OPTIMAL PINCHING CONSTANTS OF ODD DIMENSIONAL HOMOGENEOUS SPACES

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ABSTRACT. In this article we compute the pinching constants of all homogeneous Riemannian metrics on the Berger space $M^{13} = \mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$ and of all $\mathsf{U}(2)$-biinvariant homogeneous Riemannian metrics on the Aloff–Wallach space $W_{1,1} = \mathsf{SU}(3)/\mathsf{S}^1_{1,1}$. We prove that the optimal pinching constants are precisely $\frac{1}{37} \approx 0.0270$ in both cases. So far M^{13} and $W_{1,1}$ were only known to admit Riemannian metrics with pinching constants $\frac{16}{29\cdot 37} \approx 0.0149$.

We also investigate the optimal pinching constants for the homogeneous metrics on the other Aloff–Wallach spaces $W_{k,l} = \mathsf{SU}(3)/\mathsf{S}^1_{k,l}$. In fact, our computations cover the class of T^2-biinvariant homogeneous Riemannian metrics. This class contains all homogeneous Riemannian metrics unless $k/l < 1$, however, in the case of $W_{1,1}$ it does not represent the entire moduli of homogeneous metrics. It turns out that the optimal pinching constants are given by a strictly increasing function in $k/l \in [0, 1]$. In particular, all the optimal pinching constants are $\leq \frac{1}{37}$.

In order to determine the extremal values of the sectional curvature of a homogeneous Riemannian metric on $W_{k,l}$ we employ a systematic technique, which can be applied to other spaces as well. The computation of the pinching constants for M^{13} is reduced to the curvature computation for two proper totally geodesic submanifolds. One of them is diffeomorphic to $\mathbb{CP}^3/\mathbb{Z}_2$ and inherits an $\mathsf{Sp}(2)$-invariant Riemannian metric, and the other is $W_{1,1}$ embedded as recently found by Taimanov. This approach explains in particular the coincidence of the optimal pinching constants for $W_{1,1}$ and M^{13}.

INTRODUCTION

Pinching constants measure how much the local geometry of a compact Riemannian manifold (M, g) with positive sectional curvature K deviates from the geometry of a standard sphere. They are defined as quotients $\delta(M, g) = \frac{\min K}{\max K}$ of the extremal values of the sectional curvature. Since the universal covering space of such a manifold remains compact, it is reasonable to restrict to the simply connected case. By the Cartan–Hadamard theorem the pinching constant is well defined as a function on the space of all Riemannian metrics on any compact simply connected manifold, and evidently $\delta(M, g) \in\,]-\infty, 1]$. The upper bound of this interval is special; if $\delta(M, g) = 1$ then (M, g) is up to scaling isometric to the standard sphere.

The topological and the differentiable sphere theorems are among the first results where weaker pinching constants have been used to restrict the topological type of the underlying manifold M. If $\delta(M, g) > \delta_{\text{diff}} \approx 0.68$ the manifold M is

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diffeomorphic to the standard sphere, and if $\delta(M,g) > \frac{1}{4}$ the manifold M is homeomorphic to the sphere. The case $\delta(M,g) = \frac{1}{4}$ is completely covered by Berger's rigidity theorem. According to this theorem any Riemannian manifold (M,g) with $\delta(M,g) = \frac{1}{4}$ that is not homeomorphic to a sphere must be isometric to one of the compact rank-1 symmetric spaces $\mathbb{CP}^m$, $\mathbb{HP}^k$, and $\mathrm{Ca}\mathbb{P}^2$. Later Berger proved that for any dimension n there exists smallest constants $\delta_n \in]0, \frac{1}{4}[$, such that M is homeomorphic to the sphere or diffeomorphic to one of the compact rank-1 symmetric spaces provided $\delta(M,g) > \delta_n$. For a comprehensive survey including recent results we refer to [AM].

It is an interesting question which phenomena arise when the pinching constant is decreased even further. In particular, one would like to know at which values of the pinching constant further topological types appear. The even dimensional homogeneous case has been covered by Valiev [Va], who obtained the pinching constant $\frac{1}{64}$ for all three examples that exist according to Wallach's classification [Wa].

In the odd dimensional case the classification of homogeneous spaces that admit homogeneous Riemannian metrics with positive sectional curvature has been achieved by Bérard Bergery [BB]. The purpose of this paper is to study the curvature properties of these metrics. In particular, we are interested in their pinching constants. So far the only example that has been treated completely in the literature [El] is the 7-dimensional isotropy irreducible Berger space. Pinching constants for specific metrics on the 13-dimensional Berger space and on the Aloff–Wallach space $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ have been computed by Heintze [He1] and Huang [Hu]. We give a systematic treatment for all homogeneous metrics with positive sectional curvature on the Berger space and on all Aloff–Wallach spaces except for $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$. In the case of $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ our computations only cover a natural subclass of these homogeneous metrics. In Section 1 we describe the results in detail in their natural context.

In particular, our computations strongly suggest that $\frac{1}{37}$ is the interesting value in the odd dimensional case which corresponds in some sense to the constant $\frac{1}{64}$ in the even dimensional case. However, the number $\frac{1}{37}$ is more special than the numbers $\frac{1}{4}$ and $\frac{1}{64}$, as there is a dense set of pinching constants between 0 and $\frac{1}{37}$ at which new topological types appear.

Contents

1. Results for odd dimensional homogeneous spaces

Berger, Wallach, and Bérard Bergery have classified the homogeneous pairs (G,H) for which the spaces G/H admit G-invariant Riemannian metrics of positive sectional curvature[1]. Table 1 lists all of these pairs for which the dimensions of the spaces G/H are odd[2] and the corresponding pinching constants $\delta(\mathsf{G},\mathsf{H})$ according to the following definition:

Notation 1.1. The *pinching constant of the pair* (G,H) is the number

$$\delta(\mathsf{G},\mathsf{H}) \;:=\; \sup\bigl\{\,\delta\bigl(\mathsf{G}/\mathsf{H},\langle\cdot,\cdot\rangle\bigr)\;\bigm|\;\langle\cdot,\cdot\rangle\in\bar{\mathcal{M}}(\mathsf{G},\mathsf{H})\bigr\}.$$

Here, $\mathcal{M}(\mathsf{G},\mathsf{H})$ denotes the space of all G-invariant Riemannian metrics on G/H, and $\bar{\mathcal{M}}(\mathsf{G},\mathsf{H})$ denotes the quotient of $\mathcal{M}(\mathsf{G},\mathsf{H})$ by isometries and homotheties.

The pairs in Table 1 which are not separated by horizontal lines yield equivariantly diffeomorphic spaces. These correspond to the two normal homogeneous spaces discovered by Berger and to the Aloff–Wallach spaces. The sets of invariant Riemannian metrics on the Aloff–Wallach spaces are nested. In the case $k\neq l$ we have $\mathcal{M}\bigl(\mathsf{SU}(3),\mathsf{S}^1_{k,l}\bigr)=\mathcal{M}\bigl(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\bigr)$ and hence $\delta\bigl(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\bigr)=\delta\bigl(\mathsf{SU}(3),\mathsf{S}^1_{k,l}\bigr)$.

In this paper we compute for the first time the pinching constants of the pairs $\bigl(\mathsf{SU}(5),\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1\bigr)$, $\bigl(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\bigr)$. In particular, we give an explanation why these pinching constants coincide[3]. Moreover, we determine the continuous pinching function $\Delta:[0,1]\to[0,1]$ defined by $\Delta(k/l)=\delta\bigl(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\bigr)$.

The pairs and their pinching constants are discussed in the next three subsections. In the remaining subsections we relate our results to what is already known for these spaces and draw some consequences. Finally, we briefly state the known results for the even dimensional case.

Some of our results are partially based on numerical computations. In order to distinguish them from theorems we call them "Observations".

1.1. The Berger spaces.

The first two pairs of Table 1 were given by Berger [Be] in his classification of positively curved normal homogeneous spaces. The classification of Wallach [Wa] and Bérard Bergery [BB] shows that there does not exist any other homogeneous space G/H which admits a G-invariant Riemannian metric of positive sectional curvature and which has the same homotopy type as $\mathsf{Sp}(2)/\mathsf{SU}(2)$ or $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$.

1.1.1. *The Berger pair* $\bigl(\mathsf{Sp}(2),\mathsf{SU}(2)\bigr)$. The embedding $\mathsf{SU}(2)\hookrightarrow\mathsf{Sp}(2)$ is nonstandard, and the space $\mathsf{Sp}(2)/\mathsf{SU}(2)$ is isotropy irreducible. Hence, $\mathcal{M}\bigl(\mathsf{Sp}(2),\mathsf{SU}(2)\bigr)$ is 1-dimensional and there exists, up to scaling, just one $\mathsf{Sp}(2)$-invariant Riemannian metric on $\mathsf{Sp}(2)/\mathsf{SU}(2)$. The pinching constant $\frac{1}{37}$ has been computed by H.I. Eliasson [El].

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[1] The normalization in these classifications is as follows: G is assumed to be connected and simply connected, and the connected subgroup H of G and G have at most a discrete normal subgroups, so G acts almost effectively on the connected, simply connected space G/H.

[2] We do not consider those pairs (G,H) for which G/H is homeomorphic to some sphere. These pairs have been classified by D. Montgomery, H. Samelson, and A. Borel. The corresponding parameter spaces $\mathcal{M}(\mathsf{G},\mathsf{H})$ all contain the standard metrics. Hence, $\delta(\mathsf{G},\mathsf{H})=1$ (see [Zi]).

[3] Note, however, that up to now there is no explanation for the coincidence of the pinching constants of the two Berger pairs.

G	H	$\dim \mathsf{G}/\mathsf{H}$	$\dim \bar{\mathcal{M}}(\mathsf{G},\mathsf{H})$	$\delta(\mathsf{G},\mathsf{H})$
$\mathsf{Sp}(2)$	$\mathsf{SU}(2)$	7	0	$\frac{1}{37}$
$\mathsf{SU}(5)$	$\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1$	13	1	$\frac{1}{37}$
$\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1}$	$\mathsf{U}(2)$	7	1	$\frac{1}{37}$
$\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{1,1}$	T^2	7	3	$\Delta(1)\geq\frac{1}{37}$
$\mathsf{SU}(3)$	$\mathsf{S}^1_{1,1}$	7	6	$?\ \geq\Delta(1)$
$\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l}$	T^2	7	3	$\xrightarrow{k/l\to 1}\Delta(1)$
$\mathsf{SU}(3)$	$\mathsf{S}^1_{k,l}$	7	3	$\xrightarrow[k/l\neq 1]{k/l\to 1}\Delta(1)$

TABLE 1. Pinching constants of homogeneous pairs (G,H) for which G/H is odd dimensional, admits G-invariant Riemannian metrics of positive sectional curvature, and is not homeomorphic to some sphere. Pairs which are not separated by horizontal lines provide equivariantly diffeomorphic spaces. The sets of invariant Riemannian metrics on the corresponding spaces are nested. The continuous function $\Delta:[0,1]\to[0,1]$ is defined by the property that $\Delta(k/l)=\delta\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\big)$ (see also Figure 1).

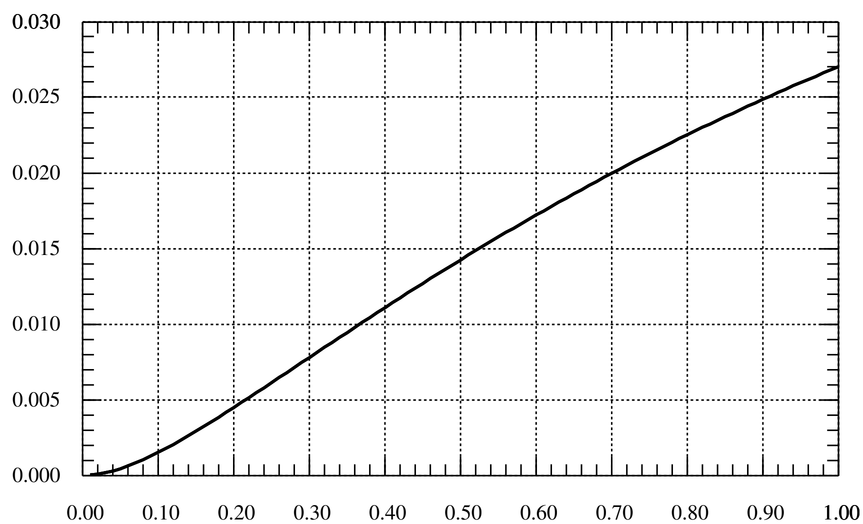

FIGURE 1. Graph of the pinching function $\Delta(k/l)=\delta\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\big)$.

1.1.2. *The Berger pair* $\big(\mathsf{SU}(5),\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1\big)$. This pair is studied in detail in Section 7. In particular, we have the following result for the 2-dimensional space of invariant Riemannian metrics $\mathcal{M}\big(\mathsf{SU}(5),\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1\big)$.

Theorem 1.2. *The pinching constant of the pair* $\big(\mathsf{SU}(5),\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1\big)$ *corresponding to the* 13*-dimensional Berger space is given by* $\delta\big(\mathsf{SU}(5),\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1\big)=\frac{1}{37}$.

So far, only the pinching constant $\frac{16}{29\cdot 37}$ of the normal homogeneous metric on $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$ has been known; it has been computed by E. Heintze [He1].

Our computation is reduced to the corresponding result for the pair $\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\big)$ that we describe in Theorem 1.5. The reduction is achieved by means of totally geodesic embeddings. The argument is outlined in Section 1.5. Full details are given in Theorem 7.10.

1.2. **The Aloff–Wallach spaces.** The pairs $\big(\mathsf{SU}(3),\mathsf{S}^1_{k,l}\big)$ are due to Aloff and Wallach [AW]. Recently, Wilking [Wi1] found the pair $\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\big)$, which yields a space diffeomorphic to $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$. In fact, this pair was missing in Berger's classification list. The pairs $\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\big)$ yield spaces that are diffeomorphic to $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$. The theorems and proofs that we outline for these pairs in this section can be found in detail in Sections 3,4, and 5.

1.2.1. *The action of gauge groups.* The pair $\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\big)$ and more generally the pairs $\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\big)$ arise from the Aloff–Wallach pairs $\big(\mathsf{SU}(3),\mathsf{S}^1_{k,l}\big)$ by a general construction (see [Zi]): Given a homogeneous pair (G,H) the group $\mathrm{Aut}(\mathsf{G},\mathsf{H})$ of automorphisms of G that map H into itself induces diffeomorphisms of G/H and acts on the space $\mathcal{M}(\mathsf{G},\mathsf{H})$ by pulling back metrics. Any subgroup $\tilde{\mathsf{H}}$ of G with $\mathsf{H}\lhd\tilde{\mathsf{H}}$ acts by conjugation on G/H and can hence be considered as a subgroup of $\mathrm{Aut}(\mathsf{G},\mathsf{H})$. In this case the fixed points of the induced action on $\mathcal{M}(\mathsf{G},\mathsf{H})$ are the $\tilde{\mathsf{H}}$-biinvariant Riemannian metrics.

Consider an $\tilde{\mathsf{H}}$-biinvariant Riemannian metric on G/H. The group $\mathsf{G}\times(\tilde{\mathsf{H}}/\mathsf{H})$ acts isometrically and transitively on G/H, the first factor on the left and the second on the right. The isotropy group of the point H with respect to this action is the diagonally embedded $\tilde{\mathsf{H}}$. Therefore, G/H is diffeomorphic to $(\mathsf{G}\times\tilde{\mathsf{H}}/\mathsf{H})/\tilde{\mathsf{H}}$. The corresponding diffeomorphism identifies the space of $\tilde{\mathsf{H}}$-biinvariant Riemannian metrics on G/H with the space $\mathcal{M}(\mathsf{G}\times\tilde{\mathsf{H}}/\mathsf{H},\tilde{\mathsf{H}})$.

1.2.2. *The Aloff–Wallach pairs* $\big(\mathsf{SU}(3),\mathsf{S}^1_{k,l}\big)$. Aloff and Wallach first showed that the homogeneous spaces $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$, where

$$\mathsf{S}^1_{k,l} \;=\; \left\{\begin{pmatrix} e^{-i(k+l)\vartheta} & 0 & 0\\ 0 & e^{ik\vartheta} & 0\\ 0 & 0 & e^{il\vartheta}\end{pmatrix} \;\middle|\; \vartheta\in\mathbb{R}\right\}, \tag{1}$$

for $k,l\in\mathbb{Z}$, $-kl(k+l)\neq 0$, admit $\mathsf{SU}(3)$-invariant Riemannian metrics of positive sectional curvature. For our purposes, it suffices[4] to consider $k,l\in\mathbb{N}$, $0<k\le l$. The countable family of Aloff–Wallach spaces can be parametrized by $k/l\in]0,1]$, which is the slope of the circle $\mathsf{S}^1_{k,l}$ in the standard maximal torus

$$\mathsf{T}^2 \;=\; \left\{\begin{pmatrix} e^{-i(\alpha+\beta)} & 0 & 0\\ 0 & e^{i\alpha} & 0\\ 0 & 0 & e^{i\beta}\end{pmatrix} \;\middle|\; \alpha,\beta\in\mathbb{R}\right\} \tag{2}$$

of $\mathsf{SU}(3)$. Hence, each rational number in $]0,1]$ corresponds to exactly one Aloff–Wallach space. Sometimes it is also interesting to consider the space $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ for $(k,l)=(0,1)$.

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[4]By the action of the Weyl group of $\mathsf{SU}(3)$ we can permute the three diagonal elements. We can also apply the complex conjugation, since we do not work in the oriented category except for Section 1.3. Moreover, we can assume $\gcd(k,l)=1$.

For $1 \le k < l$ the space $\mathcal{M}\big(\mathsf{SU}(3), \mathsf{S}^1_{k,l}\big)$ is 4-dimensional. In this case, the group $\mathrm{Aut}\big(\mathsf{SU}(3), \mathsf{S}^1_{k,l}\big)$ is the extension $\hat{\mathsf{T}}^2$ of $\mathsf{T}^2 \subset \mathsf{SU}(3)$ by the complex conjugation θ on $\mathsf{SU}(3)$. This group $\hat{\mathsf{T}}^2$ acts trivially on $\mathcal{M}\big(\mathsf{SU}(3), \mathsf{S}^1_{k,l}\big)$. In other words, all invariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ are T^2-biinvariant, i.e., $\mathcal{M}\big(\mathsf{SU}(3), \mathsf{S}^1_{k,l}\big) = \mathcal{M}\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$. The main results for this case will be described separately in Section 1.2.3.

