*DDM Preprint*
Editors: editor1, editor2, editor3, editor4 

# 1 THE MORTAR ELEMENT METHOD REVISITED – WHAT ARE THE RIGHT NORMS?

D. Braess[1], W. Dahmen[2]

## Introduction

A number of investigations have recently been devoted to the mortar method as a domain decomposition method with non-overlapping subdomains. Its attraction comes from its great flexibility due to the fact that different types of discretization are possible on different subdomains. The best experience is with $H^1$-elliptic problems. In contrast to standard conforming elements, there may be jumps across the interfaces between adjacent subdomains, and the continuity conditions are replaced by weak matching conditions the so called *mortaring conditions.* Our guiding question here will be to what extent there still remain "interdomain-constraints" on the discretizations which are possibly imposed by stability and accuracy requirements, in particular, when dealing with highly *non-quasi-uniform meshes.*

There is by now almost a standard way to treat mortar elements in the framework of nonconforming elements, where it was originally analyzed, see e.g. [BMP94]. However, since it may be technically cumbersome to eliminate the constraints imposed by the matching conditions and since fast solvers are by now available for mixed formulations, the analysis as a saddle point problem has recently attracted interest, see e.g. [BB99, BDW00, Woh99b]. Moreover, on a principal level the inf-sup condition is also often hidden in the analysis of mortar elements based on the nonconforming theory. If the inf-sup condition holds, the error of approximation by functions with and without the mortaring conditions are of the same order [Bra01, Remark III.4.10]. This tool is frequently used for estimating the term that represents the approximation error in the lemma of Berger, Scott, and Strang. Therefore we believe that the understanding of the saddle point formulation is at the heart of the matter which will be the point of view taken in this paper.

The fact that the framework for the saddle point formulation is still less well established in comparison with the nonconforming method is due to the subtle difference between (at least) two trace spaces in the scale of Sobolev spaces with index 1/2. To be specific, let $\Gamma_{kl}$ denote the (typical) interface between the subdomains $\Omega_k$ and $\Omega_\ell$. When the variational problem is considered in the Sobolev space $H^1(\Omega)$ or $H^1_0(\Omega)$, then the trace space $H^{1/2}_{00}(\Gamma_{kl})$ endowed with the norm $\|g\|_{H^{1/2}_{00}(\Gamma_{kl})} := \|\chi_{\Gamma_{kl}} g\|_{1/2,\partial\Omega_k}$ (where $\chi_{\Gamma_{kl}}$ is the standard indicator function) turns out to be an appropriate function space for the jumps over the interior boundary $\Gamma_{kl}$. In the 2D case this can be realized by forcing the trial functions to be continuous at the cross points, which is a mild constraint. However, for 3D problems the jumps would have to vanish along the

[1]Ruhr-Universität Bochum, braess@num.ruhr-uni-bochum.de
[2]RWTH Aachen, dahmen@igpm.rwth-aachen.de
The work of this author has been supported in part by the TMR network "Wavelets in Numerical Simulation" funded by the European Commission

boundaries of the interfaces, and this would entail severe restrictions on the discretizations for neighboring subdomains. Thus jumps living in the larger space $H^{1/2}(\Gamma_{kl})$, are usually admitted in actual computations with mortar elements.

This discrepancy (gap) prohibits the use of Brezzi's theory with the standard Sobolev spaces and their norms. For a rigorous treatment one had to resort to non-standard methods. One possibility is to introduce *mesh-dependent norms* as done, e.g. in [BDW00, DFG$^+$01, Woh99b]. Continuity, ellipticity, and the inf-sup condition as required by Brezzi's theory are then available. Another concept can be found in [BB99, Woh99a] where the analysis is performed in a two-stage process. In a first step merely the direct variables are estimated by the nonconforming theory. In the second step only the inf-sup condition and no ellipticity is required for achieving an error estimate of the Lagrange multipliers.

A principal objective of this paper is to narrow this gap somewhat. Specifically, we will explore to what extent and under what circumstances one can dispense with mesh-dependent norms. Some mesh-dependence still turns out to remain but only for one variable and in a weaker form no longer involving an explicit mesh size parameter. Moreover, the new norms can be bounded by $\|\cdot\|_{H_{00}^{1/2}(\Gamma_{kl})}$ if applied to a function in the space $H_{00}^{1/2}(\Gamma_{kl})$. It models a function space in which $H_{00}^{1/2}(\Gamma_{kl})$ has a finite codimension, while it differs from $H^{1/2}(\Gamma_{kl})$ by an infinite dimensional space. It is now easily understandable why all the different concepts have one point in common. They all make use of the fact — in an open or hidden way — that the subset of finite element functions whose jumps belong to $H_{00}^{1/2}(\Gamma_{kl})$, is sufficiently thick.

Aside from these theoretical considerations there is the following practical reason for addressing the above issue. Nonoverlapping domain decomposition appears to be particularly suitable for problems with complicated domains or jumping coefficients so that one expects solutions with singular behavior. Therefore the use of highly non-quasi-uniform or adaptively refined meshes in different subdomains should be covered by the theory. However, the mesh-dependent norms from [BDW00, BD98, Woh99b] only work well when using quasi-uniform meshes. In fact, in connection with error estimates the mesh sizes should not even differ too much from one subdomain to the other one, see [DFG$^+$01] for an extension to mesh-dependent norms with suitable local mesh size functions.

So the core question is how independently from each other can the discretizations on different subdomains be chosen so as to retain stability and overall accuracy even when the individual meshes are highly non-quasi-uniform.

Recently, an error analysis has been performed in [KLPV01] for the mortar method on meshes that are only *locally* quasi-uniform. The price that has been paid there is that the meshes on adjacent subdomains have to *match along the boundary* of the interface which in the three dimensional case severly imposes on the mesh generator. Our approach allows us to abundon this constraint to restore full mortar flexibility. We still obtain error estimates of the same type as in [KLPV01], where the constants now depend only on *one sided mesh size ratios.* Cleary, local refinements on or near an interface would result from a singular behavior of the approximated solution on or near that interface affecting both adjacent subdomains. Thus conditions of this type (even two sided versions) tend to be satisfied automatically by reasonable mesh adaptation strategies.

