let b be the mean value of u_x on the rectangle Ω. Now we apply Lemma C with $N = d/h$ where d is the longest side of the rectangle

$$\sum_{i,j} [u_x(x_i, y_i) - b]^2 \le 36 \frac{d^2}{h^2} \int_\Omega (\operatorname{div} \boldsymbol{v})^2 dx.$$

Once more, we make use of an orthogonality relation (5.11)

$$\int_{T_{ij}} (u_x - b)^2 dxdy = \int_{T_{ij}} (\bar{u}_x - b)^2 dxdy + \int_{T_{ij}} (u_x - \bar{u}_x)^2 dxdy.$$

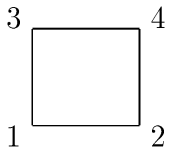

Fig. 5. Labels of points in the proof of $\Delta_{ij}u_y = \Delta_{ij}u_x$.

From (B.8) we know that the second term on the right hand side can be estimated by $\int_{T_{ij}}(\operatorname{div}\boldsymbol{v})^2dxdy$. Collecting terms, we have

$$\|u_x - b\|_0 \le 7\frac{d}{h}\|\operatorname{div}\boldsymbol{v}\|_0. \tag{B.11}$$

The estimate of u_y is reduced to that of u_x by a simple observation which refers to the discrete analogon of the relation $u_{xy} = u_{yx}$. Obviously we can arrange the differences of values of u at the corners of a square (see Fig. 5) in two ways

$$(u_4 - u_2) - (u_3 - u_1) = (u_4 - u_3) - (u_2 - u_1).$$

These equations mean $2h\Delta_{ij}u_y = 2h\Delta_{ij}u_x$ and

$$\Delta_{ij}u_y = \Delta_{ij}u_x\,. \tag{B.12}$$

The continuity of u_y at vertical edges implies

$$u_y(x_{i+1}, y_j) - u_y(x_i, y_j) = \Delta_{ij}u_y + \Delta_{i+1,j}u_y.$$

Now, from (B.8), (B.12), and Young's inequality we derive with the same argument as used before

$$|u_y(x_{i+1}, y_j) - u_y(x_i, y_j)|^2 \le 6(D_{ij} + D_{i+1,j}). \tag{B.13}$$

Next, we define the mean values on the horizontal stripes

$$\psi_j := \frac{1}{N}\sum_i u_y(x_i, y_j), \quad j = 1, 2, \ldots, M,$$

and $\psi(x, y) = \psi_j$ for $(x, y) \in \cup_j T_{ij}$. Since the definition of ψ_j asserts that $\sum_i(u(x_i, y_j) - \psi_j) = 0$, we obtain from Lemma D an estimate for the j-th strip

$$\sum_i |u_y(x_i, y_j) - \psi_j|^2 \le 12\frac{d^2}{h^2}\sum_i D_{ij}, \quad j = 1, 2, \ldots, M.$$

We sum over j, recall that $|u_y - \bar{u}_y| \le |\Delta_{ij}u_y|$ in T_{ij} and rewrite the sums as integrals

$$\int_\Omega |u_y - \psi|^2 dxdy \le 13\frac{d^2}{h^2}\int_\Omega |\operatorname{div}\boldsymbol{v}|^2 dxdy.$$

Finally, we fix $v_{1,0}$ as the piecewise linear and continuous function, such that $\frac{\partial}{\partial y} v_{1,0} = \psi$ and $\int_\Omega v_{1,0} dxdy = 0$. Hence,

$$\int\limits_\Omega |(u - v_{1,0} - bx)_y|^2 dxdy \le 13 \frac{d^2}{h^2} \int\limits_\Omega |\operatorname{div} \boldsymbol{v}|^2 dxdy.$$

Combining this and (B.11) we obtain $|u - v_{1,0}(y) - bx|_{1,\Omega} \le 8\frac{d}{h} \|\operatorname{div} \boldsymbol{v}\|_{0,\Omega}$. Since we may treat the vertical displacement in the same way, we have for some function $\boldsymbol{v}^{\text{kern}}$ of the form (5.6)

$$|\boldsymbol{v} - \boldsymbol{v}^{\text{kern}}|_{1,\Omega} \le 12 \frac{d}{h} \|\operatorname{div} \boldsymbol{v}\|_{0,\Omega}\,, \tag{B.14}$$

and the proof of Theorem 3 is complete. □

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