It remains to consider the cases $k = l = 1$ and $(k,l) = (0,1)$. The space $\mathcal{M}\big(\mathsf{SU}(3), \mathsf{S}^1_{1,1}\big)$ is 10-dimensional, and $\mathrm{Aut}\big(\mathsf{SU}(3), \mathsf{S}^1_{1,1}\big)$ is the extension $\hat{\mathsf{U}}(2)_0$ of the subgroup

$$\mathsf{U}(2)_0 = \left\{ \left(\begin{smallmatrix} \det A^{-1} & 0 \\ 0 & A \end{smallmatrix} \right) \,\middle|\, A \in \mathsf{U}(2) \right\} \tag{3}$$

by the complex conjugation θ on $\mathsf{SU}(3)$. This group $\hat{\mathsf{U}}(2)_0$ acts nontrivially on $\mathcal{M}\big(\mathsf{SU}(3), \mathsf{S}^1_{1,1}\big)$ with 3-dimensional principal orbits. Therefore, the parameter space $\bar{\mathcal{M}}\big(\mathsf{SU}(3), \mathsf{S}^1_{1,1}\big)$ is 7-dimensional. In this paper we do not investigate the entire space $\mathcal{M}\big(\mathsf{SU}(3), \mathsf{S}^1_{1,1}\big)$, but only the subspaces of T^2-biinvariant Riemannian metrics and $\mathsf{U}(2)$-biinvariant Riemannian metrics. The former can be identified with $\mathcal{M}\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{1,1}, \mathsf{T}^2\big)$ and the corresponding results can be found in Section 1.2.3. The latter can be identified with $\mathcal{M}\big(\mathsf{SU}(3) \times \mathsf{U}(2)/\mathsf{S}^1_{1,1}, \mathsf{U}(2)\big)$. This case is considered in Section 1.2.4.

If $(k,l) = (0,1)$, then $\mathrm{Aut}\big(\mathsf{SU}(3), \mathsf{S}^1_{0,1}\big)$ is also given by $\hat{\mathsf{T}}^2$, but the quotient of the 6-dimensional space $\mathcal{M}\big(\mathsf{SU}(3), \mathsf{S}^1_{0,1}\big)$ by the action of $\hat{\mathsf{T}}^2$ is 5-dimensional. Each invariant Riemannian metric on this space has zero sectional curvatures, since there is a totally geodesic submanifold which is isometric to a Riemannian product $\mathbb{S}^2 \times \mathbb{S}^1$.

1.2.3. *The pairs* $\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$. The spaces $\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}\big)/\mathsf{T}^2$ are diffeomorphic to the spaces $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ by the construction explained in Section 1.2.1. Moreover, $\mathcal{M}\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$ can be identified with the space of T^2-biinvariant Riemannian metrics on $\big(\mathsf{SU}(3), \mathsf{S}^1_{k,l}\big)$, which is always 4-dimensional. Hence, we have $\dim \bar{\mathcal{M}}\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big) = 3$ for all k, l. The detailed results for these pairs can be found in Section 5.2.

The pinching constant $\delta\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$ is a function of the parameter k/l that extends continuously to the interval $[0,1]$. More precisely, there exists a continuous function $\Delta : [0,1] \to [0,1]$ with $\Delta(k/l) = \delta\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$. We compute this function Δ at 200 points. Given (k,l) we define a function $\hat{\delta}(\mathsf{SU}(3)/\mathsf{S}^1_{k,l}, \cdot\,)$ on $\bar{\mathcal{M}}\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$ and find its maximal value in order to compute $\Delta(k/l)$. The function $\hat{\delta}(\mathsf{SU}(3)/\mathsf{S}^1_{k,l}, \cdot)$ is by construction an upper bound for $\delta(\mathsf{SU}(3)/\mathsf{S}^1_{k,l}, \cdot)$. We verify that the two function coincides at the point where $\hat{\delta}(\mathsf{SU}(3)/\mathsf{S}^1_{k,l}, \cdot\,)$ attains its maximum. We use $\hat{\delta}(\mathsf{SU}(3)/\mathsf{S}^1_{k,l}, \cdot\,)$ instead of $\delta(\mathsf{SU}(3)/\mathsf{S}^1_{k,l}, \cdot\,)$ since we can describe the function $\hat{\delta}(\mathsf{SU}(3)/\mathsf{S}^1_{k,l}, \cdot)$ on $\bar{\mathcal{M}}\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$ explicitely by a case by case definition. More precisely, by means of several algebraic inequalities we devide the domain $\bar{\mathcal{M}}\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$ into several connected closed subsets with pairwise disjoint nonvoid interior. On each of these subdomains the function $\hat{\delta}(\mathsf{SU}(3)/\mathsf{S}^1_{k,l}, \cdot)$ is given explicitely. In principle it is possible to solve the optimization problem on $\bar{\mathcal{M}}\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$ explicitely. However, there are too many cases and the low dimensional strata are given by fairly complicated polynomials. For these reasons, we perform the optimization on the 3-dimensional space $\bar{\mathcal{M}}\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\big)$ numerically. The function $\hat{\delta}(\mathsf{SU}(3)/\mathsf{S}^1_{k,l},\,\cdot\,)$ is Lipschitz and we shall see that it suffices to consider a compact subdomain of $\bar{\mathcal{M}}\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\big)$.

Observation 1.3. *The function $\Delta : [0,1]\to[0,1]$ is strictly increasing (see Figure 1 on page 4). Hence, the maximum of Δ is attained at 1, i.e., $\delta\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\big)$ is maximal for $k=l=1$.*

On $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ the space $\mathcal{M}\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\big)$ can be considered as a subspace of $\mathcal{M}\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{1,1},\mathsf{T}^2\big)$, since the elements of this space correspond to the $\mathsf{U}(2)$-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$. Therefore, we have $\delta\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{1,1},\mathsf{T}^2\big)\geq\delta\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\big)$. The numerical optimization that we perform in order to compute $\delta\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{1,1},\mathsf{T}^2\big)$ shows that the T^2-biinvariant Riemanian metric on $\big(\mathsf{SU}(3),\mathsf{S}^1_{1,1}\big)$ which realizes $\delta\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{1,1},\mathsf{T}^2\big)$ has more symmetry.

Observation 1.4. *The T^2-biinvariant Riemannian metric on $\big(\mathsf{SU}(3),\mathsf{S}^1_{1,1}\big)$ realizing $\delta\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{1,1},\mathsf{T}^2\big)$ is $\mathsf{U}(2)$-biinvariant. Hence, $\delta\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{1,1},\mathsf{T}^2\big)=\delta\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\big)$.*

1.2.4. *The Wilking pair* $\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\big)$. This pair was missing in Berger's classification. Wilking found this pair by a construction closely related to the gauge group introduced in Section 1.2.1. Moreover, he proved that $\mathcal{M}\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\big)$ can be identified with the space of $\mathsf{U}(2)$-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$. The $\mathsf{U}(2)$-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ are precisely the metrics contained in the deformation that was originally studied by of Aloff and Wallach.

Theorem 1.5. *The pinching constant of the pair $\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\big)$ corresponding to the 7-dimensional Aloff–Wallach space $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ is given by $\delta\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\big)=\frac{1}{37}$.*

For more details, see Theorem 5.3. In particular, we prove that there is just one element of $\bar{\mathcal{M}}\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1},\mathsf{U}(2)\big)$ realizing the pinching constant $\frac{1}{37}$. Wilking has shown that this Riemannian metric is induced by the product of the Killing forms on $\mathfrak{su}(3)$ and $\mathfrak{so}(3)$.

1.3. **Topological classification and pinching.** There is the natural question which pairs (G,H) of Table 1 provide spaces G/H which are homotopy equivalent, homeomorphic, or diffeomorphic. As mentioned above this problem is completely solved for the Berger spaces.

For the remaining Aloff–Wallach spaces $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ the question was answered by Lashof (see [AW]), Kreck and Stolz [KS1], and Kruggel [Kr]. Lashof proved that $H^4(\mathsf{SU}(3)/\mathsf{S}^1_{k,l},\mathbb{Z})=\mathbb{Z}_r$, where $r=k^2+l^2+kl=\frac{1}{2}(k^2+l^2+(-k-l)^2)$, provided that $\gcd(k,l)=1$. A direct consequence of this is that there exist infinitely many homotopy types in the countable family of Aloff–Wallach spaces.

Kruggel has classified the Aloff–Wallach spaces up to homotopy type, Kreck and Stolz up to homeomorphisms and diffeomorphisms. They proved that

$$\mathsf{SU}(3)/\mathsf{S}^1_{k,l} \sim_{\text{hom.equ.}} \mathsf{SU}(3)/\mathsf{S}^1_{k',l'} \quad \Leftrightarrow \begin{cases} r = r' \quad \text{and} \\ kl(k+l) = \pm k'l'(k'+l') \mod r, \end{cases}$$

$$\mathsf{SU}(3)/\mathsf{S}^1_{k,l} \sim_{\text{or.homeo}} \mathsf{SU}(3)/\mathsf{S}^1_{k',l'} \quad \Leftrightarrow \begin{cases} r = r' \quad \text{and} \\ kl(k+l) = k'l'(k'+l') \mod 24 \cdot r, \end{cases}$$

$$\mathsf{SU}(3)/\mathsf{S}^1_{k,l} \sim_{\text{or.diffeo}} \mathsf{SU}(3)/\mathsf{S}^1_{k',l'} \quad \Leftrightarrow \begin{cases} r = r' \quad \text{and} \\ kl(k+l) = k'l'(k'+l') \mod 96 \cdot 7^{\lambda} \cdot r, \end{cases}$$

where $\lambda = \begin{cases} 0, & \text{if } r = 0 \mod 7, \\ 1, & \text{otherwise.} \end{cases}$

Using the conditions derived by Kreck and Stolz, D. Zagier (see [KS1]) found Aloff–Wallach spaces $\mathsf{SU}(3)/\mathsf{S}^1_{k_1,l_1}$ and $\mathsf{SU}(3)/\mathsf{S}^1_{k_2,l_2}$ which are homeomorphic but not diffeomorphic. Moreover, he found Aloff–Wallach spaces $\mathsf{SU}(3)/\mathsf{S}^1_{k_3,l_3}$ and $\mathsf{SU}(3)/\mathsf{S}^1_{k_4,l_4}$ which are diffeomorphic though their isotropy groups $\mathsf{S}^1_{k_3,l_3}$ and $\mathsf{S}^1_{k_4,l_4}$ are not congruent under an automorphism of $\mathsf{SU}(3)$. Particular numbers given by Zagier are

$$(k_1,l_1) = (-56788, 5227) \qquad \text{and} \quad (k_2,l_2) = (-42652, 61213),$$
$$(k_3,l_3) = (-4638661, 582656) \qquad \text{and} \quad (k_4,l_4) = (-2594149, 5052965),$$

respectively.

Observation 1.6. *For the Aloff–Wallach spaces given by D. Zagier we find that*

$$\delta\big(\mathsf{SU}(3), \mathsf{S}^1_{k_1,l_1}\big) \approx 0.00154 \quad \textit{whereas} \quad \delta\big(\mathsf{SU}(3), \mathsf{S}^1_{k_2,l_2}\big) \approx 0.01226, \quad \textit{and}$$
$$\delta\big(\mathsf{SU}(3), \mathsf{S}^1_{k_3,l_3}\big) \approx 0.00272 \quad \textit{whereas} \quad \delta\big(\mathsf{SU}(3), \mathsf{S}^1_{k_4,l_4}\big) \approx 0.02598.$$

In each case the optimal pinching constants for invariant Riemannian metrics differ significantly.

It follows directly from [KS2] that the invariant Riemannian metrics realizing $\delta\big(\mathsf{SU}(3), \mathsf{S}^1_{k_3,l_3}\big)$ and $\delta\big(\mathsf{SU}(3), \mathsf{S}^1_{k_4,l_4}\big)$ are contained in different connected components of the moduli space of positive scalar curvature metrics.

The question whether there are infinitely many pairs of Aloff–Wallach spaces which are homeomorphic but not diffeomorphic is open. So is the question whether there are infinitely many pairs of Aloff–Wallach spaces which are diffeomorphic though their isotropy groups are not congruent.

1.4. **Convergence properties.** By the results of our computations it is possible to improve results of Huang [Hu] and Eschenburg [Esch].

Proposition 1.7 (cf. [Hu]). *There exist sequences of pairs* $\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$ *such that the corresponding homogeneous spaces* $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ *have distinct homotopy type and such that the corresponding pinching constants* $\delta\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$ *approach* $\delta\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{1,1}, \mathsf{T}^2\big)$*, a number that is* $\geq \frac{1}{37}$ *by Theorem 1.5 and that equals* $\frac{1}{37}$ *by Observation 1.4.*

Proof. The pinching function $\Delta : [0,1] \to [0,1]$ is continuous. Hence, it suffices to consider a sequence of Aloff–Wallach spaces of distinct homotopy type such that $k/l \to 1$ (e.g. the sequence $\mathsf{SU}(3)/\mathsf{S}^1_{k,k+1}$). □

This sequence shows that for any $\kappa < \frac{1}{37}$ there exist infinitely many homotopy types of simply connected 7-dimensional manifolds whose optimal pinching constants are contained in the interval $[\kappa, 1]$. The question remains whether there are only finitely many homotopy types of simply connected manifolds whose optimal pinching constants are contained in the interval $[\frac{1}{37}, 1]$. Provided this is true, one may ask whether these manifolds must be diffeomorphic to the obvious candidates $\mathbb{S}^7$, $\mathsf{Sp}(2)/\mathsf{SU}(2)$, and $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$.

Eschenburg [Esch] constructed inhomogeneous 7-dimensional manifolds with positive sectional curvature. These are the biquotients $\mathsf{SU}(3)/\mathsf{S}^1_{k,l;p,q}$, where $\mathsf{S}^1_{k,l;p,q}$ is a subgroup of the group $\mathsf{SU}(3) \times \mathsf{SU}(3)$, which acts by left and right translation on $\mathsf{SU}(3)$. Moreover, Eschenburg proved that for any Aloff–Wallach space $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ there exists a sequence of biquotients of distinct homotopy type such that the sectional curvatures approach the sectional curvatures of $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ uniformly (for details see [Esch]). In particular this holds for the space $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$, and hence there is a much larger sequence of 7-dimensional simply connected Riemannian manifolds whose pinching constants approach $\delta\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{1,1}, \mathsf{T}^2\big)$.

Bazaikin [Ba] used Eschenburg's method in order to construct a family of inhomogeneous 13-dimensional manifolds of positive sectional curvature. Eschenburg's argument carries over to the Berger space $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$ and the Bazaikin spaces. Consequently, in dimension 13 there also exists a family of Riemannian manifolds of distinct homotopy type whose pinching constants converge to $\frac{1}{37}$.

Note that Taimanov [Ta] recently discovered totally geodesic embeddings of some of the Eschenburg examples into some of the Bazaikin spaces.

1.5. **Totally geodesic embeddings.** We will now briefly explain the equality of $\delta\big(\mathsf{SU}(3) \times \mathsf{U}(2)/\mathsf{S}^1_{1,1}, \mathsf{U}(2)\big)$ and $\delta\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big)$. The explanation is based on two totally geodesic embeddings. The first has been found by I.A. Taimanov [Ta] and the second is constructed in this paper (see Proposition 7.3).

Proposition 1.8 (Taimanov embedding)**.** *There exists an embedding*

$$\mathsf{SU}(3)/\mathsf{S}^1_{1,1} \hookrightarrow \mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$$

which is totally geodesic for all Riemannian metrics in $\mathcal{M}\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big)$. *The canonical restriction map* $\mathcal{M}\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big) \to \mathcal{M}\big(\mathsf{SU}(3), \mathsf{S}^1_{1,1}\big)$ *induced by this embedding is a bijection between* $\mathcal{M}\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big)$ *and* $\mathcal{M}\big(\mathsf{SU}(3) \times \mathsf{U}(2)/\mathsf{S}^1_{1,1}, \mathsf{U}(2)\big)$.

Proposition 1.9. *There exists an embedding*

$$\mathbb{CP}^3/\mathbb{Z}_2 \approx \mathsf{SO}(5)/\hat{\mathsf{U}}(2) \hookrightarrow \mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$$

which is totally geodesic for all Riemannian metrics in $\mathcal{M}\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big)$. *The canonical restriction map* $\mathcal{M}\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big) \to \mathcal{M}\big(\mathsf{SO}(5), \hat{\mathsf{U}}(2)\big)$ *induced by this embedding is a bijection.*

Since the second fundamental form vanishes for a totally geodesic submanifold, the Gauss equation implies that its sectional curvature coincides with that of the ambient manifold. Hence, the pinching constant of a Riemannian metric in $\mathcal{M}\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big)$ is less than or equal to the pinching constants of the corresponding Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ and $\mathsf{SO}(5)/\hat{\mathsf{U}}(2)$.

Furthermore, we prove that the extremal sectional curvatures of $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$ are attained on planes tangent to a totally geodesic submanifold which is isometric to one of the two given above, and that the sectional curvatures of a Riemannian metric in $\mathcal{M}\big(\mathsf{SO}(5), \hat{\mathsf{U}}(2)\big)$ are less extremal than or equal to those of the corresponding metric in $\mathcal{M}\big(\mathsf{SU}(3) \times \mathsf{U}(2)/\mathsf{S}^1_{1,1}, \mathsf{U}(2)\big)$. Therefore, we have $\delta\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big) = \delta\big(\mathsf{SU}(3) \times \mathsf{U}(2)/\mathsf{S}^1_{1,1}, \mathsf{U}(2)\big)$.

Note that this explanation for the equality of $\delta\big(\mathsf{SU}(3) \times \mathsf{U}(2)/\mathsf{S}^1_{1,1}, \mathsf{U}(2)\big)$ and $\delta\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big)$ is a substantial improvement of Taimanov's original argument for the equality of the pinching constants that were found by Heintze and Huang. Taimanov considered the normal homogeneous metric on $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$ and discovered that the embedding of Proposition 1.8 induces Huang's Riemannian metric on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$. Using the formulas of Heintze and Huang, Taimanov then showed that the extremal sectional curvatures are attained on planes tangent to the totally geodesically embedded $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$.

1.6. **Einstein metrics.** The first result concerning homogeneous Einstein metrics on the Aloff–Wallach spaces is due to Wang [Wg]. He proved that on each $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ there exists at least one Einstein metric in the family of invariant Riemannian metrics. Kowalski and Vlasek then showed that there exist two homothety classes of such metrics on each $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ and exactly two provided that $k < l$.

The Riemannian metric on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ realizing the optimal pinching constant for the family of $\mathsf{U}(2)$-biinvariant metrics is Einstein. This can easily be seen by the system of equations given in [Wg]. However, our numerical investigations in Section 5.2 indicate the following.

Observation 1.10. *For $1 \le k < l$ the Riemannian metrics in $\mathcal{M}\big(\mathsf{SU}(3), \mathsf{S}^1_{k,l}\big)$ realizing the pinching constants are not Einstein.*

Kowalski and Vlasek have shown that for $k/l < 0.5148$ (note $0 \le k < l$) neither of the two Einstein metrics within $\mathcal{M}\big(\mathsf{SU}(3), \mathsf{S}^1_{k,l}\big)$ has positive sectional curvature.

Rodionov [Ro] proved that there exist exactly two homothety classes of $\mathsf{SU}(5)$-invariant Einstein metrics on $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$. Only the metrics in the first homothety class have positive sectional curvature. None of these Einstein metrics coincides with the optimal pinched metric in $\mathcal{M}\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big)$. In his paper, Rodionov also classified the Einstein metrics on the flag manifolds considered below.