The paper is organized as follows. In Section 1 we describe the continuous problem. Section 1 is concerned with the discrete counterparts. Specifically, we formulate several requirements to be met by the discretizations. These are similar in spirit (and in fact closely related) to those in [KLPV01] and have been recognized to play a pivotal role in many preceding investigations [BB99, BMP94, BDW00, BD98, Woh99b]. Section 1 is devoted to the stability analysis for this setting. In contrast to [KLPV01] we work here in a saddle point context for a choice of norms that is different from prior investigations. In Section 1 we discuss error estimates from different point of views. The concepts are then applied in Section 1 to the so called dual basis mortar method from [BR99, KLPV01, Woh99a]. In particular, we establish standard types of error estimates for locally quasi-uniform meshes without the above mentioned interface boundary matching condition from [KLPV01].

# The continuous problem

Consider the second order elliptic boundary value problem

$$\begin{array}{rcll} -\operatorname{div} a(x)\operatorname{grad} u(x) & = & f(x) & \text{in } \Omega, \\ a(x)\frac{\partial u}{\partial n} & = & g(x) & \text{on } \Gamma_N \subset \partial\Omega, \\ u & = & 0 & \text{on } \Gamma_D := \partial\Omega \setminus \Gamma_N, \end{array} \tag{1}$$

where $a(x)$ is a piecewise sufficiently smooth and uniformly positive definite matrix defined for $x$ in the bounded domain $\Omega \subset \mathbb{R}^d$, $\Gamma_D$ is a subset of the boundary $\Gamma$ of $\Omega$ (with positive measure relative to $\Gamma$), and $\Gamma_N := \Gamma \setminus \Gamma_D$. $H^1_{0,D}(\Omega)$ denotes the closure in $H^1(\Omega)$ of all $C^\infty$-functions vanishing on $\Gamma_D$.

Suppose that $\Omega$ is decomposed into non-overlapping subdomains $\Omega_k$, $k = 1, \dots, k_{\max}$, i.e.,

$$\bar\Omega = \bigcup_{k=1}^{k_{\max}} \bar\Omega_k, \quad \Omega_k \cap \Omega_l = \emptyset \qquad \text{for } k \neq l. \tag{2}$$

For simplicity we will assume throughout the rest of the paper that the domain $\Omega \subset \mathbb{R}^d$ and that the subdomains $\Omega_k$ in (2) are polyhedral. If the closures of $\Omega_k$ and $\Omega_l$ have a $(d-1)$-dimensional intersection, we set $\bar\Gamma_{kl} := \bar\Omega_k \cap \bar\Omega_l$. However, we do not insist on the partition to be geometrically conforming, i.e., $\Gamma_{kl}$ need *not* be a full common face of both subdomains. The $\Gamma_{kl}$ form the *skeleton*

$$\mathcal{S} := \bigcup_{k,l} \Gamma_{kl}.$$

$\Gamma_{kl}$, $\Gamma_N$, and $\Gamma_D$ will always be assumed to be the union of polyhedral subsets of the boundaries of the $\Omega_k$.

The mortar method is based on a variational formulation of (1) with respect to the product space

$$X_\delta := \{v \in L^2(\Omega) : v|_{\Omega_k} \in H^1(\Omega_k),\ k = 1, \dots, k_{\max},\ v|_{\Gamma_D} = 0\},$$

endowed with the norm

$$\|v\|_{1,\delta} := \left(\sum\nolimits_{k=1}^{k_{\max}} \|v\|_{H^1(\Omega_k)}^2\right)^{1/2}.$$

The space $H^1_{0,D}(\Omega)$ is characterized as a subspace of $X_\delta$ determined by appropriate constraints on jumps across interfaces.

This suggests the following weak formulation of (1): For a *suitable* pair of spaces $X, M$, *find* $(u,\lambda) \in X \times M$ *such that*

$$\begin{aligned} a(u,v) + b(v,\lambda) &= (f,v)_{0,\Omega} + (g,v)_{0,\Gamma_N} && \text{for all } v \in X, \\ b(u,\mu) &= 0 && \text{for all } \mu \in M, \end{aligned} \tag{3}$$

where $(u,v)_{0,\Omega}$ and $(g,v)_{0,\Gamma_N}$ denote the $L^2$ inner products on $\Omega$ and $\Gamma_N$, respectively.

$$\begin{aligned} a(u,v) &:= \sum\nolimits_k \int_{\Omega_k} (a(x)\nabla u(x)) \cdot \nabla v(x) dx, \\ b(v,\mu) &:= \sum\nolimits_{\Gamma_{kl} \subset \mathcal{S}} (\mu, [v])_{0,\Gamma_{kl}}. \end{aligned}$$

The *jump* $[v]$ of a function $v \in X$ is defined on $\mathcal{S}$ by $[v] := v|_{\overline{\Omega}_k} - v|_{\overline{\Omega}_l}$ on $\Gamma_{kl}$ (see [BDW00] for further background information). We note that each interface $\Gamma_{kl}$ appears *only once* in the sum over $\mathcal{S}$.

# Discretization

In order to describe next the mortar method as a *discrete* version of (3), we choose for each subdomain $\Omega_k$ a (conforming) triangulation $\mathcal{T}_k$ subject to the following assumptions:

(a) Each triangulation is completely *independent* of those on neighboring subdomains. This means that the nodes in $\mathcal{T}_k$ which belong to $\Gamma_{kl}$ need *not* match with any of the nodes of $\mathcal{T}_l$.

(b) The $\mathcal{T}_k$ will always be shape regular but only *locally* quasi-uniform, i.e., the ratios of maximal and minimal diameters of the elements in $\mathcal{T}_k$ need not remain bounded.

With each $\mathcal{T}_k$ we associate a finite element space $\mathcal{S}(\mathcal{T}_k) \subset H^1(\Omega_k) \cap H^1_{0,D}(\Omega)$. In principle, this could have any fixed polynomial order, but for simplicity we will refer in most cases to spaces of piecewise linear finite elements on $\mathcal{T}_k$. We set

$$X_h := \prod_{k=1}^{k_{\max}} \mathcal{S}_1(\mathcal{T}_k) \subset X_\delta, \tag{4}$$

where the index $h$ indicates the dependence on the discretization.

The next crucial step is to fix the Lagrange multipliers for each $\Gamma_{kl}$ (i.e. the space $M$ in (3)). In this context, we stress the following implicit notational convention to be used throughout the rest. The indexing of the interface $\Gamma_{kl}$ (as opposed to $\Gamma_{lk}$) always expresses that $\Omega_k$ has been chosen as the non-mortar side. This distinction is important because the Lagrange multipliers will only depend on the non-mortar side in a way that will be specified later in more detail. Whenever $\Gamma_{kl}$ is a full common face of both adjacent subdomains, the choice of the mortar side is completely arbitrary.