1.7. **The even dimensional case.** For the sake of completeness, we briefly deal with those pairs (G, H) for which G/H is even dimensional and admits G-invariant Riemannian metrics of positive sectional curvature[5]. By Wallach's classification [Wa] these pairs are just the pairs of Table 2, i.e., the pairs yielding the manifolds of flags in $\mathbb{CP}^2$, $\mathbb{HP}^2$, and $\mathrm{Ca}\mathbb{P}^2$, respectively.

The pinching constants of $(\mathsf{SU}(3), \mathsf{T}^2)$, $(\mathsf{Sp}(3)/\mathsf{Sp}(1)^3)$, and $(\mathsf{F}_4/\mathsf{Spin}(8))$ have been computed by Valiev [Va] in 1979. The pinching constant for $(\mathsf{SU}(3), \mathsf{T}^2)$ can be derived from our curvature computations for $(\mathsf{SU}(3), \mathsf{S}^1_{k,l})$. We explain this in

[5] We do not consider those pairs (G, H) for which G/H is homeomorphic to some sphere or some projective space. These pairs have been classified by Onishchik. Since all spaces $\mathcal{M}(\mathsf{G}, \mathsf{H})$ contain the standard metric, we have $\delta(\mathsf{G}, \mathsf{H}) = 1$ or $\delta(\mathsf{G}, \mathsf{H}) = \frac{1}{4}$, respectively (see [Zi]).

G	H	$\dim \mathsf{G}/\mathsf{H}$	$\dim \bar{\mathcal{M}}(\mathsf{G},\mathsf{H})$	$\delta(\mathsf{G},\mathsf{H})$
$\mathsf{SU}(3)$	T^2	6	2	$\frac{1}{64}$
$\mathsf{Sp}(3)$	$\mathsf{Sp}(1)^3$	12	2	$\frac{1}{64}$
F_4	$\mathsf{Spin}(8)$	24	2	$\frac{1}{64}$

TABLE 2. Pinching constants of homogeneous pairs (G,H) for which G/H is even dimensional, admits G-invariant Riemannian metrics of positive sectional curvature, and G/H is not homeomorphic to some sphere or some projective space. The spaces G/H are the manifolds of flags in $\mathbb{CP}^2$, $\mathbb{HP}^2$, and $\mathrm{Ca}\mathbb{P}^2$, respectively.

Section 5.3. The pinching constants of the remaining pairs $(\mathsf{Sp}(3)/\mathsf{Sp}(1)^3)$, and $(\mathsf{F}_4/\mathsf{Spin}(8))$ can be derived from the pinching constant of $(\mathsf{SU}(3),\mathsf{T}^2)$ by the totally geodesic embeddings $\mathsf{SU}(3)/\mathsf{T}^2 \hookrightarrow \mathsf{Sp}(3)/\mathsf{Sp}(1)^3 \hookrightarrow \mathsf{F}_4/\mathsf{Spin}(8)$ analogously to Section 1.5.

2. Extremal values of sectional curvature

The purpose of this section is to compile some results which are related to the extremal values of the sectional curvature function at some point p of an arbitrary Riemannian manifold (M,g). In Section 4 we present how these results can be turned into a concrete technique in order to determine the extremal values of the sectional curvature for the Aloff–Wallach spaces. This technique can be applied to other homogeneous spaces as well.

2.1. **Modified curvature operators.** We denote the inner product $g(\cdot,\cdot)_{|p}$ and the induced inner products on the exterior powers $\Lambda_k T_pM$ by $\langle\cdot,\cdot\rangle$. The curvature tensor at p is the 4-linear map

$$R : T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R},$$
$$R(X,Y;Z,W) \;=\; \langle \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z, W\rangle,$$

where ∇ is the covariant derivative with respect to g. The curvature tensor R has the following algebraic symmetries:

$$\text{(4a)}\qquad R(X,Y;Z,W) = -R(Y,X;Z,W) = -R(X,Y;W,Z),$$
$$\text{(4b)}\qquad R(X,Y;Z,W) = \quad R(Z,W;X,Y),$$
$$\text{(4c)}\quad 3\,b(R)(X,Y;Z,W) := R(X,Y;Z,W) + R(Y,Z;X,W) + R(Z,X;Y,W) = 0.$$

Because of (4a) the curvature tensor defines a bilinear form on Λ_2T_pM by the equation

$$R(X\wedge Y, Z\wedge W) \;:=\; -R(X,Y;Z,W)\,.$$

This bilinear form is symmetric because of (4b), i.e., $R \in S^2\Lambda_2T_pM$. Identity (4c) states that R is contained in the kernel of the Bianchi map $b : S^2\Lambda_2T_pM \to \Lambda^4T_pM$, where $\Lambda^4T_pM = (\Lambda_4T_pM)^*$ denotes the space of alternating 4-forms. Furthermore, there is the map $\iota : \Lambda^4 T_pM \to S^2\Lambda_2 T_pM$ assigning to each $T \in \Lambda^4 T_pM$ the symmetric bilinear form

$$\iota(T) : \Lambda_2 T_pM \times \Lambda_2 T_pM \to \mathbb{R}, \quad (\omega_1, \omega_2) \mapsto T(\omega_1 \wedge \omega_2).$$

It follows directly from the definitions that $b \circ \iota = \mathrm{id}_{\Lambda^4 T_pM}$. Hence, b is a projection and ι is an embedding.

The space $S^2\Lambda_2 T_pM$ is mapped isomorphically onto the space of symmetric endomorphisms of $\Lambda_2 T_pM$ assigning to each $S \in S^2\Lambda_2 T_pM$ the endomorphism $\hat{S}$ defined by $\langle \omega_1, \hat{S} \cdot \omega_2 \rangle = S(\omega_1, \omega_2)$ for all $\omega_1, \omega_2 \in \Lambda_2 T_pM$.

Definition 2.1. To each $T \in \Lambda^4 T_pM$ we associate the *modified curvature tensor* $R_T := R + \iota(T) \in S^2\Lambda_2 T_pM$. The corresponding symmetric endomorphism $\hat{R}_T$ of $\Lambda_2 T_pM$ is called the *modified curvature operator.*

It follows directly from the definition that $R_T(X \wedge Y; X \wedge Z) = R(X \wedge Y, X \wedge Z)$ for all $T \in \Lambda^4 T_pM$. In particular, the sectional curvature K, the Ricci curvature, and the scalar curvature at p can be derived from R_T for any $T \in \Lambda^4 T_pM$. Note that the scalar curvature is equal to $2 \operatorname{tr} \hat{R}_T$ for any $T \in \Lambda^4 T_pM$.

Lemma 2.2. *For linearly independent $X, Y \in T_pM$ and any $T \in \Lambda^4 T_pM$ we have*

$$K(\operatorname{span}\{X, Y\}) \;=\; \frac{R_T(X \wedge Y, X \wedge Y)}{\langle X \wedge Y, X \wedge Y \rangle},$$

where K denotes the sectional curvature at the point p.

The sectional curvature K at p is a function on the $2(n-2)$-dimensional Grassmannian $\mathrm{Gr}_2(T_pM) = \{E \subset T_pM \,|\, E \text{ 2-dim. subspace}\}$. The aim now is to characterize the extremal points of this function using modified curvature operators. We therefore summarize some facts about the Grassmannian $\mathrm{Gr}_2(T_pM)$ and the sectional curvature $K : \mathrm{Gr}_2(T_pM) \to \mathbb{R}$.

The Grassmannian $\mathrm{Gr}_2(T_pM)$ can be identified canonically with a submanifold of the projective space $\mathbb{P}(\Lambda_2 T_pM)$ by the well-known Plücker embedding

$$\mathrm{Gr}_2(T_pM) \to \mathbb{P}(\Lambda_2 T_pM), \quad E \mapsto \Lambda_2 E.$$

Moreover, $\mathrm{Gr}_2(T_pM)$ is covered twice by the Grassmannian $\mathrm{Gr}_2^+(T_pM)$ of oriented 2-dimensional subspaces which is identified canonically with the submanifold

$$\{\omega \in \Lambda_2 T_pM \,|\, \omega = X \wedge Y, \ |\omega| = 1\}$$

of the unit sphere in $\Lambda_2 T_pM$. The tangent space of the euclidean submanifold $\mathrm{Gr}_2^+(T_pM) \subset \Lambda_2 T_pM$ is given by

$$T_\omega \,\mathrm{Gr}_2^+(T_pM) \;=\; \{\varphi \in \Lambda_2 T_pM \,|\, \langle \varphi, \omega \rangle = 0, \ \varphi \wedge \omega = 0\}, \tag{5}$$

since $\omega \in \Lambda_2 T_pM$ is decomposable (i.e., $\omega = X \wedge Y$) if and only if $\omega \wedge \omega = 0$ (these are the well-known Plücker relations). This tangent space can more directly be described by the plane $E \subset T_pM$ onto which ω projects. The plane E and its orthogonal complement $E^\perp \subset T_pM$ provide the decomposition

$$\Lambda_2 T_pM \;=\; \Lambda_2(E \oplus E^\perp) \;\approx\; \Lambda_2 E \oplus \Lambda_2 E^\perp \oplus (E \otimes E^\perp). \tag{6}$$

Using an appropriate orthonormal base of T_pM one sees from (5) that

$$T_E \,\mathrm{Gr}_2 \, T_pM \;\approx\; T_\omega \,\mathrm{Gr}_2^+(T_pM) \;\approx\; E \otimes E^\perp.$$

The quadratic forms $\tilde{K}_T : \Lambda_2 T_pM \to \mathbb{R}$, $\omega \mapsto R_T(\omega,\omega)$ all extend the lift of K to $\mathrm{Gr}_2^+(T_pM) \subset \mathbb{S}(\Lambda_2 T_pM)$. It follows that $\omega \in \mathbb{S}(\Lambda_2 T_pM)$ is a critical point for the lift if and only if $d\tilde{K}_{T|\omega} \cdot T_\omega \,\mathrm{Gr}_2^+(T_pM) = 0$, i.e., $R_T(\omega, T_\omega \,\mathrm{Gr}_2^+(T_pM)) = 0$ for some and hence all $T \in \Lambda^4 T_pM$. Therefore, a plane $E \subset T_pM$ is critical for K if and only if $R_T|_{\Lambda_2 E \times E\otimes E^\perp} = 0$ for some and hence all $T \in \Lambda^4 T_pM$.

In the following proposition we give another characterization for the extremal points of the sectional curvature K. The first part of this proposition is actually a result of J.A. Thorpe.

Proposition 2.3 (cf. [Th1],[Th2])**.** *A plane $E \subset T_pM$ is a critical point for the sectional curvature $K : \mathrm{Gr}_2(T_pM) \to \mathbb{R}$ if and only if there exists a $T \in \Lambda^4 T_pM$ such that $\Lambda_2 E$ is an eigenspace of $\hat{R}_T$. If E is a local minimum (resp. maximum) for K then $R_T(\varphi,\varphi) - K(E)\langle\varphi,\varphi\rangle \geq 0$ (resp. ≤ 0) for all $\varphi \in E \otimes E^\perp \subset \Lambda_2 T_pM$.*

Proof. Consider the decompositions (6) and

$$\Lambda^4 T_pM \;\approx\; \Lambda^4 E \oplus \Lambda^4 E^\perp \oplus (\Lambda^3 E \otimes E^\perp) \oplus (E \otimes \Lambda^3 E^\perp) \oplus (\Lambda^2 E \otimes \Lambda^2 E^\perp).$$

If $\Lambda_2 E$ is an eigenspace of $\hat{R}_T$ then $R_T|_{\Lambda_2 E \times E \otimes E^\perp} = 0$. Hence, E is a critical point of K. Conversely, suppose that E is critical. Consider the linear form $T : \Lambda_2 E \otimes \Lambda_2 E^\perp \to \mathbb{R}$, $\omega \otimes \varphi \mapsto -R(\omega,\varphi)$ and extend this form arbitrarily to a linear form on $\Lambda_4(E \oplus E^\perp)$. Then $\Lambda_2 E$ is an eigenspace of $\hat{R}_T$, since $R_T|_{\Lambda_2 E \times \Lambda_2 E^\perp} = 0$ and also $R_T|_{\Lambda_2 E \times E \otimes E^\perp} = 0$.

Let E be a local minimum for K, let $\omega \in \mathrm{Gr}_2^+(T_pM)$ project to E, and let $c : (-\varepsilon,\varepsilon) \to \mathrm{Gr}_2^+(T_pM)$ be an arbitrary curve with $c(0) = \omega$, $\dot{c}(0) = \varphi$, and $\ddot{c}(0) = \psi$. Then $\langle\psi,\omega\rangle = -\langle\varphi,\varphi\rangle$, and we have

$$\begin{aligned} 0 \;\leq\; \tfrac{d^2}{dt^2}(\tilde{K}_T \circ c)_{|t=0} \;&=\; d^2\tilde{K}_{T|\omega}(\varphi,\varphi) + d\tilde{K}_{T|w} \cdot \psi \\ &=\; 2R_T(\varphi,\varphi) + 2R_T(\omega,\psi) \;=\; 2R_T(\varphi,\varphi) + 2K(E)\langle\varphi,\varphi\rangle, \end{aligned}$$

since ω is an eigenvector of $\hat{R}_T$. □

The $T \in \Lambda^4 T_pM$ from Proposition 2.3 is nothing but the Lagrange multiplier for the extremal value problem $\hat{K} : \mathrm{Gr}_2^+(T_pM) \to \mathbb{R}$.

In dimension 3 the sectional curvature at some point p can only have three critical values. Note that in dimension 4 there exist curvature tensors such that the derived sectional curvature has 18 critical values. Such examples can be constructed by perturbing the curvature tensor of a generic Einstein metric.

As an application of Proposition 2.3 we prove the following lemma.

Lemma 2.4. *Let N be a totally geodesic submanifold of a Riemannian manifold M and $p \in N$. Then every critical point of the sectional curvature $\mathrm{Gr}_2(T_pN) \to \mathbb{R}$ of N at p is a critical point of the sectional curvature $\mathrm{Gr}_2(T_pM) \to \mathbb{R}$ of M at p.*

Proof. Let $V = T_pN$ and $W = V^\perp$. Then

$$\begin{aligned} \Lambda_2 T_pM \;&\approx\; \Lambda_2 V \oplus \Lambda_2 W \oplus (V \otimes W), \\ \Lambda^4 T_pM \;&\approx\; \Lambda^4 V \oplus \Lambda^4 W \oplus (\Lambda^3 V \otimes W) \oplus (V \otimes \Lambda^3 W) \oplus (\Lambda^2 V \otimes \Lambda^2 W). \end{aligned}$$

We can choose a $T \in \Lambda^2 V \oplus \Lambda^2 W \subset \Lambda^4 T_pM$ such that $R_T|_{\Lambda_2 V \times \Lambda_2 W} = 0$. Moreover, we have $R_T|_{\Lambda_2 V \times V \otimes W} = R|_{\Lambda_2 V \times V \otimes W} = 0$ because of the Codazzi equation. Hence, $\hat{R}_T$ is block diagonal, and we can apply the Gauss equation and Proposition 2.3. □

The modified curvature operators also provide natural bounds for the sectional curvature by their smallest and largest eigenvalues. There is the interesting question when these inequalities are sharp, i.e., the question in which cases the smallest eigenvalue of some $\hat{R}_T$ is the minimal sectional curvature.

Proposition 2.5. *For any $T \in \Lambda^4 T_p M$ the sectional curvature K at p is bounded from below by the smallest eigenvalue $\mu(\hat{R}_T)$ of the modified curvature operator $\hat{R}_T$ and bounded from above by the largest eigenvalue $\nu(\hat{R}_T)$. In particular,*

$$\sup\{\mu(\hat{R}_T) \,|\, T \in \Lambda^4 T_p M\} \;\leq\; K \;\leq\; \inf\{\nu(\hat{R}_T) \,|\, T \in \Lambda^4 T_p M\}, \tag{7}$$

and, moreover, both the supremum and the infimum are attained at some points $T_{\max}, T_{\min} \in \Lambda^4 T_p M$. Inequality (7) is sharp for $n = \dim M \leq 4$.

For $n = 4$ the result was already known to Thorpe (cf. [Th1], [Th2]). Inequality (7) fails to be sharp in dimension ≥ 5. Suitable counterexamples have been constructed by S.M. Zoltek [Zo].

Proof. For any $T \in \Lambda^4 T_p M$ we find that

$$\begin{aligned} K \;&\leq\; \sup\{R_T(\omega,\omega) \,|\, \omega \in \mathrm{Gr}_2^+(T_p M)\} \\ &\leq\; \sup\{R_T(\omega,\omega) \,|\, \omega \in \Lambda_2 T_p M \text{ with } \langle\omega,\omega\rangle = 1\} \;=\; \nu(\hat{R}_T). \end{aligned}$$

There is a corresponding chain of inequalities for the lower bound.

The claim that the infimum is attained is nontrivial if and only if $n \geq 4$. Since $\nu(\hat{R}_T)$ depends continuously on T, it is sufficient to show that we have $\nu(\hat{R}_T) > \nu(\hat{R})$ outside a compact domain in $\Lambda^4 T_p M$. This follows from the fact that the endomorphisms $\widehat{\iota(T)}$ are traceless, which means that a large norm of T implies that the absolute values of the smallest and the largest eigenvalue of $\widehat{\iota(T)}$ are both also large.

For $n = 3$ the Grassmannian $\mathrm{Gr}_2^+(\mathbb{R}^3)$ is the whole sphere $\mathbb{S}^2 \subset \Lambda_2\mathbb{R}^3$. Therefore every unit eigenvector belonging to $\mu(\hat{R})$ resp. $\nu(\hat{R})$ is an element of $\mathrm{Gr}_2^+(\mathbb{R}^3)$, and we have $K_{\max} = \mu(\hat{R})$ and $K_{\min} = \nu(\hat{R})$.

In the case $n = 4$ we consider the the quadratic map $q : \Lambda_2\mathbb{R}^4 \to \Lambda_4\mathbb{R}^4$, $\omega \mapsto \omega\wedge\omega$ and the eigenspace W belonging to the eigenvalue $\nu(\hat{R}_{T_{\min}})$ in order to prove the sharpness of inequality (7). The image $q(W \setminus \{0\})$ either contains an element of the Grassmannian (an ω with $\omega \wedge \omega = 0$) or is included in an open half space of $\Lambda_4\mathbb{R}^4 \approx \mathbb{R}$. The latter case contradicts the minimality of $\nu(\hat{R}_{T_{\min}})$. ☐

Let H be a closed subgroup of the compact isotropy group $\mathrm{Isom}(M^n, g)_p$. The following Lemma 2.6 states that it suffices to modify the curvature tensor by an H-invariant form $T \in \Lambda^4 T_p M$ in order to give bounds for the sectional curvature by inequality (7). This has two great advantages: First, the dimensions of the optimization problems $T \mapsto \mu(\hat{R}_T)$ and $T \mapsto \nu(\hat{R}_T)$ are reduced. Second, the modified curvature operator is H-equivariant. This allows us to determine the eigenspaces of $\hat{R}_T$ using Schur's lemma. We call these operators *invariantly modified curvature operators.*

Lemma 2.6. *Let H be a closed subgroup of the compact isotropy group $\mathrm{Isom}(M^n, g)_p$. For any $T \in \Lambda^4 T_p M$ set $\bar{T} := \frac{1}{|\mathsf{H}|}\int_{\mathsf{H}} hT$. Then*

$$\mu(\hat{R}_T) \;\leq\; \mu(\hat{R}_{\bar{T}}) \quad \textit{and} \quad \nu(\hat{R}_T) \;\geq\; \nu(\hat{R}_{\bar{T}}).$$

In particular, if $\min K = \mu(\hat{R}_T)$ *for some* $T \in \Lambda^4 T_pM$ *then also* $\min K = \mu(\hat{R}_{\bar{T}})$, *and if* $\max K = \nu(\hat{R}_T)$ *for some* $T \in \Lambda^4 T_pM$ *then also* $\max K = \nu(\hat{R}_{\bar{T}})$.

Proof. Because of the invariance of R we have $\nu(\hat{R}_{hT}) = \nu(\hat{R}_T)$ for all $h \in \mathsf{H}$. Hence, this value remains unchanged if we average over H. Since $\nu(\hat{R}_T)$ is a convex function of T, Jensen's inequality proves the lemma. □

2.2. **Invariantly modified curvature operators of homogeneous spaces.** We describe how the invariantly modified curvature operators introduced in Section 2.1 can be computed for a G-invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on a homogeneous space G/H, where H is a closed subgroup of the compact group G.

Let us fix a biinvariant Riemannian metric $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ on G in the following, and let $\mathfrak{p}$ be the orthogonal comlement of $\mathfrak{h}$ with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. The adjoint representation Ad of G turns $\mathfrak{g}$ into an G-module and hence into an H-module. The space $\mathfrak{p}$ is an H-submodule of $\mathfrak{g}$. The H-representation obtained in this way is known as the isotropy representation.

The space $\mathcal{M}(\mathsf{G},\mathsf{H})$ of G-invariant metrics on G/H can be identified canonically with the space of H-invariant inner products on $\mathfrak{p}$. The space $\mathcal{M}_{\mathsf{G}}(\mathsf{G},\mathsf{H})$ of left invariant, H-biinvariant Riemannian metrics on G can be identified canonically with the space of H-invariant inner products on $\mathfrak{g}$.

There is a natural projection $\mathcal{M}_{\mathsf{G}}(\mathsf{G},\mathsf{H}) \to \mathcal{M}(\mathsf{G},\mathsf{H})$ which comes from the restriction of H-invariant inner products of $\mathfrak{g}$ to $\mathfrak{p}$. In fact, for corresponding metrics on G and G/H the canonical projection $\mathsf{G} \to \mathsf{G}/\mathsf{H}$ is a Riemannian submersion.

We extend a given H-invariant inner product on $\mathfrak{p}$ to an H-invariant inner product on $\mathfrak{g}$ by taking the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ for the components in $\mathfrak{h}$. This defines a section for the projection $\mathcal{M}_{\mathsf{G}}(\mathsf{G},\mathsf{H}) \to \mathcal{M}(\mathsf{G},\mathsf{H})$. In the following, we always denote by $\langle\cdot,\cdot\rangle$ an element of $\mathcal{M}(\mathsf{G},\mathsf{H})$ and also its image in $\mathcal{M}_{\mathsf{G}}(\mathsf{G},\mathsf{H})$ under this section.

The space $\mathcal{M}(\mathsf{G},\mathsf{H})$ can also be identified with the set of H-equivariant positive definite endomorphisms G of $\mathfrak{g}$ which restrict to the identity on $\mathfrak{h}$ in such a way that $\langle X,Y\rangle = \langle\!\langle GX,Y\rangle\!\rangle$ for all $X,Y \in \mathfrak{g}$.

Invariantly modified curvature operators $\hat{R}_T$ are H-equivariant symmetric endomorphisms of $\Lambda_2\mathfrak{p}$. Hence, Schur's lemma provides some block structure of each $\hat{R}_T$. We have to determine the irreducible representation spaces in $\Lambda_2\mathfrak{p}$ and the trivial representation spaces in $\Lambda^4\mathfrak{p}$. The equivariant map $\Lambda_2\mathfrak{p} \to \mathfrak{g}$ obtained by restriction of the Lie bracket of $\mathfrak{g}$ can be used to determine the irreducible representation spaces in $\Lambda_2\mathfrak{p}$.

In order to compute the remaining entries of the modified curvature operator we derive a new formula for the curvature tensor. This formula splits the curvature tensor in three parts each of which is contained in $S^2\Lambda_2T_pM$. The submersion $\mathsf{G} \to \mathsf{G}/\mathsf{H}$ is emphasized and O'Neill's formula just modifies the second of the three parts. The inversion of the metric is explicit and enters only the third part. This is a great advantage over the previously known formulas when one deals with spaces whose isotropy representations contain several equivalent subrepresentations.

Lemma 2.7. *The curvature tensor R of the compact homogeneous space G/H endowed with the invariant metric $\langle\cdot,\cdot\rangle = \langle\!\langle GX, Y\rangle\!\rangle$ is given by*

$$\begin{aligned} R(X\wedge Y; Z\wedge W) \;&=\; \tfrac{1}{2}\big(\langle\!\langle B_-(X,Y),[Z,W]\rangle\!\rangle + \langle\!\langle [X,Y], B_-(Z,W)\rangle\!\rangle\big) \\ &\quad + \tfrac{1}{4}\Big(\langle [X,W], P\,[Y,Z]\rangle - \langle [X,Z], P\,[Y,W]\rangle - 2\,\langle [X,Y], P\,[Z,W]\rangle\Big) \\ &\quad + \;\Big(\langle\!\langle B_+(X,W), G^{-1}B_+(Y,Z)\rangle\!\rangle - \langle\!\langle B_+(X,Z), G^{-1}B_+(Y,W)\rangle\!\rangle\Big), \end{aligned}$$

where the symmetric resp. skew symmetric bilinear maps B_+ and B_- are defined by $B_+(X,Y) = \frac{1}{2}\big([X,GY]+[Y,GX]\big)$, $B_-(X,Y) = \frac{1}{2}\big([GX,Y]+[X,GY]\big)$ and $P:\mathfrak{g}\to\mathfrak{g}$ denotes the H-equivariant projection onto the H-invariant subspace $\mathfrak{p}$.

Proof. By O'Neill's formula (see [Bs]) we have to add the term

$$\tfrac{1}{4}\big(\langle [X,W],(\mathbb{1}-P)[Y,Z]\rangle - \langle [X,Z],(\mathbb{1}-P)[Y,W]\rangle - 2\,\langle [X,Y],(\mathbb{1}-P)[Z,W]\rangle\big)$$

to the curvature tensor of the left invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on the Lie group G. The curvature tensor of $(\mathsf{G},\langle\cdot,\cdot\rangle)$ is given in Lemma B.1 in Appendix B. $\square$

The isotropy representation $\mathfrak{p}$ of each of the spaces we consider in the rest of the paper decomposes into several mutually inequivalent irreducible representation spaces $\mathfrak{p}_1,\dots\mathfrak{p}_N$. By Schur's Lemma the equivariant positive definite endomorphisms of $\mathfrak{p}$ are given by diagonal matrices with positive entries $t_1,\dots t_n$. In this case, it is easy to invert these endomorphisms explicitly, and we have $B_+(X,Y) = \frac{t_k - t_j}{2}\,[X,Y]$ and $B_-(X,Y) = \frac{t_k+t_j}{2}\,[X,Y]$ for $X\in\mathfrak{p}_j$ and $Y\in\mathfrak{p}_k$.

3. Curvature of the Aloff–Wallach spaces

In this section and the next two sections we derive the statements for the Aloff–Wallach spaces $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ that were outlined in Section 1.2.2. The major object of this section is the computation of the curvature operator for the T^2-biinvariant Riemannian metrics on each $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$. We determine the extremal values of the sectional curvature by means of the curvature operator in the next section.

In order to stress that the action of the Weyl group of $\mathsf{SU}(3)$ induces diffeomorphisms between the Aloff–Wallach spaces, we denote the subgroup $\mathsf{S}^1_{k,l}\subset\mathsf{SU}(3)$ given in equation (1) in Section 1.2.2 also by

$$\mathsf{S}^1_{\vec{k}} \;=\; \left\{\begin{pmatrix} e^{-ik_0\vartheta} & 0 & 0\\ 0 & e^{ik_1\vartheta} & 0\\ 0 & 0 & e^{ik_2\vartheta}\end{pmatrix} \,\middle|\, \vartheta\in\mathbb{R}\right\},$$

where $\vec{k} = (k_0,k_1,k_2)$, i.e., $k_0 = -(k+l)$, $k_1 = k$, and $k_2 = l$. We shall see that this notation simplifies the computations in this section a lot. Nevertheless, we continue to assume $0<k\le l$ in the statements, since this assumption simplifies the premises.

3.1. **Invariant Riemannian metrics.** Each group $\mathsf{S}^1_{\vec{k}}$ is a normal subgroup of the maximal torus $\mathsf{T}^2 = \{\mathrm{diag}(e^{i\vartheta_0},e^{i\vartheta_1},e^{i\vartheta_2})\,|\,\vartheta_0+\vartheta_1+\vartheta_2 = 0\}$ of $\mathsf{SU}(3)$. The three singular directions in this hexagonal torus T^2 correspond to the subgroups

$$\begin{aligned}
\mathsf{U}(2)_0 &= \left\{ \left(\begin{smallmatrix} \det A^{-1} & 0 \\ 0 & A \end{smallmatrix} \right) \,\middle|\, A \in \mathsf{U}(2) \right\} &&= \mathrm{Centralizer}(\mathsf{S}^1_{(-2,1,1)}),\\
\mathsf{U}(2)_1 &= \left\{ \left(\begin{smallmatrix} A_{11} & 0 & A_{12} \\ 0 & \det A^{-1} & 0 \\ A_{21} & 0 & A_{22} \end{smallmatrix} \right) \,\middle|\, A \in \mathsf{U}(2) \right\} &&= \mathrm{Centralizer}(\mathsf{S}^1_{(1,-2,1)}),\\
\mathsf{U}(2)_2 &= \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & \det A^{-1} \end{smallmatrix} \right) \,\middle|\, A \in \mathsf{U}(2) \right\} &&= \mathrm{Centralizer}(\mathsf{S}^1_{(1,1,-2)})
\end{aligned}$$

of $\mathsf{SU}(3)$, which all contain T^2 and which are permuted by the action of the Weyl group.

The involution $\theta : \mathsf{SU}(3) \to \mathsf{SU}(3)$, $A \mapsto \bar{A}$, induces the inversions of the groups $\mathsf{S}^1_{\vec{k}}$, T^2 and therefore involutions $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}} \to \mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$, which we also denote by θ. Moreover, θ maps each $\mathsf{U}(2)_j$ to itself. We extend the groups $\mathsf{S}^1_{\vec{k}}$, T^2, and $\mathsf{U}(2)_j$ by θ and denote these extensions by $\hat{\mathsf{S}}^1_{\vec{k}}$, $\hat{\mathsf{T}}^2$, and $\hat{\mathsf{U}}(2)_j$. All these groups are subgroups of the corresponding index 2 extesions $\widehat{\mathsf{SU}}(3)$ of $\mathsf{SU}(3)$ itself by θ.

The Lie algebra of $\mathsf{S}^1_{\vec{k}}$ is spanned by the vector

$$\zeta \;:=\; \sqrt{2}\, i \begin{pmatrix} \hat{k}_0 & 0 & 0 \\ 0 & \hat{k}_1 & 0 \\ 0 & 0 & \hat{k}_2 \end{pmatrix},$$

where $\hat{k} = \vec{k}/|\vec{k}|$ is the unit vector belonging to $\vec{k}$. Note that ζ is a unit vector with respect to the $\mathrm{Ad}_{\mathsf{SU}(3)}$-invariant inner product $\langle\!\langle X, Y \rangle\!\rangle := -\frac{1}{2} \operatorname{Re} \operatorname{tr}(XY)$ on $\mathfrak{su}(3)$.

The tangent space of $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$ can be identified with the orthogonal complement of ζ in $\mathfrak{su}(3)$. We shall see in the following that the $\mathrm{Ad}_{\mathsf{T}^2}$-invariant inner products on this orthogonal complement can be described by four numbers $t, s_0, s_1, s_2 > 0$.

The orthogonal complement $\bar{\varepsilon}$ of ζ in the Lie algebra $\mathfrak{t}_2$ of T^2 with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is spanned by the vector

$$\epsilon \;:=\; \frac{2i}{\sqrt{6t}} \begin{pmatrix} \hat{k}_1 - \hat{k}_0 & 0 & 0 \\ 0 & \hat{k}_2 - \hat{k}_0 & 0 \\ 0 & 0 & \hat{k}_0 - \hat{k}_1 \end{pmatrix}$$

We identify the orthogonal complement of $\mathfrak{t}_2$ in $\mathfrak{su}(3)$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ with three copies of $\mathbb{C}$ by the map

$$\mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2 \to \mathfrak{su}(3), \quad (z_0, z_1, z_2) \mapsto \begin{pmatrix} 0 & -\frac{z_2}{\sqrt{s_2}} & \frac{\bar{z}_1}{\sqrt{s_1}} \\ \frac{\bar{z}_2}{\sqrt{s_2}} & 0 & -\frac{z_0}{\sqrt{s_0}} \\ -\frac{z_1}{\sqrt{s_1}} & \frac{\bar{z}_0}{\sqrt{s_0}} & 0 \end{pmatrix}. \tag{8}$$

Now we restrict the $\mathrm{Ad}_{\mathsf{SU}(3)}$-invariant inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ to the spaces $\bar{\varepsilon}$, $\mathbb{C}_0$, $\mathbb{C}_1$, $\mathbb{C}_2$, and we define an inner product $\langle \cdot, \cdot \rangle_{t,\vec{s}}$, where $\vec{s} = (s_0, s_1, s_2)$, by scaling these restrictions by t, s_0, s_1, s_2, respectively.

Lemma 3.1. *The Lie algebra $\mathfrak{su}(3)$ decomposes as $\mathfrak{t}_2 \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2$ with respect to the adjoint action of $\mathsf{T}^2 \subset \mathsf{SU}(3)$. This action is trivial on $\mathfrak{t}_2 = \operatorname{span}\{\zeta, \epsilon\}$, and we have $\mathrm{Ad}_{\mathrm{diag}(e^{i\vartheta_0}, e^{i\vartheta_1}, e^{i\vartheta_2})}\, z = e^{i(\vartheta_{j+1} - \vartheta_{j-1})} z$ on each $\mathbb{C}_j$. The involution θ acts as $-\mathrm{id}$ on $\mathfrak{t}_2$ and as complex conjugation on each $\mathbb{C}_j$.*

Corollary 3.2. *The cone of T^2-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$ can be identified with the cone of inner products $\langle \cdot, \cdot \rangle_{t,\vec{s}}$ on $\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2$. Under this identification the cone of $\mathsf{U}(2)$-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ is mapped bijectively onto the cone of inner products $\langle \cdot, \cdot \rangle_{t,s} := \langle \cdot, \cdot \rangle_{t,(t,s,s)}$ on $(\bar{\varepsilon} \oplus \mathbb{C}_0) \oplus$*

$(\mathbb{C}_1 \oplus \mathbb{C}_2) \approx \mathbb{R}^2 \oplus \mathbb{C}_2$. The involution θ preserves the decomposition $\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2$. Hence, each T^2-biinvariant Riemannian metric is $\hat{\mathsf{T}}^2$-biinvariant and each $\mathsf{U}(2)$-biinvariant Riemannian metric is $\hat{\mathsf{U}}(2)$-biinvariant.

The statements in Section 1.2.2 concerning the $\mathsf{S}^1_{1,1}$-biinvariant Riemannian metrics can be easily deduced from Lemma 3.1. In particular it is easy to see that for $1 \le k < l$ the $\mathsf{S}^1_{1,1}$-biinvariant Riemannian metrics are T^2-biinvariant.

We denote by e_j the standard base of the $\mathbb{C}_j$ and by $\langle \cdot, \cdot \rangle$ the inner product on $\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2$ such that ϵ, e_0, ie_0, e_1, ie_1, e_2, ie_2 is an orthonormal base. Because of our identification (8) this standard inner product corresponds to the metric $\langle \cdot, \cdot \rangle_{t,\vec{s}}$ throughout. We denote the inner products induced by $\langle \cdot, \cdot \rangle$ on the exterior products $\Lambda^k(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2)$ also by $\langle \cdot, \cdot \rangle$.

The Lie brackets are equivariant under simultaneous cyclic permutations of the three triples $\hat{k}$, $\vec{s}$, (e_0, e_1, e_2), and (ie_0, ie_1, ie_2), since we have chosen the identification (8) properly.

$$
\begin{aligned}
&[\zeta, e_j] = \sqrt{2}(\hat{k}_{j+1} - \hat{k}_{j-1})\, ie_j, \qquad [\zeta, ie_j] = -\sqrt{2}(\hat{k}_{j+1} - \hat{k}_{j-1})\, e_j,\\
&[\epsilon, e_j] = -\sqrt{\tfrac{6}{t}}\,\hat{k}_j\, ie_j, \qquad [\epsilon, ie_j] = \sqrt{\tfrac{6}{t}}\,\hat{k}_j\, e_j,\\
&[e_{j+1}, e_{j-1}] = -[ie_{j+1}, ie_{j-1}] = \sqrt{\tfrac{s_j}{s_{j+1}s_{j-1}}}\, e_j,\\
&[ie_{j+1}, e_{j-1}] = [e_{j+1}, ie_{j-1}] = -\sqrt{\tfrac{s_j}{s_{j+1}s_{j-1}}}\, ie_j,\\
&[e_j, ie_j] = \tfrac{2}{s_0}\big(\tfrac{1}{\sqrt{2}}(\hat{k}_{j+1} - \hat{k}_{j-1})\,\zeta - \tfrac{\sqrt{6t}}{2}\hat{k}_0\,\epsilon\big).
\end{aligned}
\tag{9}
$$

3.2. **Fibrations and totally geodesic submanifolds.** We now describe some totally geodesic submanifolds of the Aloff–Wallach spaces.

Proposition 3.3. *The Aloff–Wallach spaces $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$ admit the following four fibrations:*

$$
\begin{aligned}
\mathsf{S}^1 = \mathsf{T}^2/\mathsf{S}^1_{\vec{k}} &\to \mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}} \to \mathsf{SU}(3)/\mathsf{T}^2,\\
\mathbb{S}^3/\mathbb{Z}_{|k_j|} = \mathsf{U}(2)_j/\mathsf{S}^1_{\vec{k}} &\to \mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}} \to \mathsf{SU}(3)/\mathsf{U}(2)_j = \mathbb{CP}^2 \text{ for } j = 0, 1, 2.
\end{aligned}
$$

Any T^2-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$ induces Berger metrics on each of the fibres $\mathbb{S}^3/\mathbb{Z}_{|k_j|}$. The fibres of all four fibrations are totally geodesic for every T^2-biinvariant Riemannian metric $\langle \cdot, \cdot \rangle_{t,s}$ on $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$.