If $\Gamma_{kl}$ is strictly contained in at least one of the faces, the following provision has to be taken. We will always assume that $\partial\Gamma_{kl}$ is covered by the faces of the cells in $\Gamma_{kl}$ induced by at least one of the triangulations $\mathcal{T}_k$ or $\mathcal{T}_l$. If only one of these triangulations has this property, the corresponding subdomain has to be chosen as the non-mortar side and hence will be denoted by $\Omega_k$.

Meanwhile several types of Lagrange multiplier spaces have been considered in the literature, see e.g. [BMP94, BD98, KLPV01, Woh99a]. Instead of considering any specification we formulate first some requirements on the multiplier spaces that can be extracted from the above mentioned studies. To this end, let $\mathcal{T}_{kl}$ denote the restriction of the mesh $\mathcal{T}_k$ to $\Gamma_{kl}$ and set $S^0_{kl} := \mathcal{S}(\mathcal{T}_{kl}) \cap H^1_0(\Gamma_{kl}) \subseteq H^{1/2}_{00}(\Gamma_{kl})$. Given $S^0_{kl}$, we will employ finite dimensional spaces $M_{kl} \subset L_2(\Gamma_{kl})$ with the following properties:

**P.1** The spaces $S^0_{kl}$ and $M_{kl}$ have the same dimension

$$\dim S^0_{kl} = \dim M_{kl}. \tag{5}$$

**P.2** Whenever $\mathcal{S}(\mathcal{T}_k)$ has *approximation order* $n$, then $M_{kl}$ should have approximation order at least $n-1$, i.e.,

$$\inf_{w \in M_{kl}} \|v - w\|_{0,\Gamma_{kl}} \le c\bar{h}^{n-1} |v|_{n-1,\Gamma_{kl}}, \tag{6}$$

where $\bar{h}$ is the maximal mesh size of $\mathcal{T}_{kl}$. More precisely, defining for every vertex $i$ of $\mathcal{T}_k$ the local mesh size $h_i := \max\{\operatorname{diam}\tau : \tau \in \mathcal{T}_{kl},\ i \in \tau\}$, we set

$$\bar{h} := \max_{i \in \mathcal{T}_{kl}} h_i, \qquad \underline{h} := \min_{i \in \mathcal{T}_{kl}} h_i.$$

Thus for piecewise linear finite elements on $\Omega_k$ one has $n = 2$ and (6) requires first order convergence.

**P.3** The pair $(S^0_{kl}, M_{kl})$ is $L_2$-*stable*, i.e.,

$$\inf_{w \in M_{kl}} \sup_{v \in S^0_{kl}} \frac{(v, w)_{0,\Gamma_{kl}}}{\|v\|_{0,\Gamma_{kl}} \|w\|_{0,\Gamma_{kl}}} \ge c_0 \tag{7}$$

for some fixed constant $c_0$ (depending on $\Gamma_{kl}$).

**P.4** It is well-known that (7) implies that

$$(Q_{kl} w, v)_{0,\Gamma_{kl}} = (w, v)_{0,\Gamma_{kl}}, \quad \forall \quad v \in S^0_{kl}, \tag{8}$$

uniquely defines a projector $Q_{kl} : L_2(\Gamma_{kl}) \to M_{kl}$ such that

$$\|Q_{kl} w\|_{0,\Gamma_{kl}} \le c_0^{-1} \|w\|_{0,\Gamma_{kl}}, \quad w \in L_2(\Gamma_{kl}). \tag{9}$$

Here we require in addition that the adjoint $Q^*_{kl} : L_2(\Gamma_{kl}) \to S^0_{kl}$ of $Q_{kl}$ is also bounded on $H^{1/2}_{00}(\Gamma_{kl})$

$$\|Q^*_{kl} v\|_{H^{1/2}_{00}(\Gamma_{kl})} \le c_1 \|v\|_{H^{1/2}_{00}(\Gamma_{kl})}. \tag{10}$$

The pair $(S^0_{kl}, M_{kl})$ is called *admissible* if **P.1** – **P.4** hold.

**Remark 1** *When $\mathcal{T}_{kl}$ is quasi-uniform,* **P.4** *is a consequence of (9) and the approximation property (6) in* **P.2** *provided that the spaces $M_{kl}$ also satisfy a standard inverse property. Only if the meshes are merely locally quasi-uniform, requirement* **P.4** *requires attention.*

The space of discrete multipliers is now defined as

$$M_h := \prod_{\Gamma_{kl}\subset S} M_{kl} \tag{11}$$

where, again, the index $h$ indicates the dependence on $\mathcal{T}_{kl}$ and should not be viewed as mesh size parameter when used as a subscript. Moreover, the finite element functions that satisfy the mortaring conditions, form the space

$$V_h := \{v_h \in X_h,\, b(v_h, \mu) = 0 \quad \forall \mu_h \in M_h\}. \tag{12}$$

The discrete counterpart to (3) now reads

$$\begin{array}{rcll} a(u_h, v_h) + b(v_h, \lambda_h) & = & (f, v_h)_{0,\Omega} + (g, v_h)_{0,\Gamma_N}, & v_h \in X_h, \\ b(u_h, \mu_h) & = & 0, & \mu_h \in M_h. \end{array} \tag{13}$$

We will show that, (13) is a stable and accurate discretization of (3), if the pairs $(S^0_{kl}, M_{kl})$, $\Gamma_{kl} \in \mathcal{S}$ are (uniformly) admissible in the above sense.

## Stability

First we address the stability of (13). In contrast to [KLPV01] we treat (13) as a saddle point problem. Thus, one has to show that the operators

$$\mathcal{L}_h := \begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \;:\; X_h \times M_h \to X_h' \times M_h' \tag{14}$$

induced by (13) are *uniformly* bounded and have *uniformly bounded inverses* with respect to the underlying meshes. Of course, this depends on the norms for $X_h$ and $M_h$ which have yet to be specified. As explained in [BDW00], due to the subtle differences between the trace spaces $H^{1/2}(\Gamma_{kl})$ and $H^{1/2}_{00}(\Gamma_{kl})$, standard (broken Sobolev norms) turn out to be inappropriate. While for quasi-uniform grids appropriate *mesh-dependent norms* offer a cure [BDW00, BD98, Woh99b, Woh99a] we wish to reduce the mesh-dependence of norms in favor of mesh flexibility.