Proof. The centralizer of a generic element of the maximal torus T^2 in $\mathsf{SU}(3)$ is T^2 itself. Lemma A.3 shows that $\mathsf{T}^2/\mathsf{S}^1_{\vec{k}} = \mathsf{S}^1$ is a totally geodesic subspace of $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$. In order to see that the other fibres are totally geodesic consider the singular directions $\mathrm{diag}(1, -1, -1)$, $\mathrm{diag}(-1, 1, -1)$, and $\mathrm{diag}(-1, -1, 1)$ of T^2 and proceed analogously. It is easy to see that the each fibre $\mathsf{U}(2)_j/\mathsf{S}^1_{\vec{k}}$ is diffeomorphic to the lense space $\mathbb{S}^3/\mathbb{Z}_{|k_j|}$. Since $\mathsf{U}(2)_j$ acts isometrically on these spaces the induced metrics are Berger metrics. □

Note that for an arbitrary invariant Riemannian metric on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ only the fibre $\mathsf{U}(2)_0/\mathsf{S}^1_{1,1} = \mathbb{RP}^3$ is neccessarily totally geodesic.

Lemma 3.4. *The connected component of the fixed point set of* $\theta : \mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}} \to \mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$ *containing* $[\mathbb{1}]$ *is the subspace* $\mathsf{SO}(3)/\mathbb{Z}_2$. *Hence,* $\mathsf{SO}(3)/\mathbb{Z}_2$ *is totally geodesic for any* T^2*-biinvariant Riemannian metric on* $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$.

Proof. The group $\widehat{\mathsf{SU}}(3)$ acts transitively on $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$. The connected component of the centralizer of θ that contains $\mathbb{1}$ can easily be recognized as the subgroup $\mathsf{SO}(3) \subset \mathsf{SU}(3)$. We now apply Lemma A.3. □

The space $\mathsf{SU}(3)/\mathsf{S}^1_{0,1}$ is exceptional since one of the fibrations of Proposition 3.3 takes a slightly different form:

Proposition 3.5 (cf. [BB])**.** *The space* $\mathsf{SU}(3)/\mathsf{S}^1_{0,1}$ *admits the fibration*

$$\mathbb{S}^1 \times \mathbb{S}^2 = \mathsf{U}(2)_1/\mathsf{S}^1_{0,1} \;\to\; \mathsf{SU}(3)/\mathsf{S}^1_{0,1} \;\to\; \mathsf{SU}(3)/\mathsf{U}(2)_1 = \mathbb{CP}^2.$$

The fibre is totally geodesic for all invariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{0,1}$. *Hence, the sectional curvature function of each of these metrics has zeros.*

Proof. The unique order 2 element in $\mathsf{S}^1_{0,1}$ is $\mathrm{diag}(-1,1,-1)$. The centralizer of this element in $\mathsf{SU}(3)$ is the subgroup $\mathsf{U}(2)_1$. Lemma A.3 yields that $\mathsf{U}(2)_1/\mathsf{S}^1_{0,1}$ is a totally geodesic subspace of $\mathsf{SU}(3)/\mathsf{S}^1_{0,1}$. This subspace is isometric to a Riemannian product $\mathbb{S}^1 \times \mathbb{S}^2$. □

The fact that $\mathsf{SU}(3)/\mathsf{S}^1_{0,1}$ does not admit invariant Riemannian metrics of positive sectional curvature has been already proved by Bérard Bergery. In fact, he proves that $\mathbb{S}^1 \times \mathbb{S}^2$ is a totally geodesic submanifold of $\mathsf{SU}(3)/\mathsf{S}^1_{0,1}$ in a different way (see [BB, 8(B) and lemme 6]).

3.3. **The invariantly modified curvature operators.** We compute the invariantly modified curvature operators defined in Section 2 for the T^2-biinvariant Riemannian metrics $\langle\cdot,\cdot\rangle_{t,\vec{s}}$ on $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$. Recall that the involution θ also acts isometrically for each of these metrics and that we have identified each of these metrics in dependence of $(t,\vec{s})$ with the standard inner product $\langle\cdot,\cdot\rangle$ on $\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2$. Thus each of the invariantly modified curvature operators is an $\hat{\mathsf{T}}^2$-equivariant symmetric linear map $\hat{R}_T : \Lambda_2(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2) \to \Lambda_2(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2)$ and it is useful to determine the irreducible $\hat{\mathsf{T}}^2$-representations in $\Lambda_2(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2)$ and the invariant elements in $\Lambda^4(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2)$.

Lemma 3.6. *The second exterior power of the tangent space* $T_{[\mathbb{1}]}\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$ *decomposes with respect to the* $\hat{\mathsf{T}}^2$*-action as*

$$\Lambda_2(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2) \;\approx\; 3\bar{\varepsilon} \,\oplus\, \big(\bigoplus_{j=0}^{2} V_j\big) \,\oplus\, \big(\bigoplus_{j=0}^{2}(\mathbb{C}^a_j \oplus \mathbb{C}^b_j)\big),$$

where $3\bar{\varepsilon}$ *is a trivial* 3*-dimensional* T^2*-representation, on which the involution* θ *acts as* $-\mathrm{id}$. *Each space* V_j *is a complex* 1*-dimensional* $\hat{\mathsf{T}}^2$*-representation, and for each* $j = 0,1,2$ *the spaces* $\mathbb{C}^a_j$ *and* $\mathbb{C}^b_j$ *are equivalent complex* 1*-dimensional* $\hat{\mathsf{T}}^2$*-representations. Apart from the equivalence of* $\mathbb{C}^a_j$ *and* $\mathbb{C}^b_j$ *all summands are mutually inequivalent.*

Proof. By the standard decomposition formula for $\Lambda_2(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2)$ we have

$$\Lambda_2(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2) \;\approx\; \big(\bigoplus_{j=0}^{2} \Lambda_2\mathbb{C}_j\big) \oplus \big(\bigoplus_{j=0}^{2}(\bar{\varepsilon} \otimes \mathbb{C}_j)\big) \oplus \big(\bigoplus_{j=0}^{2}(\mathbb{C}_{j+1} \otimes \mathbb{C}_{j-1})\big),$$

where the T^2-representation $3\bar{\varepsilon} := \oplus_{j=0}^{2} \Lambda_2 \mathbb{C}_j$ is trivial, and $\mathbb{C}_j^a := \bar{\varepsilon} \otimes \mathbb{C}_j$ is equivalent to $\mathbb{C}_j$ for $j \in \{0, 1, 2\}$. We define the unit vectors $\epsilon_j := e_j \wedge ie_j$, $e_j^a := \epsilon \wedge ie_j$, and $ie_j^a := -\epsilon \wedge e_j$. In order to decompose $\mathbb{C}_{j+1} \otimes \mathbb{C}_{j-1}$ into irreducible representations we consider the equivariant map $\mathbb{C}_{j+1} \otimes \mathbb{C}_{j-1} \to \mathbb{C}_j$ which is obtained as a restriction of the Lie bracket. Up to complex conjugation, this map is just the usual cross product. Its kernel V_j is therefore spanned (over $\mathbb{R}$) by the unit vectors

$$\begin{aligned} v_j &:= \tfrac{1}{\sqrt{2}}(e_{j+1} \wedge e_{j-1} + ie_{j+1} \wedge ie_{j-1}) \quad \text{and} \\ iv_j &:= \tfrac{1}{\sqrt{2}}(e_{j+1} \wedge ie_{j-1} - ie_{j+1} \wedge e_{j-1}). \end{aligned}$$

By Schur's lemma the orthogonal complement $\mathbb{C}_j^b$ of the kernel is equivalent to $\mathbb{C}_j$. It is spanned by the vectors

$$\begin{aligned} e_j^b &:= \tfrac{1}{\sqrt{2}}(e_{j+1} \wedge e_{j-1} - ie_{j+1} \wedge ie_{j-1}) \quad \text{and} \\ ie_j^b &:= -\tfrac{1}{\sqrt{2}}(e_{j+1} \wedge ie_{j-1} + ie_{j+1} \wedge e_{j-1}) \end{aligned}$$

We have $\mathrm{Ad}_{\mathrm{diag}(e^{i\vartheta_0}, e^{i\vartheta_1}, e^{i\vartheta_2})}\, v_j = e^{i(2\vartheta_j - \vartheta_{j+1} - \vartheta_{j-1})}\, v_j = e^{3i\vartheta_j}\, v_j$, which proves that the summands are inequivalent. $\square$

Corollary 3.7. *Any $\hat{\mathsf{T}}^2$-equivariant homomorphism $A : \Lambda_2(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2) \to \Lambda_2(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2)$ consists of the following seven blocks:*

$$A|_{V_j}, \quad A|_{\mathbb{C}_j^a \oplus \mathbb{C}_j^b}, \quad \textit{and} \quad A|_{3\bar{\varepsilon}}.$$

If A is symmetric, each $A|_{V_j}$ is given by a real number, each $A|_{\mathbb{C}_j^a \oplus \mathbb{C}_j^b}$ by a symmetric 2×2-matrix, and $A|_{3\bar{\varepsilon}}$ by a real symmetric 3×3-matrix.

Proof. Schur's lemma implies the block structure. If A is symmetric, the restrictions $A|_{V_j}$, $A|_{\mathbb{C}_j^a \oplus \mathbb{C}_j^b}$, and $A|_{3\bar{\varepsilon}}$ are also symmetric. Since V_j is irreducible, $A|_{V_j}$ has exactly one real eigenvalue, which is given by $\langle v_j, Av_j \rangle$. The map $A|_{\mathbb{C}_j^a \oplus \mathbb{C}_j^b}$ is $\mathbb{C}$-linear, because it is an equivariant $\mathbb{R}$-linear map, and the multiplication by i on each $\mathbb{C}_j$ can be realized by an element of T^2. The involution θ acts as complex conjugation on $\mathbb{C}_j^a \oplus \mathbb{C}_j^b$. Therefore $A|_{\mathbb{C}_j^a \oplus \mathbb{C}_j^b}$ is represented by the real 2×2-matrix

$$A|_{\mathbb{C}_j^a \oplus \mathbb{C}_j^b} = \begin{pmatrix} \langle Ae_j^a, e_j^a \rangle & \langle Ae_j^b, e_j^a \rangle \\ \langle Ae_j^a, e_j^b \rangle & \langle Ae_j^b, e_j^b \rangle \end{pmatrix},$$

which is a symmetric matrix because of the symmetry of A. The endomorphism $A|_{3\bar{\varepsilon}}$ is given by the real symmetric 3×3-matrix $(\langle A\epsilon_l, \epsilon_m \rangle)_{l,m=0,1,2}$. $\square$

Lemma 3.8. *The $\hat{\mathsf{T}}^2$-invariant elements of $\Lambda^4(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2)$ are precisely the duals of*

$$\begin{aligned} &\eta_0 \cdot e_1 \wedge ie_1 \wedge e_2 \wedge ie_2 + \eta_1 \cdot e_2 \wedge ie_2 \wedge e_0 \wedge ie_0 + \eta_2 \cdot e_0 \wedge ie_0 \wedge e_1 \wedge ie_1 \\ &+ \xi \cdot \tfrac{1}{2}\epsilon \wedge (ie_0 \wedge ie_1 \wedge ie_2 - ie_0 \wedge e_1 \wedge e_2 - e_0 \wedge ie_1 \wedge e_2 - e_0 \wedge e_1 \wedge ie_2) \end{aligned} \tag{10}$$

with respect to $\langle \cdot, \cdot \rangle$, where $\eta_0, \eta_1, \eta_2, \xi \in \mathbb{R}$.

Proof. The standard decomposition formula for $\Lambda_4(\bar\varepsilon \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2)$ yields

$$\begin{aligned}\Lambda_4(\bar\varepsilon \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2) \;\approx\;& (\bar\varepsilon \otimes \mathbb{C}_0 \otimes \mathbb{C}_1 \otimes \mathbb{C}_2)\\ &\oplus \big(\bigoplus_{j=0}^{2}(\Lambda_2\mathbb{C}_{j+1} \otimes \Lambda_2\mathbb{C}_{j-1})\big)\\ &\oplus \big(\bigoplus_{j=0}^{2}(\Lambda_2\mathbb{C}_{j} \otimes \mathbb{C}_{j+1} \otimes \mathbb{C}_{j-1})\big)\\ &\oplus \big(\bigoplus_{j=0}^{2}\big((\bar\varepsilon \otimes \Lambda_2\mathbb{C}_j \otimes \mathbb{C}_{j+1}) \oplus (\bar\varepsilon \otimes \Lambda_2\mathbb{C}_j \otimes \mathbb{C}_{j-1})\big)\big).\end{aligned}$$

Using Lemma 3.6 and standard equivalences it is easy to see that there are no trivial representations in the last two rows. The space $\oplus_{j=0}^{2}(\Lambda_2\mathbb{C}_{j+1} \otimes \Lambda_2\mathbb{C}_{j-1})$ is apparently a trivial representation. Therefore, we just have to find the trivial representations in $\bar\varepsilon \otimes \mathbb{C}_0 \otimes \mathbb{C}_1 \otimes \mathbb{C}_2 \cong \mathbb{C}_0 \otimes \mathbb{C}_1 \otimes \mathbb{C}_2$. A direct computation yields

$$\begin{aligned}&\ker[\zeta,\cdot] \cap \ker[\epsilon,\cdot] \;=\\ &\quad \operatorname{span}\{\epsilon \wedge (ie_0 \wedge ie_1 \wedge ie_2 - ie_0 \wedge e_1 \wedge e_2 - e_0 \wedge ie_1 \wedge e_2 - e_0 \wedge e_1 \wedge ie_2),\\ &\qquad \epsilon \wedge (e_0 \wedge e_1 \wedge e_2 - e_0 \wedge ie_1 \wedge ie_2 - ie_0 \wedge e_1 \wedge ie_2 - ie_0 \wedge ie_1 \wedge e_2)\},\end{aligned}$$

where only the first vector is invariant under the involution θ. □

In order to give the modified curvature operators in such a way that they are easy to use in the following sections we perform an affine linear transformation of the triple (η_0, η_1, η_2). Let

$$\sigma_1 = s_0 + s_1 + s_2, \qquad \sigma_2 = s_1 s_2 + s_0 s_2 + s_0 s_1, \qquad \sigma_3 = s_0 s_1 s_2$$

denote the elementary symmetric polynomials in $\vec{s}$,

$$\sigma \;:=\; 4\sigma_2 - \sigma_1^2 \;=\; 2s_1s_2 + 2s_0s_2 + 2s_0s_1 - s_0^2 - s_1^2 - s_2^2,$$

and T be the dual of (10) with respect to $\langle\cdot,\cdot\rangle$, where

$$\eta_j \;=\; -\tfrac{\sigma}{2\sigma_3} + \tfrac{s_{j-1} - s_j + s_{j+1}}{s_{j-1}s_{j+1}} - \tfrac{3t\hat{k}_{j-1}\hat{k}_{j+1}}{2s_{j-1}s_{j+1}} - \hat\eta_j.$$

We denote the modified curvature operator $\hat{R}_T$ also by $\hat{R}(\hat\eta_0, \hat\eta_1, \hat\eta_1, \xi)$.

Proposition 3.9. *The modified curvature operator $\hat{R}_T$ of the Riemannian metric $\langle\cdot,\cdot\rangle_{t,\vec{s}}$ on the Aloff–Wallach space $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$ is given by*

$$\begin{aligned}\hat{R}_T|_{V_j} \;&=\; \hat\eta_j,\\ \hat{R}_T|_{3\bar\varepsilon} \;&=\; \begin{pmatrix} a_0 & b_2 - \hat\eta_2 & b_1 - \hat\eta_1\\ b_2 - \hat\eta_2 & a_1 & b_0 - \hat\eta_0\\ b_1 - \hat\eta_1 & b_0 - \hat\eta_0 & a_2\end{pmatrix},\\ \hat{R}_T|_{\mathbb{C}_j^a \oplus \mathbb{C}_j^b} \;&=\; \begin{pmatrix} c_j & \sqrt{2}(\xi_j - \xi)\\ \sqrt{2}(\xi_j - \xi) & 2d_j - \hat\eta_j\end{pmatrix},\end{aligned}$$

where

$$a_j = \tfrac{4}{s_j} - \tfrac{9t\hat{k}_j^2}{2s_j^2}, \qquad b_j = -\tfrac{\sigma}{\sigma_3} + \tfrac{s_{j-1}-s_j+s_{j+1}}{s_{j-1}s_{j+1}} - \tfrac{9t\hat{k}_{j-1}\hat{k}_{j+1}}{2s_{j-1}s_{j+1}},$$
$$c_j = \tfrac{3t\hat{k}_j^2}{2s_j^2}, \qquad d_j = -\tfrac{\sigma}{4\sigma_3} + \tfrac{s_{j-1}-s_j+s_{j+1}}{s_{j-1}s_{j+1}},$$
$$\xi_j = \sqrt{\tfrac{3t}{8\sigma_3}}\big(\hat{k}_j + \hat{k}_{j-1}\tfrac{s_j-s_{j+1}}{s_{j-1}} + \hat{k}_{j+1}\tfrac{s_j-s_{j-1}}{s_{j+1}}\big).$$

Proof. We first compute the curvature tensor on $\mathbb{C}_{j+1}\otimes\mathbb{C}_{j-1}\times\mathbb{C}_{j+1}\otimes\mathbb{C}_{j-1}$. Specializing the formula of Lemma 2.7 for $X, Z \in \mathbb{C}_{j+1}$ and $Y, W \in \mathbb{C}_{j-1}$ yields (note that $[\mathbb{C}_{j+1}, \mathbb{C}_{j-1}] = \mathbb{C}_j$, $[\mathbb{C}_j, \mathbb{C}_j] = \mathfrak{t}_2$)

$$\begin{aligned} R(X\wedge Y; Z\wedge W) \;=\;& \tfrac{s_{j-1}-s_j+s_{j+1}}{2}\langle\!\langle [X,Y],[Z,W]\rangle\!\rangle \\ &+ \tfrac{s_j^2-(s_{j-1}-s_{j+1})^2}{4s_j}\langle\!\langle [X,W],[Y,Z]\rangle\!\rangle - \tfrac{t}{4}\langle\!\langle [X,Z],P[Y,W]\rangle\!\rangle, \end{aligned}$$

where P is the equivariant projection $\mathfrak{su}(3)\to\bar{\varepsilon}\oplus\mathbb{C}_0\oplus\mathbb{C}_1\oplus\mathbb{C}_2$. We use this formula to compute $R_T(v_j, v_j)$, which directly gives $\hat{R}_T|_{V_j}$, and $R_T(e_j^b, e_j^b)$.

Using Lemma 2.7 directly we then compute $R_T(e_j^a, e_j^a)$.

Next, we specialize the curvature tensor to $\bar{\varepsilon}\otimes\mathbb{C}_j\times\mathbb{C}_{j+1}\otimes\mathbb{C}_{j-1}$. By the formula of Lemma 2.7 we obtain for $X\in\bar{\varepsilon}$, $Y\in\mathbb{C}_j$, $Z\in\mathbb{C}_{j+1}$, and $W\in\mathbb{C}_{j-1}$

$$\begin{aligned} R(X\wedge Y; Z\wedge W) \;=\;& \tfrac{t}{4}\Big(\langle\!\langle [X,Y],[Z,W]\rangle\!\rangle + \tfrac{s_j-s_{j+1}}{s_{j-1}}\langle\!\langle [X,W],[Y,Z]\rangle\!\rangle \\ &+ \tfrac{s_{j-1}-s_j}{s_{j+1}}\langle\!\langle [X,Z],[Y,W]\rangle\!\rangle\Big). \end{aligned}$$

Applying this formula gives $R_T(e_j^a, e_j^b)$. By Corollary 3.7 we have finished the computation of $\hat{R}_T|_{\mathbb{C}_j^a\oplus\mathbb{C}_j^b}$.

For $\hat{R}_T|_{3\bar{\varepsilon}}$ we use Lemma 2.7 in order to compute $R_T(\epsilon_j, \epsilon_j)$ directly and in order to specialize the curvature tensor formula of Lemma 2.7 to $\Lambda_2\mathbb{C}_{j+1}\times\Lambda_2\mathbb{C}_{j-1}$, which yields

$$R(X\wedge Y; Z\wedge W) \;=\; \tfrac{\sigma}{4s_j}\langle\!\langle [X,Y],[Z,W]\rangle\!\rangle - \tfrac{t}{2}\langle\!\langle [X,Y],P[Z,W]\rangle\!\rangle$$

for $X, Y\in\mathbb{C}_{j+1}$ and $Z, W\in\mathbb{C}_{j-1}$. With this latter formula it is straightforward to compute $R_T(\epsilon_{j+1}, \epsilon_{j-1})$. □

4. Extremal sectional curvatures on $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$

In this section we reduce the extremal value problem for the sectional curvature of a T^2-biinvariant Riemannian metric on an Aloff–Wallach space $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$ to several low dimensional extremal value problems. The relation between the solutions of the low dimensional problems and the extremal values of the sectional curvature is established by means of the invariantly modified curvature operators $\hat{R}_T$. In this way we employ the $\hat{\mathsf{T}}^2$-equivariance of these operators as much as possible. Moreover, we continue to keep all our computations equivariant with respect to the induced actions of the Weyl group of $\mathsf{SU}(3)$.

The key ingredient in determining the extremal values of the sectional curvature is Proposition 2.5: The sectional curvature is bounded from below by the smallest eigenvalue and bounded from above by the largest eigenvalue of any invariantly modified curvature operator $\hat{R}_T$. One might be tempted to search for forms T_1 and T_2 corresponding to the optimal bounds and to prove that there is a Grassmann element in the eigenspace belonging to the smallest resp. largest eigenvalue of $\hat{R}_{T_1}$ resp. $\hat{R}_{T_2}$. However, in this form the problem seems to be quite untractable.

Therefore, we rather proceed in the opposite way: We consider the intersection of certain invariant subspaces of $\Lambda_2(T_{[\mathbb{1}]}\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}})$ with the Grassmannian $\mathrm{Gr}_2^+(T_{[\mathbb{1}]}\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}})$ and determine the vectors in these intersections which provide the extremal sectional curvatures. We then prove that these vectors become eigenvectors belonging to the smallest resp. largest eigenvalues of properly chosen modified curvature operators $\hat{R}_T$.

4.1. **Sectional curvature and invariant subspaces of $\Lambda_2(T_{[\mathbb{1}]}\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}})$.** To begin with, we introduce the quantities that play the central role in all subsequent estimates in this and the next section.

Definition 4.1. Let K denote the sectional curvature of a T^2-biinvariant Riemannian metric $\langle\cdot,\cdot\rangle_{t,\vec{s}}$ on an Aloff–Wallach space $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$. We set

$$\begin{aligned}\hat{\lambda} &:= \inf\bigl\{K(\omega)\,\big|\,\omega\in\mathrm{Gr}_2^+(\bar{\varepsilon}\oplus\mathbb{C}_0\oplus\mathbb{C}_1\oplus\mathbb{C}_2)\cap\bigl(\bigoplus_{j=0}^{2}V_j\bigr)\oplus\bigl(\bigoplus_{j=0}^{2}(\mathbb{C}^a_j\oplus\mathbb{C}^b_j)\bigr)\bigr\},\\ \bar{\lambda} &:= \inf\bigl\{K(\omega)\,\big|\,\omega\in\mathrm{Gr}_2^+(\bar{\varepsilon}\oplus\mathbb{C}_0\oplus\mathbb{C}_1\oplus\mathbb{C}_2)\cap 3\bar{\varepsilon}\oplus V_0\oplus V_1\oplus V_2\bigr\},\ \text{and}\\ \Lambda_j &:= \sup\bigl\{K(\omega)\,\big|\,\omega\in\mathrm{Gr}_2^+(\bar{\varepsilon}\oplus\mathbb{C}_0\oplus\mathbb{C}_1\oplus\mathbb{C}_2)\cap 3\bar{\varepsilon}\oplus\mathbb{C}^a_j\oplus\mathbb{C}^b_j\bigr\}.\end{aligned}$$

We shall use properly modified curvature operators $\hat{R}_T$ in order to find computationally more suitable characterizations of the numbers $\hat{\lambda}$, $\bar{\lambda}$, and Λ_j. Moreover, we use these characterizations to show that $\min K = \min\{\hat{\lambda},\bar{\lambda}\}$ resp. $\max K = \max\{\Lambda_0,\Lambda_1,\Lambda_2\}$ for closed sets $\bar{\Omega}_{\min}$ resp. $\bar{\Omega}_{\max}$ of T^2-biinvariant Riemannian metrics with nonvoid interior $\Omega_{\min}$ resp. $\Omega_{\max}$. By the very definition

$$\delta(\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}},\langle\cdot,\cdot\rangle_{t,\vec{s}} \le \tfrac{\min\{\hat{\lambda},\bar{\lambda}\}}{\max\{\Lambda_0,\Lambda_1,\Lambda_2\}}$$

everywhere. In other words, we have introduced an upper bound for the pinching constant even outside $\bar{\Omega}_{\min}\cap\bar{\Omega}_{\max}$. This bound plays a crucial role in the next section, where we maximize the pinching constants over the space $\mathcal{M}\bigl(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{\vec{k}},\mathsf{T}^2\bigr)$ of T^2-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$.

Next we discuss the intersection of the invariant subspaces used in the definition of $\hat{\lambda}$, $\bar{\lambda}$, and Λ_j with the Grassmannian of oriented 2-planes. We first deal with invariant subspaces of $V_0\oplus V_1\oplus V_2\oplus\bigl(\oplus_{j=0}^2(\mathbb{C}^a_j\oplus\mathbb{C}^b_j)\bigr)$.

Lemma 4.2. *Each invariant subspace of $\mathbb{C}^a_j\oplus\mathbb{C}^b_j$ is either $\{0\}$, $\mathbb{C}^a_j\oplus\mathbb{C}^b_j$ itself, or of the form $\mathbb{C}^\gamma_j := \mathbb{C}\cdot e^\gamma_j$, where $e^\gamma_j = \cos(\gamma)\,e^a_j+\sin(\gamma)\,e^b_j$ for $\gamma\in\,]-\frac{\pi}{2},\frac{\pi}{2}]$. The space $\mathbb{C}^\gamma_j$ intersects the Grassmannian of oriented 2-planes if and only if $\gamma=0$, i.e., if and only if $\mathbb{C}^\gamma_j=\mathbb{C}^a_j$.*

Proof. This first part of the lemma is an immediate consequence of Lemma 3.1. It is easy to verify that only the elements of $\mathbb{C}^a_j\approx\bar{\varepsilon}\otimes\mathbb{C}_j$ are decomposable. □

Lemma 4.3. *Each space $V_j\oplus\mathbb{C}^b_j$ intersects $\mathrm{Gr}_2^+(\bar{\varepsilon}\oplus\mathbb{C}_0\oplus\mathbb{C}_1\oplus\mathbb{C}_2)$.*

Proof. It is easy to check that $\frac{1}{\sqrt{2}}(v_j+e^b_j)$ is contained in the intersection. □

Lemma 4.4. *The invariant subspace $V_{j_1}\oplus V_{j_2}\oplus\mathbb{C}^{\gamma_{j_1}}_{j_1}\oplus\mathbb{C}^{\gamma_{j_2}}_{j_2}$ intersects the Grassmannian of oriented 2-planes provided that $\gamma_{j_1}\in\,]0,\frac{\pi}{2}[$ and $\gamma_{j_2}\in\,]-\frac{\pi}{2},0[$.*

Proof. A straightforward computation shows that the vector

$$\sqrt{-\cos\gamma_{j_2}\sin\gamma_{j_2}}\,(e_{j_1}^{\gamma_{j_1}}+\sin(\gamma_{j_1})\,v_{j_1})+\sqrt{\cos\gamma_{j_1}\sin\gamma_{j_1}}\,(ie_{j_2}^{\gamma_{j_2}}-\sin(\gamma_{j_2})\,iv_{j_2}).$$

satisfies the Plücker equation. Hence, the corresponding unit vector belongs to the Grassmannian. □

Next we deal with the spaces $3\bar{\varepsilon}\oplus V_0\oplus V_1\oplus V_2$ and $3\bar{\varepsilon}\oplus\mathbb{C}_j^b$.

Lemma 4.5. *The intersection of the invariant subspace* $3\bar{\varepsilon}\oplus V_0\oplus V_1\oplus V_2$ *with* $\mathrm{Gr}_2^+(\bar{\varepsilon}\oplus\mathbb{C}_0\oplus\mathbb{C}_1\oplus\mathbb{C}_2)$ *is precisely the union of* $\hat{\mathsf{T}}^2$*-orbits through the points*

$$\omega_x \;=\; x_0\epsilon_0+x_1\epsilon_1+x_2\epsilon_2-\sqrt{2x_1x_2}\,iv_0+\sqrt{2x_2x_0}\,v_1+\sqrt{2x_0x_1}\,v_2,$$

where $x_0,x_1,x_2\geq 0$ *and* $x_0+x_1+x_2=1$.

Proof. Consider an arbitrary vector $\omega=x_0\epsilon_0+x_1\epsilon_1+x_2\epsilon_2+z_0v_0+z_1v_1+z_2v_2$, where x_0,x_1,x_2 are real and z_0,z_1,z_2 are complex. The equation $\omega\wedge\omega=0$ yields the conditions $z_{j+1}z_{j-1}=-\sqrt{2}\,i\,x_j\bar{z}_j$. These imply $|z_j|^2=2x_{j+1}x_{j-1}$ and $z_0z_1z_2=-\sqrt{8}\,i\,x_0x_1x_2$. Hence, $z_j=\sqrt{2x_{j+1}x_{j-1}}\,e^{i\beta_j}$, where $\beta_0+\beta_1+\beta_2=-\frac{\pi}{2}+2\pi\mathbb{Z}$. The torus T^2 acts transitively on these elements. □

Lemma 4.6. *For each* $j\in\{0,1,2\}$ *the intersection of the invariant subspace* $3\bar{\varepsilon}\oplus\mathbb{C}_j^b$ *with* $\mathrm{Gr}_2^+(\bar{\varepsilon}\oplus\mathbb{C}_0\oplus\mathbb{C}_1\oplus\mathbb{C}_2)$ *is precisely the union of the* $\hat{\mathsf{T}}^2$*-orbits through the points*

$$\omega_x \;=\; x_{j+1}\epsilon_{j+1}+x_{j-1}\epsilon_{j-1}+\sqrt{2x_{j+1}x_{j-1}}\,e_j^b,$$

where $x_{j+1}\geq 0$, $x_{j-1}\leq 0$, *and* $x_{j+1}-x_{j-1}=1$.

Proof. Proceed as in the preceding proof. □

The Grassmann elements of the two lemmas above can nicely be described in the following way: The subspace $3\bar{\varepsilon}$ intersects the Grassmannian of oriented 2-planes in the set $\{\pm\epsilon_0,\pm\epsilon_1,\pm\epsilon_2\}$. Consider the octahedron

$$O:=\big\{x=x_0\epsilon_0+x_1\epsilon_1+x_2\epsilon_2\;\big|\;|x_0|+|x_1|+|x_2|=1\big\}$$

in $3\bar{\varepsilon}$ whose vertices are given by the elements $\pm\epsilon_0$, $\pm\epsilon_1$, $\pm\epsilon_2$. We denote the faces of O by $O_{1,1,1}$, $O_{-1,1,1}$, and so on. The involution θ interchanges two opposite faces. Lemma 4.5 states that there is a bijection between $O_{1,1,1}\cup O_{-1,-1,-1}$ and the T^2-orbits of Grassmann elements in $3\bar{\varepsilon}\oplus V_0\oplus V_1\oplus V_2$. Analogous statements holds for the other faces. For example there is a bijection between the union of the two faces $O_{-1,1,1}$ and $O_{1,-1,-1}$ and the T^2-orbits of Grassmann elements in $3\bar{\varepsilon}\oplus V_0\oplus\mathbb{C}_1^b\oplus\mathbb{C}_2^b$. In this case, the bijection is obtained by means of the vectors

$$\omega_x \;=\; x_0\epsilon_0+x_1\epsilon_1+x_2\epsilon_2-\sqrt{2x_1x_2}\,iv_0+\sqrt{-2x_2x_0}\,e_1^b+\sqrt{-2x_0x_1}\,e_2^b.$$

for $x_0\leq 0$, $x_1,x_2\geq 0$, and $-x_0+x_1+x_2=1$.

4.2. **Minimal sectional curvature.** We now restrict to those invariant modifications $T\in\Lambda^4(\bar{\varepsilon}\oplus\mathbb{C}_0\oplus\mathbb{C}_1\oplus\mathbb{C}_2)$ of the curvature operator $\hat{R}$ which are adapted to the spaces $\big(\oplus_{j=0}^2V_j\big)\oplus\big(\oplus_{j=0}^2(\mathbb{C}_j^a\oplus\mathbb{C}_j^b)\big)$ and $3\bar{\varepsilon}\oplus V_0\oplus V_1\oplus V_2$. We set $\hat{\eta}_0=\hat{\eta}_1=\hat{\eta}_2=\lambda$ in Proposition 3.9. Thereby we turn $V_0\oplus V_1\oplus V_2$ into an eigenspace belonging to the eigenvalue λ of the resulting modified curvature operator. Let us denote this operator by $\hat{R}(\lambda,\xi)$, and let us write $\hat{R}(\lambda)$ instead of $\hat{R}(\lambda,\xi)|_{3\bar{\varepsilon}}$, since this block does not depend on the ξ_j. Moreover, we denote by $\mu(\,\cdot\,)$ and $\nu(\,\cdot\,)$ the smallest and the largest eigenvalue of a matrix, respectively.

We first give an alternative characterization of $\hat{\lambda}$ introduced in Definition 4.1:

Proposition 4.7. *Let $\lambda_j(\xi)$ denote the smallest root of the quadratic polynomial $p_\xi(\lambda) = \det(\hat{R}(\lambda,\xi)|_{\mathbb{C}_j^a \oplus \mathbb{C}_j^b} - \lambda 1\!\!1)$. Then*

$$\hat{\lambda} \;=\; \sup_\xi \min\{\lambda_0(\xi), \lambda_1(\xi), \lambda_2(\xi)\}.$$

The supremum is attained at some point $\xi_{\mathrm{m}} \in [\min\{\xi_j\}, \max\{\xi_j\}]$. Moreover,

$$\lambda = \mu\big(\hat{R}(\lambda, \xi_{\mathrm{m}})|_{(\oplus_{j=0}^2 V_j)\oplus(\oplus_{j=0}^2(\mathbb{C}_j^a \oplus \mathbb{C}_j^b))}\big)$$

holds for $\lambda \le \hat{\lambda}$.

Proof. We use the notation of Proposition 3.9. For any ξ,

$$\lambda_j(\xi) \;=\; \tfrac{c_j + d_j}{2} - \sqrt{(\tfrac{c_j - d_j}{2})^2 + (\xi_j - \xi)^2}$$

is the smallest root of the polynomial $p_\xi(\lambda)$. Let $\lambda_{\mathrm{m}} = \sup_\xi \min\{\lambda_0(\xi), \lambda_1(\xi), \lambda_2(\xi)\}$. The supremum is attained at some point $\xi_{\mathrm{m}} \in [\min\{\xi_j\}, \max\{\xi_j\}]$, since each $\lambda_j(\xi)$ is a hyperbola with maximum at ξ_j. Moreover, the supremum is given by the maximum of one hyperbola or by the intersection of two of them, i.e., there exists a $j_0 \in \{0,1,2\}$ with $\xi_{\mathrm{m}} = \xi_{j_0}$ and $\lambda_{\mathrm{m}} = \lambda_{j_0}(\xi_{\mathrm{m}})$, or there exist j_1, j_2 with $\xi_{j_1} < \xi_{\mathrm{m}} < \xi_{j_2}$ and $\lambda_{\mathrm{m}} = \lambda_{j_1}(\xi_{\mathrm{m}}) = \lambda_{j_2}(\xi_{\mathrm{m}})$.

We first prove that $\hat{\lambda} \le \lambda_{\mathrm{m}}$. Consider the modified curvature operator $\hat{R}(\lambda_{\mathrm{m}}, \xi_{\mathrm{m}})$. If $\xi_{\mathrm{m}} = \xi_{j_0}$ and $\lambda_{\mathrm{m}} = \lambda_{j_0}(\xi_{\mathrm{m}})$ then by the definition of λ_{j_0} the space $\mathbb{C}_{j_0}^a$ or the space $V_{j_0} \oplus \mathbb{C}_{j_0}^b$ is an eigenspace belonging to the eigenvalue λ_{m} of $\hat{R}(\lambda_{\mathrm{m}}, \xi_{\mathrm{m}})$. Both these spaces intersect $\mathrm{Gr}_2^+(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2)$ by Lemma 4.2 and Lemma 4.3. Hence, Proposition 2.3 implies that λ_{m} is a critical value of the sectional curvature at $[1\!\!1]$. If $\xi_{j_1} < \xi_{\mathrm{m}} < \xi_{j_2}$ and $\lambda_{\mathrm{m}} = \lambda_{j_1}(\xi_{\mathrm{m}}) = \lambda_{j_2}(\xi_{\mathrm{m}})$ then the off-diagonal elements in $\hat{R}(\lambda_{\mathrm{m}}, \xi_{\mathrm{m}})|_{\mathbb{C}_{j_1}^a \oplus \mathbb{C}_{j_1}^b}$ resp. $\hat{R}(\lambda_{\mathrm{m}}, \xi_{\mathrm{m}})|_{\mathbb{C}_{j_2}^a \oplus \mathbb{C}_{j_2}^b}$ are negative resp. positive. Therefore, the components of the eigenvectors belonging to the smallest eigenvalues of these operators have equal resp. different signs. In other words, there exists $\gamma_{j_1} \in \,]0, \frac{\pi}{2}[$ and $\gamma_{j_2} \in \,]-\frac{\pi}{2}, 0[$ such that $V_{j_1} \oplus V_{j_2} \oplus \mathbb{C}_{j_1}^{\gamma_{j_1}} \oplus \mathbb{C}_{j_2}^{\gamma_{j_2}}$ is an eigenspace of $\hat{R}(\lambda_{\mathrm{m}}, \xi_{\mathrm{m}})$ belonging to the eigenvalue λ_{m}. This space intersects $\mathrm{Gr}_2^+(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2)$ by Lemma 4.4.