Our main deviation from previous studies therefore lies in the choice of the norms. Recall that the jumps $[v_h]$ are *not* required to lie in the spaces $H^{1/2}_{00}(\Gamma_{kl})$ which naturally arise in the analysis of the continuous problem. However, it will be seen that it suffices to measure their *projection* into the trace spaces $S^0_{kl} \subset H^{1/2}_{00}(\Gamma_{kl})$ in the norm $\|\cdot\|_{H^{1/2}_{00}(\Gamma_{kl})}$. In fact, for any $v_h \in X_h$ we define

$$\|v_h\|^2_{1,h} := \|v_h\|^2_{1,\delta} + \sum_{\Gamma_{kl}\in\mathcal{S}} \|Q^*_{kl}[v_h]\|^2_{H^{1/2}_{00}(\Gamma_{kl})}, \tag{15}$$

while for $\mu \in M_h \subset M := \prod_{\Gamma_{kl}\subset\mathcal{S}} (H_{00}^{1/2}(\Gamma_{kl}))'$ we take the natural dual norm

$$\|\mu\|_{-1/2}^2 := \sum_{\Gamma_{kl}\subset\mathcal{S}} \|\mu\|^2_{(H_{00}^{1/2}(\Gamma_{kl}))'}. \tag{16}$$

Note that any mesh-dependence of $\|\cdot\|_{1,h}$ enters only implicitly through the projectors $Q^*_{kl}$.

First we address the continuity of the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ with respect to these norms. Since for $v_h \in X_h$ and $\mu_h \in M_h$

$$|(v_h,\mu_h)_{0,\Gamma_{kl}}| = |(Q^*_{kl}v_h,\mu_h)_{0,\Gamma_{kl}}| \leq \|Q^*_{kl}v_h\|_{H_{00}^{1/2}(\Gamma_{kl})}\|\mu_h\|_{(H_{00}^{1/2}(\Gamma_{kl}))'},$$

one has, in view of (15), that

$$|a(u_h,v_h)| \lesssim \|u_h\|_{1,h}\|v_h\|_{1,h}, \quad |b(v_h,\mu_h)| \lesssim \|v_h\|_{1,h}\|\mu_h\|_{-1/2}, \tag{17}$$

holds for any $v_h, u_h \in X_h$, $\mu_h \in M_h$, where the constants depend on the constant $c_0$ in **P.3**.

The first step towards confirming stability of the discretization is to confirm the ellipticity of the bilinear form $a(\cdot,\cdot)$ on the kernel

$$V_h := \{v \in X_h : b(v,\mu) = 0 \quad \text{for } \mu \in M_h\}$$

of the constraints.

**Proposition 1** *The bilinear form $a(\cdot,\cdot)$ is elliptic on $V_h$, i.e.,*

$$a(v,v)\|v\|_{1,h}^2 \qquad \text{for all } v \in V_h. \tag{18}$$

**Proof** The inequality $a(v,v)\|v\|_{1,\delta}^2$ for $v \in V_h$, can be inferred by the analysis in [BMP94]. So the desired ellipticity estimate stated in the theorem follows as soon as we have proved that also $\sum_{\Gamma_{kl}\subset S} \|[v]\|^2_{1/2,h,\Gamma_{kl}} \lesssim \|v\|^2_{1,\delta}$ for $v \in V_h$. But this is obviously true since by definition of $Q_{kl}$ one has for $v_h \in V_h$ and any $w \in L_2(\Gamma_{kl})$ that $(Q^*_{kl}[v_h], w)_{0,\Gamma_{kl}} = ([v_h], Q_{kl}w)_{0,\Gamma_{kl}} = 0$. Thus $Q^*_{kl}[v_h] = 0$ which completes the proof. ■

Since the continuity (17) and ellipticity (18) have already been established, it remains to verify the validity of the *LBB-condition* to ensure the stability of the discretization (13), i.e., the uniform bounded invertibility of the mappings $\mathcal{L}_h$ from (14); see, e.g. [BF91].

**Theorem 1** *Assume that the pairs $(S^0_{kl}, M_{kl})$ are admissible (i.e., that* **P.1** – **P.4** *hold) and that the meshes $\mathcal{T}_k$ are shape regular and locally quasi-uniform. Then there exists a constant $\beta > 0$ depending only on the mesh parameters and on the constants $c_0, c_1$ in* **P.3** *and* **P.4***, respectively, such that the pairs of spaces $X_h, M_h$ defined above satisfy the LBB-condition*

$$\inf_{\mu\in M_h} \sup_{v\in X_h} \frac{b(v,\mu)}{\|v\|_{1,h}\|\mu\|_{-1/2}} \geq \beta. \tag{19}$$

The main ingredient in the proof of Theorem 1 is the following observation.

**Lemma 1** *Under the hypotheses of Theorem 1 there exists for every* $\mu \in M_{kl}$ *an element* $v^* \in S^0_{kl}$ *such that*

$$(v^*, \mu)_{0,\Gamma_{kl}} \geq c \left( \|v^*\|^2_{H^{1/2}_{00}(\Gamma_{kl})} + \|\mu\|^2_{(H^{1/2}_{00}(\Gamma_{kl}))'} \right) \tag{20}$$

*holds for some constant* $c > 0$ *independent of* $v^*$ *and* $\mu$.

**Proof** Given any $\mu \in M_{kl}$, one can find, by definition, a $v \in H^{1/2}_{00}(\Gamma_{kl})$ such that

$$\|\mu\|_{(H^{1/2}_{00}(\Gamma_{kl}))'} \leq 2\frac{(v,\mu)_{0,\Gamma_{kl}}}{\|v\|_{H^{1/2}_{00}(\Gamma_{kl})}} = 2\frac{(v, Q_{kl}\mu)_{0,\Gamma_{kl}}}{\|v\|_{H^{1/2}_{00}(\Gamma_{kl})}} = 2\frac{(Q^*_{kl}v,\mu)_{0,\Gamma_{kl}}}{\|v\|_{H^{1/2}_{00}(\Gamma_{kl})}}.$$

Thus, setting $v^* := Q^*_{kl}v \in S^0_{kl}$, we conclude, in view of (10),

$$c_1^{-1}\|v^*\|_{H^{1/2}_{00}(\Gamma_{kl})}\|\mu\|_{(H^{1/2}_{00}(\Gamma_{kl}))'} \leq \|v\|_{H^{1/2}_{00}(\Gamma_{kl})}\|\mu\|_{(H^{1/2}_{00}(\Gamma_{kl}))'} \leq 2(v^*,\mu)_{0,\Gamma_{kl}},$$

which completes the proof. ■

We are now ready to complete the

**Proof** of Theorem 1. Given $\mu \in M_h$ let $\mu_{kl}$ denote its component corresponding to $\Gamma_{kl} \subset \mathcal{S}$. We define a suitable $v \in X_h$ as follows. For each $\Gamma_{kl}$ let $v_{kl} \in S^0_{kl}$ be the function constructed in Lemma 1 satisfying (20). Bearing in mind, that, by our notational convention, $\Omega_k$ denotes the non-mortar side of $\Gamma_{kl}$, we define $\hat{v}_{kl}$ to be the harmonic extension of the boundary data

$$\hat{v}_{kl}(x) = \begin{cases} v_{kl}(x) & \text{if } x \in \Gamma_{kl}, \\ 0 & \text{if } x \in \partial\Omega_k \setminus \Gamma_{lk}, \end{cases}$$

and define $v|_{\Omega_k} := \sum_{\Gamma_{kl}\subset\partial\Omega_k} \hat{v}_{kl}$ as the superposition of these extensions. In particular, $v$ vanishes identically on any subdomain $\Omega_l$ that is never a non-mortar side. Hence

$$([v],\mu)_{0,\Gamma_{kl}} = (v_{kl},\mu)_{0,\Gamma_{kl}} \gtrsim \|v_{kl}\|^2_{H^{1/2}_{00}(\Gamma_{kl})} + \|\mu\|^2_{(H^{1/2}_{00}(\Gamma_{kl}))'}. \tag{21}$$

This therefore implies also

$$\begin{aligned} \sum_{\Gamma_{kl}\subset\partial\Omega_k} ([v],\mu)_{0,\Gamma_{kl}} &\gtrsim \sum_{\Gamma_{kl}\subset\partial\Omega_k} \|\hat{v}_{kl}\|^2_{H^1(\Omega_k)} + \|\mu\|^2_{(H^{1/2}_{00}(\Gamma_{kl}))'} \\ &\gtrsim \|v\|^2_{H^1(\Omega_k)} + \sum_l \|\mu\|^2_{(H^{1/2}_{00}(\Gamma_{kl}))'}. \end{aligned} \tag{22}$$

Since clearly $\|Q^*_{kl}v\|_{H^{1/2}_{00}(\Gamma_{kl})} = \|v_{kl}\|_{H^{1/2}_{00}(\Gamma_{kl})}$ it follows that

$$\left(\sum_{\Gamma_{kl}\subset\mathcal{S}} \|v\|^2_{H^1(\Omega_k)}\right)^{1/2} + \left(\sum_{\Gamma_{kl}\subset\mathcal{S}} \|v_{kl}\|^2_{H^{1/2}_{00}(\Gamma_{kl})}\right)^{1/2} \geq \|v\|_{1,h}.$$

Combining (21) and (22), we have

$$\left(\sum_{\Gamma_{kl}\subset\mathcal{S}} \|\mu\|^2_{(H^{1/2}_{00}(\Gamma_{kl}))'}\right)^{1/2} \left\{\left(\sum_{\Gamma_{kl}\subset\mathcal{S}} \|v\|^2_{H^1(\Omega_k)}\right)^{1/2} + \left(\sum_{\Gamma_{kl}\subset\mathcal{S}} \|v_{kl}\|^2_{H^{1/2}_{00}(\Gamma_{kl})}\right)^{1/2}\right\} \lesssim \sum_{\Gamma_{kl}\subset\mathcal{S}} ([v],\mu)_{0,\Gamma_{kl}},$$

and conclude that $b(v,\mu) \gtrsim \|v\|_{1,h}\|\mu\|_{-1/2}$. This establishes the validity of the LBB-condition. ■

# Error Estimates

We wish to discuss next the accuracy of the above discretizations. For simplicity we confine the discussion to piecewise linear trial spaces on the subdomains so that the approximation order is $n = 2$. Accordingly the approximation order of the multipliers is assumed to be $n-1 = 1$. The higher order case can be treated analogously provided the solution $u$ of the continuous problem (3) has enough regularity on each $\Omega_k$. Moreover, we will always assume that the pairs $(S^0_{kl}, M_{kl})$ are admissible and that the meshes $\mathcal{T}_k$ are shape regular and locally quasi-uniform. Let us denote by $\bar{h}_k$ the maximal mesh size in $\Omega_k$. If $u|_{\Omega_k} \in H^2(\Omega_k)$, then one hopes that the discrete solution $u_h$ of (13) satisfies an estimate of the type

$$\|u-u_h\|^2_{1,h} \le c\sum_{k=1}^{k_{max}} \bar{h}_k^2 \|u\|^2_{2,\Omega_k}. \tag{23}$$

We want to identify the essential obstructions encountered when going about an estimate of the type (24). The usual point of departure is Strang's second lemma, see e.g. [Bra01], p. 107 or [BDW00], which says that

$$\|u-u_h\|_{1,h} \le c\left(\inf_{v_h\in V_h}\|u-v_h\|_{1,h} + \sup_{v_h\in V_h}\frac{\int_{\mathcal{S}} a\frac{\partial u}{\partial n}[v_h]ds}{\|v_h\|_{1,h}}\right). \tag{24}$$

Since $v_h \in V_h$, due to the orthogonality relation (12) we may subtract an arbitrary element $\mu_h \in M_h$ from the conormal derivative of $u$ in the consistency error so that

$$\begin{aligned}\sum_{\Gamma_{kl}\in\mathcal{S}} (a\frac{\partial u}{\partial n},[v_h])_{0,\Gamma_{kl}} &= \sum_{\Gamma_{kl}\in\mathcal{S}} (a\frac{\partial u}{\partial n}-\mu_h,[v_h])_{0,\Gamma_{kl}} \\ &\le \sum_{\Gamma_{kl}\in\mathcal{S}} \|a\frac{\partial u}{\partial n}-\mu_h\|_{(H^{1/2}(\Gamma_{kl}))'}\|[v_h]\|_{1/2,\Gamma_{kl}}. \end{aligned}\tag{25}$$