For the reverse inequality $\lambda_{\mathrm{m}} \le \hat{\lambda}$ notice that for any given ξ the operator $\hat{R}(\lambda, \xi)|_{\mathbb{C}_j^a \oplus \mathbb{C}_j^b}$ is positive definite if $\lambda \to -\infty$. Hence we have $\lambda \le \mu\big(\hat{R}(\lambda,\xi)|_{\mathbb{C}_j^a \oplus \mathbb{C}_j^b}\big)$ for $\lambda \le \lambda_j(\xi)$. In particular, if $\xi = \xi_{\mathrm{m}}$ we obtain

$$\lambda \le \mu\big(\hat{R}(\lambda, \xi_{\mathrm{m}})|_{\oplus_{j=0}^2(\mathbb{C}_j^a \oplus \mathbb{C}_j^b)}\big) \quad \text{for any } \lambda \le \lambda_{\mathrm{m}}.$$

Hence, we have

$$\lambda_{\mathrm{m}} = \mu\big(\hat{R}(\lambda_{\mathrm{m}}, \xi_{\mathrm{m}})|_{(\oplus_{j=0}^2 V_j)\oplus(\oplus_{j=0}^2(\mathbb{C}_j^a \oplus \mathbb{C}_j^b))}\big).$$

Proposition 2.5 now implies that $\lambda_{\mathrm{m}} \le K(\omega)$ for all $\omega \in \mathrm{Gr}_2^+(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2) \cap (\oplus_{j=0}^2 V_j) \oplus \big(\oplus_{j=0}^2(\mathbb{C}_j^a \oplus \mathbb{C}_j^b)\big)$. □

The next lemma can be used to compute the number $\hat{\lambda}$ algorithmically.

Lemma 4.8. *The number $\hat{\lambda}$ is the maximum of $\min\{\lambda_0(\xi), \lambda_1(\xi), \lambda_2(\xi)\}$, where ξ is one of the ξ_j or an intersection point of two hyperbolas. Any intersection point* *of the hyperbolas* $\lambda_1(\xi)$ *and* $\lambda_2(\xi)$ *is a root of the quadratic polynomial*

$$\begin{aligned}
&2\xi^2\big(4(\xi_1-\xi_2)^2-(c_1+d_1-c_2-d_2)^2\big)\\
&\quad-2\xi\Big((\xi_1-\xi_2)\big((c_1-d_1)^2-(c_2-d_2)^2\big)\\
&\qquad\quad+(\xi_1+\xi_2)\big(4(\xi_1-\xi_2)^2-(c_1+d_1-c_2-d_2)^2\big)\Big)\\
&\quad+2(\xi_1^2-\xi_2^2)^2-(\xi_1^2+\xi_2^2)(c_1+d_1-c_2-d_2)^2\\
&\quad+(\xi_1^2-\xi_2^2)\big((c_1-d_1)^2-(c_2-d_2)^2\big)\\
&\quad+(c_1d_1-c_2d_2)(-c_1^2-d_1^2+c_2^2+d_2^2)\\
&\qquad\quad+(c_1d_1+c_2d_2)(c_1+d_1-c_2-d_2)^2.
\end{aligned}$$

Proof. Straightforward. □

Next we characterize the number $\bar\lambda$ in terms of the modified curvature operators. For this purpose we set $A := \hat R(0) = \left(\begin{smallmatrix} a_0 & b_2 & b_1\\ b_2 & a_1 & b_0\\ b_1 & b_0 & a_2\end{smallmatrix}\right)$, where we use the notation of Proposition 3.9.

Lemma 4.9. *We have*

$$\bar\lambda = \inf\big\{x^tA\,x \,\big|\, x_0,x_1,x_2\ge 0,\ x_0+x_1+x_2=1\big\}.$$

Proof. A straightforward computation shows that the sectional curvature of the plane ω_x given in Lemma 4.5 is $R(\omega_x,\omega_x) = x^t\hat R(0)\,x$. □

In order to determine $\bar\lambda$ we first ignore the restriction $x_0,x_1,x_2\ge 0$.

Lemma 4.10. *Let* $\tilde f$ *denote the restriction of the quadratic form* $f:\mathbb{R}^3\to\mathbb{R}$, $x\mapsto x^tAx$ *to the affine hyperplane* $\sum_{j=0}^2 x_j = 1$. *Suppose* $\sum_{j=0}^2(a_j-b_j)\ge 0$ *and* $D_0+D_1+D_2>0$, *where*

$$D = \begin{pmatrix} D_0\\ D_1\\ D_2\end{pmatrix} = A^{\mathrm{ad}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix} a_1a_2-b_0^2-a_1b_1-a_2b_2+b_0b_1+b_2b_0\\ a_2a_0-b_1^2-a_2b_2-a_0b_0+b_1b_2+b_0b_1\\ a_0a_1-b_2^2-a_0b_0-a_1b_1+b_2b_0+b_1b_2\end{pmatrix}.$$

Then $\tilde f$ *attains a minimum at the point* $\tilde x = \frac{D}{D_0+D_1+D_2}$. *This minimum is given by* $\tilde\lambda = \frac{\det A}{D_0+D_1+D_2}$.

Proof. Let Q denote the orthogonal projection onto the orthogonal complement of $(1,1,1)^t$. The Hessian $d^2\tilde f_{|x}$ is positive definite if the coefficients α_1 and α_2 in the characteristic polynomial $\det(Q^tAQ-\lambda\mathbb{1}) = \sum_{l=0}^3(-1)^l\,\alpha_{n-l}\,\lambda^l$ satisfy $\alpha_1\ge 0$ and $\alpha_2>0$. It is easy to check $\alpha_1 = \frac{2}{3}\sum_{j=0}^2(a_j-b_j)$ and $\alpha_2 = (D_0+D_1+D_2)/3$. Therefore, $\tilde f$ is strictly convex.

A point $\tilde x$ is critical for $\tilde f$ if and only if there exists a constant $\kappa\in\mathbb{R}$ such that $A\tilde x = \kappa(1,1,1)^t$. Multiplication of the latter equation with A^{ad} yields $(\det A)\tilde x = \kappa D$. If $\det A\ne 0$ there exists a unique point where the minimum of $\tilde f$ is attained, which is given by $\tilde x = \frac{D}{D_0+D_1+D_2}$. Since this point depends continuously on the matrix A, it is also in the case $\det A = 0$ the point where $\tilde f$ attains its minimum. □

We are now able to formulate one of the central results of this section.

Proposition 4.11. *If* $\sum_{j=0}^{2}(a_j - b_j) \geq 0$*,* $D_0 + D_1 + D_2 > 0$*, and* $D_0, D_1, D_2 \geq 0$*, then*

$$\bar{\lambda} = \tilde{\lambda} = \tfrac{\det A}{D_0+D_1+D_2} \quad \textit{and} \quad \min K = \min\{\hat{\lambda}, \bar{\lambda}\}.$$

Proof. Since $D_0, D_1, D_2 \geq 0$ the function f of Lemma 4.10 attains its minimum at the point $\tilde{x} = \frac{D}{D_0+D_1+D_2}$ with nonnegative components. Hence, $\bar{\lambda} = \tilde{\lambda}$. For each $\lambda \in \mathbb{R}$ the restriction $\tilde{f}_\lambda$ of the quadratic form

$$f_\lambda(x) \;=\; x^t(\hat{R}(\lambda) - \lambda \mathbb{1})x \;=\; x^t\hat{R}(0)x - \lambda(x_0 + x_1 + x_2)^2,$$

to the affine plane $x_0 + x_1 + x_2 = 1$ attains its minimum $\tilde{\lambda}$ also at $\tilde{x}$, and we have $\tilde{f}_\lambda(\tilde{x}) = \tilde{\lambda} - \lambda$. Since the set $\{x \in \mathbb{R}^3 \mid x_0 + x_1 + x_2 \neq 0\}$ is dense in $\mathbb{R}^3$, f_λ is positive semidefinite if and only if $\tilde{f}_\lambda(x) \geq 0 \;\forall x \in \mathbb{R}^3$ with $x_0 + x_1 + x_2 = 1$. Therefore,

$$\lambda = \mu\big(\hat{R}(\lambda)|_{3\bar{\varepsilon} \oplus V_0 \oplus V_1 \oplus V_2}\big)$$

for $\lambda \leq \bar{\lambda}$, hence we obtain $\min\{\hat{\lambda}, \bar{\lambda}\} \leq \mu\big(\hat{R}(\min\{\hat{\lambda}, \bar{\lambda}\}, \xi_{\mathrm{m}})\big)$ by Proposition 4.7. Now Proposition 2.5 shows $\min K = \min\{\hat{\lambda}, \bar{\lambda}\}$. ☐

Corollary 4.12. *On each space* $\mathsf{SU}(3)/\mathsf{S}^1_{\bar{k}}$ *the equation* $\min K = \min\{\hat{\lambda}, \bar{\lambda}\}$ *in particular holds for those Riemannian metrics* $\langle\cdot,\cdot\rangle_{t,\bar{s}}$ *with* $s_0 + s_1 + s_2 = 1$*, and sufficiently small* t*. On* $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ *the equality* $\min K = \min\{\hat{\lambda}, \bar{\lambda}\}$ *in particular holds for an open neighborhood of the* $\mathsf{U}(2)$*-biinvariant Riemannian metrics with positive sectional curvature.*

Proof. This corollary follows from the proofs of Lemma 5.10 and Lemma 5.1. ☐

4.3. **Maximal sectional curvature.** We write $\hat{R}(\hat{\eta}_0, \hat{\eta}_1, \hat{\eta}_2, \xi)$ for the invariantly modified curvature operator $\hat{R}_T$ of Proposition 3.9 and $\nu(\cdot)$ for the largest eigenvalue of a matrix.

Lemma 4.13. *We have*

$$\begin{aligned}\Lambda_0 \;&=\; \inf_{\check{\eta} \geq 0} \nu\big(\hat{R}(b_0 + \check{\eta}, b_1, b_2, \xi_0)|_{3\bar{\varepsilon} \oplus \mathbb{C}_0^a \oplus \mathbb{C}_0^b}\big)\\ &=\; \begin{cases} \max\{a_0, c_0, \frac{(2d_0 - b_0)^2 - a_1 a_2}{2(2d_0 - b_0) - a_1 - a_2}\} & \textit{if } 2d_0 - b_0 \geq \max\{a_1, a_2\},\\ \max\{a_0, a_1, a_2, c_0\} & \textit{otherwise,}\end{cases}\end{aligned}$$

and analogous equations for Λ_1 *and* Λ_2*.*

Proof. The inequality $\Lambda_0 \leq \inf_{\check{\eta} \geq 0} \nu\big(\hat{R}(b_0 + \check{\eta}, b_1, b_2, \xi_0)|_{3\bar{\varepsilon} \oplus \mathbb{C}_0^a \oplus \mathbb{C}_0^b}\big)$ follows from Proposition 2.5. In order to show the reverse inequality we have to show that the infimum is the sectional curvature of some Grassmann element in $3\bar{\varepsilon} \oplus \mathbb{C}_0^a \oplus \mathbb{C}_0^b$. Let $f_1(\check{\eta}) = 2d_0 - b_0 - \check{\eta}$ and $f_2(\check{\eta}) = \nu\left(\begin{smallmatrix} a_1 & -\check{\eta} \\ -\check{\eta} & a_2 \end{smallmatrix}\right)$. Then

$$\nu\big(\hat{R}(b_0 + \check{\eta}, b_1, b_2, \xi_0)|_{3\bar{\varepsilon} \oplus \mathbb{C}_0^a \oplus \mathbb{C}_0^b}\big) = \max\big\{a_0, c_0, f_1(\check{\eta}), f_2(\check{\eta})\big\}.$$

The numbers a_0 and c_0 are eigenvalues of $\hat{R}(b_0 + \check{\eta}, b_1, b_2, \xi_0)$. The corresponding eigenspaces include the spaces $\mathbb{R} \cdot \epsilon_0$ and e_0^a, respectively. These spaces intersect $\mathrm{Gr}_2^+(\bar{\varepsilon} \oplus \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2)$. The function f_1 decreases strictly and the function f_2 increases strictly. We have to consider two different cases. Suppose first that $2d_0 - b_0 = f_1(0) \leq f_2(0) = \max\{a_1, a_2\}$. Then $\inf_{\check{\eta} \geq 0} \max\{f_1(\check{\eta}), f_2(\check{\eta})\} = \max\{a_1, a_2\}$, and we are done. Now suppose $f_1(0) > f_2(0)$. The functions f_1 and f_2 intersect at the unique point

$$\check\eta_{\mathrm{m}} \;=\; \tfrac{(2d_0-b_0-a_1)(2d_0-b_0-a_2)}{2(2d_0-b_0)-a_1-a_2} > 0.$$

If $(x_1,x_2)^t$ is an eigenvector of $\left(\begin{smallmatrix} a_1 & -\check\eta \\ -\check\eta & a_2\end{smallmatrix}\right)$ then we can without loss of generality assume $x_1 \geq 0$, $x_2 \leq 0$, and $x_1 - x_2 = 1$. The space $\mathbb{R}(x_1\epsilon_1 + x_2\epsilon_2) \oplus \mathbb{C}^b_0$ intersects the Grassmannian by Lemma 4.6. $\square$

Lemma 4.14. *We have* $\max K = \Lambda_0$ *provided that*

$$\begin{aligned}\Lambda_0 \;&\geq\; \nu\big(\hat R(b_0+\check\eta_{\mathrm{m}}, b_1, b_2, \xi_0)|_{V_0\oplus V_1\oplus V_2\oplus\mathbb{C}^a_1\oplus\mathbb{C}^b_1\oplus\mathbb{C}^a_2\oplus\mathbb{C}^b_2}\big)\\ &\leq\; \max\Big\{b_0+\check\eta_{\mathrm{m}}, b_1, b_2,\, \nu\Big(\begin{smallmatrix} c_1 & \sqrt2(\xi_1-\xi_0)\\ \sqrt2(\xi_1-\xi_0) & 2d_1-b_1\end{smallmatrix}\Big),\, \nu\Big(\begin{smallmatrix} c_2 & \sqrt2(\xi_2-\xi_0)\\ \sqrt2(\xi_2-\xi_0) & 2d_2-b_2\end{smallmatrix}\Big)\Big\},\end{aligned}$$

where $\check\eta_{\mathrm{m}} = 0$ *if* $2d_0 - b_0 \leq \max\{a_1,a_2\}$ *and* $\check\eta_{\mathrm{m}} = \frac{(2d_0-b_0-a_1)(2d_0-b_0-a_2)}{2(2d_0-b_0)-a_1-a_2}$ *otherwise.*

Proof. The statement follows directly from the definitions of Λ_j and $\check\eta_{\mathrm{m}}$ and from Proposition 2.5. $\square$

Corollary 4.15. *On each* $\mathsf{SU}(3)/\mathsf{S}^1_{\vec k}$ *the equation* $\max K = \Lambda_0$ *in particular holds for those Riemannian metrics* $\langle\cdot,\cdot\rangle_{t,\vec s}$ *with* $s_0+s_1+s_2 = 1$*,* $s_0 \leq s_1 \leq s_2$ *and sufficiently small* t*. On* $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ *the equation* $\max K = \Lambda_0$ *in particular holds for an open neighborhood of the* $\mathsf{U}(2)$*-biinvariant Riemannian metrics with positive sectional curvature.*

Proof. The corollary follows from the proof of Lemma 5.11 and from the proof of Lemma 5.2. $\square$

4.4. **The cone of positive sectional curvature.** For any Aloff–Wallach space $\mathsf{SU}(3)/\mathsf{S}^1_{\vec k}$ we describe the cone Ω of quadruples $(t,\vec s)$ such that the T^2-biinvariant Riemannian metric $\langle\cdot,\cdot\rangle_{t,\vec s}$ has positive sectional curvature.

Lemma 4.16. *Let* $\Omega_{\hat\lambda} := \{(t,\vec s)\,|\,\hat\lambda > 0\}$*. Then* $\Omega \subset \Omega_{\hat\lambda}$*. The condition* $\hat\lambda > 0$ *does not depend on* t*. Hence we have* $\Omega_{\hat\lambda} = \mathbb{R}_+ \times \Omega'_{\hat\lambda}$ *for some* $\Omega'_{\hat\lambda}$*.*

Proof. Since $\min K \leq \hat\lambda$ the inclusion $\Omega \subset \Omega_{\hat\lambda}$ is apparent. In order to prove that the condition $\hat\lambda > 0$ is independent of t, we go back to the characterization of $\hat\lambda$ in Proposition 4.7. Multiplication of $\hat R(\lambda,\xi)|_{\mathbb{C}^a_j\oplus\mathbb{C}^b_j} - \lambda\mathbb{1}$ by $\left(\begin{smallmatrix} 1 & 0\\ 0 & \sqrt2\end{smallmatrix}\right)^{-1}$ on the left and on the right shows that the smallest zero $\lambda_j(\xi)$ of $\lambda \mapsto \det\big(\hat R(\lambda,\xi)|_{\mathbb{C}^a_j\oplus\mathbb{C}^b_j} - \lambda\mathbb{1}\big)$ is the smallest eigenvalue of the matrix $\left(\begin{smallmatrix} c_j & \xi_j-\xi\\ \xi_j-\xi & d_j\end{smallmatrix}\right)$. Therefore, we have $\lambda_j(\xi) > 0$ if and only if $f_j(\xi) > 0$, where $f_j(\xi) := c_jd_j - (\xi_j-\xi)^2$. In particular,

$$\hat\lambda > 0 \;\Leftrightarrow\; \exists\,\xi\in\mathbb{R} : f_0(\xi), f_1(\xi), f_2(\xi) > 0. \tag{11}$$

This proves that the condition $\hat\lambda > 0$ is independent of t. $\square$

Lemma 4.17. *Let* Ω'_d*,* Ω'_σ *be the cones of triples* $\vec s$ *such that* $d_0, d_1, d_2 > 0$*, and* $\sigma > 0$*, respectively. We have the chain of inclusions*

$$\Omega'_{\hat\lambda} \subset \Omega'_d \subset \Omega'_\sigma.$$

The inclusion $\Omega'_d \subset \Omega'_\sigma$ *in particular implies that* $s_{j-1} + s_{j+1} > s_j$ *for* $\vec s \in \Omega'_d$*.*

Proof. We have $\sigma = 4\sigma_3(d_0 + d_1 + d_2)$. This proves $\Omega'_d \subset \Omega'_\sigma$. The inclusion $\Omega'_{\hat\lambda} \subset \Omega'_d$ follows from the characterization of $\hat\lambda$ in Proposition 4.7. $\square$

Lemma 4.18. *The following condition is equivalent to $\hat\lambda > 0$ on Ω'_d:*

$$(\xi_{j+1} - \xi_{j-1})^2 < c_{j+1}d_{j+1} + c_{j-1}d_{j-1} + 2\sqrt{c_{j+1}d_{j+1}c_{j-1}d_{j-1}} \quad \text{for } j = 0, 1, 2.$$

Proof. We use the condition in (11) for the proof. The two parabolas f_1 and f_2 intersect at some (unique) point ξ_s in the interval $]\min\{\xi_1, \xi_2\}, \max\{\xi_1, \xi_2\}[$ if and only if $|c_1d_1 - c_2d_2| < (\xi_1 - \xi_2)^2$. In this case we have $\sup_{\xi\in\mathbb{R}} \min\{f_1(\xi), f_2(\xi)\} = f_1(\xi_s)$, and we have $f_1(\xi_s) > 0$ if and only if

$$c_1d_1 + c_2d_2 + 2\sqrt{c_1d_1c_2d_2} < (\xi_1 - \xi_2)^2 < c_1d_1 + c_2d_2 + 2\sqrt{c_1d_1c_2d_2}.$$

The first inequality already follows from $|c_1d_1 - c_2d_2| < (\xi_1 - \xi_2)^2$. In the case where f_1 and f_2 do not intersect in $]\min\{\xi_1, \xi_2\}, \max\{\xi_1, \xi_2\}[$ we have $f_1(\xi_1), f_2(\xi_1) > 0$ or $f_1(\xi_2), f_2(\xi_2) > 0$. The condition $(\xi_1 - \xi_2)^2 < c_1d_1 + c_2d_2 + 2\sqrt{c_1d_1c_2d_2}$ is void, since this inequality follows from $(\xi_1 - \xi_2)^2 \le |c_1d_1 - c_2d_2|$. $\square$

Aloff and Wallach proved that the metrics $\langle\cdot,\cdot\rangle_{t,t,s,s}$ have positive sectional curvature provided that $t < s$ and $0 < k_1 \le k_2$. By Lemma 4.16 this means that $\Omega'_{\hat\lambda}$ contains the set $\{\vec{s}\,|\, s_0 < s_1,\ s_1 = s_2\}$. This fact is also an immediate consequence of Lemma 4.18.