We know from the trace theorem that $\|[v_h]\|_{1/2,\Gamma_{kl}} \le c(\|v_h\|_{1,\Omega_k}+\|v_h\|_{1,\Omega_l})$. Moreover, when the $M_{kl}$ have approximation order $n-1=1$, a standard duality argument ensures that we can find a $\mu_h \in M_{kl}$ such that

$$\left\|a\frac{\partial u}{\partial n}-\mu_h\right\|_{(H^{1/2}(\Gamma_{kl}))'} \le c\bar{h}_k\left\|a\frac{\partial u}{\partial n}\right\|_{1/2,\Gamma_{kl}} \le c\bar{h}_k\|\nabla u\|_{1,\Omega_k} \le c\bar{h}_k\|u\|_{2,\Omega_k},$$

where we have used the trace theorem again. Therefore, by using the Cauchy–Schwarz inequality, one obtains

$$\begin{aligned}\sum_{\Gamma_{kl}\in\mathcal{S}}(a\frac{\partial u}{\partial n},[v_h])_{0,\Gamma_{kl}} &\leq c\left(\sum_{\Gamma_{kl}\in\mathcal{S}}\bar{h}_k^2\|u\|_{2,\Omega_k}^2\right)^{1/2}\|v_h\|_{1,\delta}\\ &\leq c\left(\sum_{\Gamma_{kl}\in\mathcal{S}}\bar{h}_k^2\|u\|_{2,\Omega_k}^2\right)^{1/2}\|v_h\|_{1,h},\end{aligned}$$

so that the quotient in (24) is bounded by $c\left(\sum_{\Gamma_{kl}\in\mathcal{S}}\bar{h}_k^2\|u\|_{2,\Omega_k}^2\right)^{1/2}$.

It remains to establish an analogous bound for the approximation error $\inf_{v_h\in V_h}\|u-v_h\|_{1,h}$ in (24). To this end, note first that $[u-v_h]=-[v_h]$ and for $v_h\in V_h$ one has $Q_{kl}^*[u-v_h]=0$. Hence,

$$\inf_{v_h\in V_h}\|u-v_h\|_{1,h}=\inf_{v_h\in V_h}\|u-v_h\|_{1,\delta}. \tag{26}$$

The right hand side of (26) has indeed been shown in [KLPV01] to be bounded by the right hand side of (23) under a certain assumption **M1**. This condition requires that the meshes induced by $\mathcal{T}_k$ and $\mathcal{T}_l$ *match along* the boundary $\partial\Gamma_{kl}$ of the interface $\Gamma_{kl}$. Condition **M1** allows one to employ local extensions from $H_{00}^{1/2}(\Gamma_{kl})$ to deal with the constraints.

In order to avoid this constraint we prefer an alternative and start with an unconstrained approximation of $u$ on each subdomain $\Omega_k$. In fact, from the inf-sup condition in Theorem 1 above and Fortin's general argument [Bra01, Remark III.4.10] we conclude that the estimate in $X_h$ yields an upper bound for the approximation in the kernel $V_h$,

$$\inf_{v_h\in V_h}\|u-v_h\|_{1,h}\leq c\inf_{v_h\in X_h}\|u-v_h\|_{1,h}. \tag{27}$$

In this case, however, the full norm has to be used, i.e., the terms $\|Q_{kl}^*[u-v_h]\|_{H_{00}^{1/2}(\Gamma_{kl})}$ have to be estimated as well (in particular, when $[u-v_h]\notin H_{00}^{1/2}(\Gamma_{kl})$). Since $u\in H^2(\Omega)$, there are many ways to construct an approximation $v_h$ in $X_h$ such that

$$\|u-v_h\|_{1,\delta}^2\leq c\sum_{k=1}^{k_{max}}\bar{h}_k^2\|u\|_{2,\Omega_k}^2, \tag{28}$$

e.g. Lagrange interpolants or Clément's quasi-interpolants would do. Thus, it remains to estimate the terms $\|Q_{kl}^*[u-v_h]\|_{H_{00}^{1/2}(\Gamma_{kl})}$ which are actually more problematic. Of course, the problem is that under the above assumptions $[u-v_h]$ is not necessarily in $H_{00}^{1/2}(\Gamma_{kl})$ so that one cannot directly bound $\|Q_{kl}^*[u-v_h]\|_{H_{00}^{1/2}(\Gamma_{kl})}$ by $\|[u-v_h]\|_{H_{00}^{1/2}(\Gamma_{kl})}$ (see (10) in **P.4**) and use then the trace theorem in order to ensure ultimately that

$$\|Q_{kl}^*[u-v_h]\|_{H_{00}^{1/2}(\Gamma_{kl})}\leq c\left(\|u-v_h\|_{1,\Omega_k}+\|u-v_h\|_{1,\Omega_l}\right), \tag{29}$$

thereby obtaining again the same bound as in (28) and thus confirming (23). Therefore we will discuss next some instances where (29) is indeed true which incidentally will shed some light on the type of obstructions arising in the general case.

First of all, since $[u]=0$ we have $[u-v_h] \in H_{00}^{1/2}(\Gamma_{kl})$ if and only if $[v_h] \in H_{00}^{1/2}(\Gamma_{kl})$. For $d=2$ this can always be arranged by choosing $v_h$ to interpolate $u$ at the cross points (without requiring the whole mortar discretization to enforce continuity at cross points!) In the case $d=3$ this is not possible. This is exactly where condition **M1** in [KLPV01] comes into play which requires that the meshes in $\Omega_k$ and $\Omega_l$ match along $\partial\Gamma_{kl}$ so that $[v_h]$ can indeed be arranged to be in $H_{00}^{1/2}(\Gamma_{kl})$, e.g., by choosing $v_h$ as the nodal interpolant. In this case (10) in **P.4** can be invoked to estimate

$$\|Q_{kl}^*[u-v_h]\|_{H_{00}^{1/2}(\Gamma_{kl})} \leq c\|[u-v_h]\|_{H_{00}^{1/2}(\Gamma_{kl})}$$

which indeed leads to (29) and thus is an instance where (23) can be confirmed. Hence in summary, one way to ensure an estimate of the type (23) is to sacrifice some of the *mortar flexibility* by enforcing interface boundary matching condition **M1**.

On the other hand, one could hope that $[u-v_h]$ fails to be in $H_{00}^{1/2}(\Gamma_{kl})$ by such a small deviation so that the smoothing caused by the application of $Q_{kl}^*$ keeps $\|Q_{kl}^*[u-v_h]\|_{H_{00}^{1/2}(\Gamma_{kl})}$ comparable to $\|[u-v_h]\|_{1/2,\Gamma_{kl}}$ which would again lead to (29). One way to pursue this line is to apply an inverse inequality

$$\|g_h\|_{H_{00}^{1/2}(\Gamma_{kl})} \leq c\|h^{-1/2}g_h\|_{0,\Gamma_{kl}}, \quad g_h \in S_{kl}^0, \tag{30}$$

which is to be understood as follows. Following [DFG+01] we denote by $h$ a *mesh function*, namely the unique piecewise linear function that interpolates the maximal diameter of all triangles sharing the corresponding nodal point. Then estimates of the form (30) are established in [DFG+01]. Now denoting by $h_k, h_l$ the mesh size functions induced on $\Gamma_{kl}$ by the mesh on $\Omega_k$ and $\Omega_l$, respectively, we have

$$\|Q_{kl}^*[u-v_h]\|_{H_{00}^{1/2}(\Gamma_{kl})} \leq c\|h_k^{-1/2}Q_{kl}^*[u-v_h]\|_{0,\Gamma_{kl}}. \tag{31}$$

Now if $Q_{kl}^*$ were sufficiently local in the sense that the following condition
**P.5**:

$$\|h_k^{-s}Q_{kl}^*v\|_{0,\Gamma_{kl}} \leq c\|h_k^{-s}v\|_{0,\Gamma_{kl}}, \tag{32}$$

holds, this combined with (32) would allow us to infer from (31) that

$$\|Q_{kl}^*[u-v_h]\|_{H_{00}^{1/2}(\Gamma_{kl})} \leq c\|h_k^{-1/2}[u-v_h]\|_{0,\Gamma_{kl}}. \tag{33}$$

We recall that $\Omega_k$ is the non-mortar side. Arranging now for $v_k := v_h|_{\Omega_k}$, $v_l := v_h|_{\Omega_l}$ that the restrictions $v_k|_{\Gamma_{kl}}$, $v_l|_{\Gamma_{kl}}$ to $\Gamma_{kl}$ are suitable local Clément approximations, standard arguments yield

$$\begin{aligned} \|h_k^{-1/2}[u-v_h]\|_{0,\Gamma_{kl}}^2 \quad &\leq \quad c\sum_{\tau\in\mathcal{T}_{kl}} \Big\{ (h_{k,\tau}^{-1/2}h_{k,\tau}^{3/2})^2|u|_{3/2,\hat{\tau}}^2 + \\ &\qquad c\sum_{\tau'\in\mathcal{T}_l|_{\Gamma_{kl}},\tau'\cap\tau\neq\emptyset} (h_{k,\tau}^{-1/2}h_{l,\tau'}^{3/2})^2|u|_{3/2,\hat{\tau}'}^2 \Big\}, \end{aligned} \tag{34}$$

where $\hat{\tau}$ is the union of supports of basis functions overlapping $\tau$. Thus introducing

$$\rho_{kl} := \max_{x\in\Gamma_{kl}} \sqrt{h_l(x)/h_k(x)}, \tag{35}$$

we obtain from (34) by summing over $\tau$

$$\|h_k^{-1/2}[u - v_h]\|^2_{0,\Gamma_{kl}} \leq \rho^2_{kl}\, (\bar{h}_k^2 + \bar{h}_l^2)\, \|u\|^2_{3/2,\Gamma_{kl}}.$$

This estimate combined with the trace theorem

$$\|Q^*_{kl}[u - v_h]\|_{H^{1/2}_{00}(\Gamma_{kl})} \leq c\rho_{kl}\, \left(\bar{h}_k\|u\|_{2,\Omega_k} + \bar{h}_l\|u\|_{2,\Omega_l}\right), \tag{36}$$

also yields the estimate (23) upon summing over $k$.

**Theorem 2** *Suppose that all the meshes $\mathcal{T}_k$ are shape regular and locally (not globally) quasi-uniform. If* **P.5** *holds, then the matching condition* **M1** *from [KLPV01] can indeed be abundoned to obtain still an estimate of the form (23), provided that there exist a uniform bound*

$$\rho_{kl} \leq c. \tag{37}$$

Of course, if the meshes are quasi-uniform, the above argument simplifies and one arrives at the situation considered in [BDW00, BD98]. Note also that (37) is a weak constraint that tends to be satisfied automatically when the meshes are determined by reasonable error estimators since a possible singularity on or near an interface will affect a neighborhood on both sides.

Finally, it should be noted that estimates of the form (23) are ultimately of limited value when dealing with highly non-quasi-uniform meshes. In fact, they would be only useful when the solution $u$ has deficient regularity so that the local $H^2$ norms (or even $H^s$-norms for $s < 2$) are not appropriate. This issue is beyond the scope of the present discussion and will be addressed elsewhere.

# Dual Bases

We wish to apply our approach to the so called *dual bases mortar method* that has been proposed in [KLPV01] for $d = 3$ and in [Woh99a] for $d = 2$, see also [BR99] for a similar approach in the wavelet context. Note that the assumptions in [KLPV01] are phrased in a somewhat different way but Lemma 3.2 in [KLPV01] relates the requirements there closely to the present formulation **P.4**. Specifically, in [KLPV01] two types of Lagrange multiplier spaces $M_h$ are discussed for piecewise linear trial functions in $X_h$. Finite volume discretizations on dual meshes are shown to satisfy **P.1** – **P.4** where, however, **P.4** can only be ensured to hold under certain restrictions on local mesh size ratios.

In contrast, the so called dual bases mortar method realizes **P.1** – **P.4** for *any* locally quasi-uniform shape regular meshes without any quantitative mesh constraints. Let us briefly recall the main ingredients.

The multiplier space $M_{kl}$ is most conveniently defined with the aid of the following mapping $F_{kl}$. Let $\tau$ be any triangle in $\mathcal{T}_{kl}$ and let for any $v \in S^0_{kl}$ the values of $v$ at the nodes $x_i$ of $\tau$ be denoted by $v_i$. Then $F_{kl}v = w$ is defined as the unique piecewise linear function on $\mathcal{T}_{kl}$ whose restriction to $\tau$ is determined by its nodal values $w_i$, for $d = 3$ as follows:

(i) $w_i := 3v_i - v_r - v_s$ for all vertices $i \neq r \neq s$ of $\tau$ when none of these vertices belongs to $\partial\Gamma_{kl}$;

(ii) If exactly one vertex, say $x_i$ lies on $\partial\Gamma_{kl}$ set $w_i := (v_r+v_s)/2$, $w_r := (5v_r-3v_s)/2$, $w_s := (5v_s - 3v_r)/2$;

(iii) If exactly two vertices $x_r, x_s$ belong to $\partial\Gamma_{kl}$ let $w_i = w_r = w_s := v_i$;

(iv) If all vertices of $\tau$ belong to $\partial\Gamma_{kl}$ set $w_i = w_r = w_s = v_q$ where $x_q$ is the nearest interior node to $\tau$.

Now let us denote by $\phi_i$, $x_i \in \mathcal{N}_{kl}$, the standard piecewise linear basis functions for $S^0_{kl}$ normalized by $\phi_i(x_r) = \delta_{i,r}$, where $\mathcal{N}_{kl}$ is the set of *interior* nodes of $\mathcal{T}_{kl}$. Let

$$\psi_i := F_{kl}\phi_i, \quad x_i \in \mathcal{N}_{kl}$$

and define $M_{kl} := \operatorname{span}\{\psi_i : x_i \in \mathcal{N}_{kl}\}$. This yields

$$(\phi_i, \psi_j)_{0,\Gamma_{kl}} = 0, \quad i \neq j, \quad (\phi_i, \psi_i)_{0,\tau} = \frac{|\tau|}{3}, \quad x_i \in \tau, \tag{38}$$

so that

$$\dim M_{kl} = \dim \mathcal{S}^0_{kl}, \tag{39}$$

which is **P.1**. More precisely, one concludes from the above relations

$$(\phi_i, \psi_j)_{0,\Gamma_{kl}} = a_i\delta_{i,j}, \quad a_i := \frac{1}{3}\sum_{\tau : x_i \in \tau} |\tau| \sim h_i^{d-1}. \tag{40}$$

Thus one has the explicit representations

$$Q_{kl}w = \sum_{x_i \in \mathcal{N}_{kl}} (w, \phi_i)_{0,\Gamma_{kl}} a_i^{-1}\psi_i, \quad Q^*_{kl}v = \sum_{x_i \in \mathcal{N}_{kl}} (v, \psi_i)_{0,\Gamma_{kl}} a_i^{-1}\phi_i. \tag{41}$$

Since constants are easily seen to be locally reproduced in $M_{kl}$, it is now easy to verify the approximation property **P.2**. Likewise biorthogonality (40) easily leads to **P.3**, see also [KLPV01] while **P.4** has also been established already in Lemma 3.2 of [KLPV01]. Hence all the requirements **P.1** – **P.4** hold in this case. Thus to apply the above reasoning we only have to discuss (32). For any $\tau \in \mathcal{T}_{kl}$ one has for $\sigma_i := \operatorname{supp}\phi_i = \operatorname{supp}\psi_i$, $\hat{\tau} := \bigcup\{\sigma_i : x_i \in \tau\}$

$$\|h^{-1/2}Q^*_{kl}g\|_{0,\tau} \leq ch_\tau^{-1/2}\sum_{x_i \in \tau} a_i^{-1}\|g\|_{0,\sigma_i}\|\psi_i\|_{0,\sigma_i}\|\phi_i\|_{0,\sigma_i} \leq ch_\tau^{-1/2}\|g\|_{0,\hat{\tau}},$$

where we have used that, in view of the $L_\infty$ normalization of the basis functions $\phi_i, \psi_i$, the local quasi-uniformity of $\mathcal{T}_{kl}$ and (40), $a_i^{-1}\|\psi_i\|_{0,\sigma_i}\|\phi_i\|_{0,\sigma_i} \sim 1$. Hence (32) follows from summing over $\tau \in \mathcal{T}_{kl}$. This confirms that **P.5** holds as well.

**Corollary 1** *The error estimate (23) holds for the dual basis mortar method provided the meshes satisfy (37).*

If the Lagrange multiplier spaces did not have local dual bases either $\phi_i$ or $\psi_i$ in (40) would have global support but would, by Demko's theorem, exhibit a certain exponential decay. This would still entail the validity of condition **P5** but under certain constraints on the local mesh size ratios, as expected.

# References

[BB99]Faker Ben Belgacem. The mortar finite element method with Lagrange multipliers. *Numer. Math.*, 84(2):173–197, 1999.

[BD98]D. Braess and W. Dahmen. Stability estimates of the mortar finite element method for 3-dimensional problems. *East-West J. Numer. Math.*, 6(4):249–264, 1998.

[BDW00]D. Braess, W. Dahmen, and C. Wieners. A multigrid algorithm for the mortar finite element method. *SIAM J. Numer. Anal.*, 37:48–69, 2000.

[BF91]F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods.* Springer-Verlag, New-York, 1991.

[BMP94]Christine Bernardi, Yvon Maday, and Anthony T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In Haim Brezis and Jacques-Louis Lions, editors, *Collège de France Seminar.* Pitman, 1994. This paper appeared as a technical report about five years earlier.

[BR99]Silvia Bertoluzza and Valerie Rerrier. The mortar wavelet method. Technical Report 99–17, Istituto di Analisi Numerica del C.N.R. Pavia, 1999. ENUMATH 99, to appear.

[Bra01]D. Braess. *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics.* Cambridge University Press, Cambridge, 2001.

[DFG$^+$01]Wolfgang Dahmen, Birgit Faermann, Ivan Graham, Wolfgang Hackbusch, and Stefan Sauter. Inverse inequalities for non-quasiuniform meshes and applications to the mortar element method. Technical Report 201, IGPM, RWTH Aachen, 2001.

[KLPV01]Chisup Kim, Raytcho Lazarov, Joseph Pasciak, and Panayot Vassilevski. Multiplier spaces for the mortar finite element method in three dimensions. *SIAM J. Numer. Anal.*, 39:519–538, 2001.

[Woh99a]B. Wohlmuth. Discretization methods and iterative solvers based on domain decomposition. Technical report, Habilitation, Department of Mathematics, Augsburg, 1999.

[Woh99b]B. Wohlmuth. Hierarchical a posteriori error estimators for mortar finite element methods with Lagrange multipliers. *SIAM J. Numer. Anal.*, 36:1636–1658, 1999.