Let $A := \hat{R}(0) = \begin{pmatrix} a_0 & b_2 & b_1 \\ b_2 & a_1 & b_0 \\ b_1 & b_0 & a_2 \end{pmatrix}$, and denote by $\tilde{A} = \begin{pmatrix} \tilde a_0 & \tilde b_2 & \tilde b_1 \\ \tilde b_2 & \tilde a_1 & \tilde b_0 \\ \tilde b_1 & \tilde b_0 & \tilde a_2 \end{pmatrix}$ the matrix obtained from A by setting $t = 0$. For a geometric meaning of the numbers $\tilde a_j$ and $\tilde b_j$ see Section 5.3.

Lemma 4.19. *The matrix $\tilde A$ is positive definite for $\vec{s} \in \Omega'_\sigma \setminus \{\vec{s}\,|\, s_0 = s_1 = s_2\}$. In the case $s_0 = s_1 = s_2$ the matrix $\tilde A$ is positive semidefinite and its kernel is spanned by $(1,1,1)^t$.*

Proof. We make the successive coordinate transformations

$$\begin{aligned}(s_0, s_1, s_2) &\mapsto (u_0, u_1, u_2) = \left(\sqrt{\tfrac{s_0}{s_1s_2}}, \sqrt{\tfrac{s_1}{s_2s_0}}, \sqrt{\tfrac{s_2}{s_0s_1}}\right) \\ &\mapsto (y_0, y_1, y_2) = \tfrac{1}{2}(-u_0 + u_1 + u_2, u_0 - u_1 + u_2, u_0 + u_1 - u_2).\end{aligned} \tag{12}$$

Note that $s_0 = s_1 = s_2$ if and only if $y_0 = y_1 = y_2$. We obtain

$$\tilde a_j = 4(y_j^2 + y_1y_2 + y_2y_0 + y_0y_1) \quad \text{and} \quad \tilde b_j = -2(2y_1y_2 + y_2y_0 + y_0y_1).$$

Hence,

$$\tilde A = \begin{pmatrix} y_0 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & y_2 \end{pmatrix}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix}\begin{pmatrix} y_0 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & y_2 \end{pmatrix} + (y_1y_2 + y_2y_0 + y_0y_1)\begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix}.$$

The matrix $\begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix}$ is positive semidefinite and its kernel is spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Moreover, we have $\frac{\sigma}{4\sigma_3} = y_1y_2 + y_2y_0 + y_0y_1$. This proves the lemma. $\square$

Let $v := \left(\frac{\hat k_0}{s_0}, \frac{\hat k_1}{s_1}, \frac{\hat k_2}{s_2}\right)^t$. We have $A = \tilde A - \frac{9t}{2}\, vv^t$, i.e., the difference of A and $\tilde A$ is a positive semidefinite rank 1 matrix.

Lemma 4.20. *For $\vec{s} \in \Omega'_\sigma$ we set*

$$t_A(\vec{s}) := \begin{cases} 0, & \text{if } s_0 = s_1 = s_2, \\ \frac{2}{9}\langle v, \tilde{A}^{-1} v\rangle^{-1}, & \text{otherwise.} \end{cases}$$

Then A is positive definite if and only $t < t_A(\vec{s})$. In this case we in particular have $\sum_{j=0}^{2}(a_j - b_j) > 0$ and $D_0 + D_1 + D_2 > 0$, where $D = (D_0, D_1, D_2)^t = A^{\mathrm{ad}}(1,1,1)^t$.

Proof. We fix $\vec{s} \in \Omega'_\sigma \setminus \{\vec{s} \,|\, s_0 = s_1 = s_2\}$. $\tilde{A}$ is positive definite and the function $t \mapsto \det A = \big(1 - \frac{9t}{2}\langle v, \tilde{A}^{-1} v\rangle^{-1}\big) \det \tilde{A}$ has exactly one zero $t_A(\vec{s})$. If A is positive definite then the proof of Lemma 4.10 shows that $\sum_{j=0}^{2}(a_j - b_j) > 0$ and $D_0 + D_1 + D_2 > 0$. $\square$

Next we consider the vector D along the graph of the map $\Omega'_\sigma \to \mathbb{R}$, $\vec{s} \mapsto t_A(\vec{s})$, i.e., we consider the mapping

$$\hat{D} : \Omega'_\sigma \to \mathbb{R}, \quad \vec{s} \mapsto D_{|(t_A(\vec{s}),\vec{s})}.$$

In terms of this mapping we define the cones

$$\Omega'_{\hat{D}} := \{\vec{s} \in \Omega'_\sigma \,|\, \hat{D}_0(\vec{s}), \hat{D}_1(\vec{s}), \hat{D}_2(\vec{s}) > 0\},$$
$$\Omega^* := \{(t, \vec{s}) \in \mathbb{R}_+ \times \Omega'_{\hat{D}} \,|\, t < t_A(\vec{s})\}.$$

Proposition 4.21. *The cone Ω^* has the following properties:*

1. $\min K = \min\{\hat{\lambda}, \bar{\lambda}\}$ *for* $(t, \vec{s}) \in \Omega^*$ *and*
2. $\Omega \cap (\mathbb{R}_+ \times \Omega'_{\hat{D}}) = \Omega^* \cap \Omega_{\hat{\lambda}}$.

Proof. We consider a fixed $\vec{s} \in \Omega'_{\hat{D}}$. Since

$$A^{\mathrm{ad}} = \Big((1 - \tfrac{9t}{2}\langle v, \tilde{A}^{-1} v\rangle)\tilde{A}^{-1} + \tfrac{9t}{2}\tilde{A}^{-1} v v^t \tilde{A}^{-1}\Big) \det \tilde{A},$$

we have

$$\langle v, \tilde{A}^{\mathrm{ad}} v\rangle\, D = \tfrac{2(t_A(\vec{s}) - t)}{9 t_A(\vec{s})^2}(\det \tilde{A})\tilde{D} + \tfrac{t}{t_A(\vec{s})}(w_0 + w_1 + w_2) w, \tag{13}$$

where $w = (w_0, w_1, w_2)^t = \tilde{A}^{\mathrm{ad}} v$. In particular, each function $t \mapsto D_j$ is linear and has at most one zero. Let $\tilde{D} = \tilde{A}^{\mathrm{ad}}(1,1,1)^t$. Since $\tilde{D}_j = 6\frac{\sigma}{\sigma_3}\, s_j(s_{j+1} + s_{j-1}) > 0$ and $\hat{D}_j(\vec{s}) > 0$ there is no zero in the interval $]0, t_A(\vec{s})[$. We have $\min K = \min\{\hat{\lambda}, \bar{\lambda}\}$ for $(t, \vec{s}) \in \Omega^*$ by Proposition 4.11 and Lemma 4.20.

The second inclusion is an immediate consequence of the first one, since $\hat{\lambda} = (\det A)/(D_0 + D_1 + D_2)$ by Proposition 4.11. $\square$

Lemma 4.22. *The cone $\Omega'_{\hat{D}}$ can be characterized in terms of the vector $w = (w_0, w_1, w_2)^t = \tilde{A}^{\mathrm{ad}} v$ as*

$$\Omega'_{\hat{D}} \;=\; \{\vec{s} \in \Omega'_\sigma \,|\, (w_0 + w_1 + w_2) w_j > 0 \text{ for } j = 0,1,2\} \;=\; \Omega'_w \cup \Omega'_{-w},$$

where $\Omega'_w := \{\vec{s} \in \Omega'_\sigma \,|\, w_0, w_1, w_2 > 0\}$ and $\Omega'_{-w} := \{\vec{s} \in \Omega'_\sigma \,|\, w_0, w_1, w_2 < 0\}$.

Proof. The claim follows immediately from equation (13). $\square$

In order to inspect the cones $\Omega'_{\hat{\lambda}}$ and $\Omega'_{\hat{D}}$ in more detail, we use the y-variables defined in (12) instead of the s-variables. Moreover, we use the normalization $y_0 + y_1 + y_2 = 1$. All y_j are positive on Ω'_d, since $0 < y_0 + y_1 + y_2$ and $0 < d_j = 2y_{j+1}y_{j-1}$ for all j.

The components w_j of the vector w are polynomials of degree 6 in y_0, y_1, y_2. For example, w_0 is given by

$$\begin{aligned} w_0 \;&= 2\,(\hat k_1 + \hat k_2)\big(6y_0^4(y_1+y_2)^2 + 5y_0^3(y_1+y_2)^3 - 8y_0^3y_1y_2(y_1+y_2) \\ &\qquad + y_0^2(y_1+y_2)^4 - 8y_0^2y_1y_2(y_1+y_2)^2 - 8y_0^2y_1^2y_2^2 \\ &\qquad + 3y_0y_1y_2(y_1+y_2)^3 - 24y_0y_1^2y_2^2(y_1+y_2) + 2y_1^2y_2^2(y_1-y_2)^2\big) \\ &\quad - 2\,(k_1-k_2)(y_1-y_2)(y_1+y_2)\big(y_0^3(y_1+y_2) + 3y_0^2(y_1+y_2)^2 - 2y_0^2y_1y_2 \\ &\qquad + 5y_0y_1y_2(y_1+y_2) + 2y_1^2y_2^2\big) \end{aligned}$$

The curve $w_0 = 0$ contains the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(\frac13,\frac13,\frac13)$, and $(0,1,1)$. The tangent line at $(\frac13,\frac13,\frac13)$ to $w_0 = 0$ is given by

$$k_0y_0 + k_1y_1 + k_2y_2 \;=\; k_1(y_1-y_0) + k_2(y_2-y_0) \;=\; 0. \tag{14}$$

By symmetry the tangent line at $(\frac13,\frac13,\frac13)$ to the curves $w_1 = 0$ and $w_2 = 0$ is also given by (14). Notice that the polynomial $w_0+w_1+w_2$ admits the nice factorisation

$$\begin{aligned} w_0 + w_1 + w_2 = 24\,&(y_1+y_2)(y_2+y_0)(y_0+y_1) \\ &\cdot(y_1y_2 + y_2y_0 + y_0y_1)(\hat k_0y_0 + \hat k_1y_1 + \hat k_2y_2). \end{aligned}$$

The set $\Omega'_{\hat\lambda}$ has been described in Lemma 4.18 in terms of the numbers

$$\psi_j \;:=\; c_{j+1}d_{j+1} + c_{j-1}d_{j-1} + 2\sqrt{c_{j+1}d_{j+1}c_{j-1}d_{j-1}} - (\xi_{j+1}-\xi_{j-1})^2.$$

The boundary of the set $\{\vec s\,|\,\psi_0,\psi_1,\psi_2 > 0\}$ is contained in the union of the zero sets of

$$p_j \;:=\; 4c_{j+1}d_{j+1}c_{j-1}d_{j-1} - \big(c_{j+1}d_{j+1} + c_{j-1}d_{j-1} - (\xi_{j+1}-\xi_{j-1})^2\big).$$

These numbers p_j are polynomials of degree 12 in y_0, y_1, y_2. An explicit computation shows that p_0 contains the factor $(y_1-y_2)^2(y_0+y_1)^2(y_0+y_2)^2$. Hence, the relevant part of p_0 is the following polynomial of degree 6:

$$\begin{aligned} \tilde p_0 \;&= \hat k_0^2y_0^2(y_1-y_2)^2\big(-\hat k_0^2y_0^2 + 2(\hat k_1y_1 - \hat k_2y_2)^2 - \hat k_1\hat k_2(y_1+y_2)^2\big) \\ &\quad - (\hat k_1y_1 - \hat k_2y_2)^2(\hat k_1y_1 + \hat k_2y_2)^2(y_1-y_2)^2 \\ &\quad + \hat k_1\hat k_2\big((\hat k_1-\hat k_2)^2y_0^2(y_1+y_2)^4 - 16(\hat k_1y_1 - \hat k_2y_2)^2y_1^2y_2^2\big) \end{aligned}$$

The tangent space of the curve $\tilde p_0 = 0$ at $(\frac13,\frac13,\frac13)$ is given by

$$k_1(y_1-y_0) - k_2(y_2-y_0) \;=\; 0, \tag{15}$$

provided that $k_1 \neq k_2$. Notice that we continue to assume that $0 < k_1 \le k_2$. In the case $k_1 = k_2 = 1$ the point $(\frac13,\frac13,\frac13)$ is a singular point of the curve $\tilde p_0 = 0$. In the same way, we obtain polynomials $\tilde p_1$ and $\tilde p_2$ with corresponding properties. Each curve $\tilde p_j = 0$ has only tangential intersections with the curves $d_{j+1} = 0$ and $d_{j-1} = 0$. In fact, it is contained in the region where d_{j+1} and d_{j-1} are both nonnegative. These facts follow when comparing the definitions of ψ_j and p_j.

Observation 4.23. *With our normalisation of $\vec k$ there is the inclusion $\Omega'_{\hat\lambda} \subset \Omega'_{-w}$, and hence the cone Ω can be characterized as follows:*

$$\Omega \;=\; \Omega^* \cap \Omega_{\hat\lambda} \;=\; \big\{(t,\vec s) \in \mathbb{R}_+ \times \Omega'_{\hat\lambda} \,|\, t < t_A(\vec s)\big\}.$$

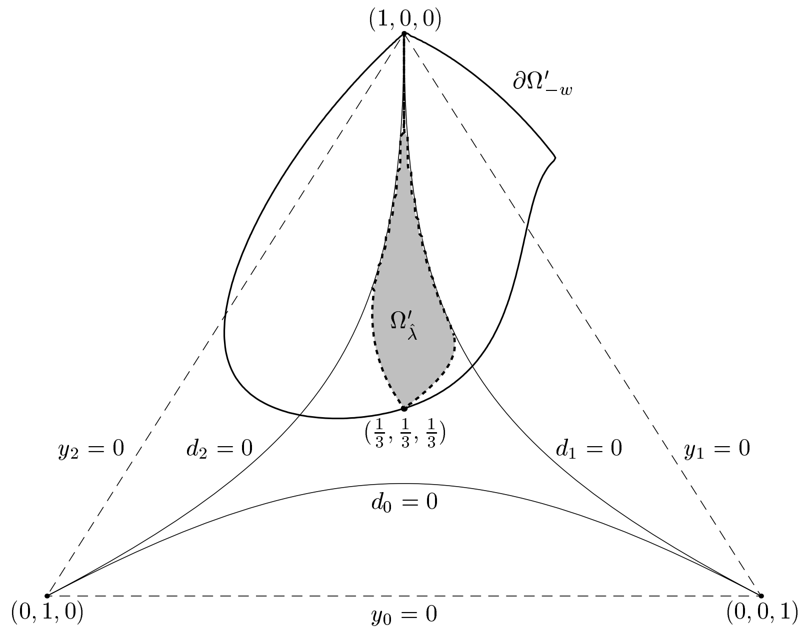

FIGURE 2. The inclusion $\Omega'_{\hat\lambda} \subset \Omega'_{-w}$ in the case $k = 1$, $l = 4$. This figure is representative for all pairs (k, l) with $0 < k \le l$. The line from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to $(1, 0, 0)$ is contained in $\Omega'_{\hat\lambda}$. The boundary $\partial\Omega'_{-w}$ consists of parts of the algebraic curves $d_1 = 0$ and $d_2 = 0$ and of parts of algebraic curves which intersect $d_1 = 0$ and $d_2 = 0$ tangentially.

Comparing the slopes in (14) and (15) we see that this statement holds infinitesimally at the boundary point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ provided that $k_1 < k_2$. Numerical investigations strongly suggest that the statement is globally true. In fact, it should be possible to use the facts above to write down a rigorous proof. Notice that in the exceptional case $k_1 = 0$ we have $\Omega_{\hat\lambda} = \emptyset$ since $c_1 = 0$ and hence $f_1(\xi) \le 0$ for all $\xi \in \mathbb{R}$. We illustrate Observation 4.23 in Figure 2.

5. THE PINCHING CONSTANTS OF $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ AND $\mathsf{SU}(3)/\mathsf{T}^2$

We use the results of Section 4 to determine the precise pinching constants of all $\mathsf{U}(2)$-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$. We consider the T^2-biinvariant Riemannian metrics on the spaces $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ and compute in particular the pinching function $\Delta : [0,1] \to [0,1]$, $\Delta(k/l) = \delta\big(\big(\mathsf{SU}(3), \mathsf{S}^1_{k,l}\big)\big)$. Finally, we determine the precise pinching constants for all invariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{T}^2$.

5.1. **The pinching constants of $\mathsf{U}(2)$-biinvariant metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$.** We have parametrized the $\mathsf{U}(2)$-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ in Corollary 3.2, where we have defined the Riemannian metric $\langle\cdot,\cdot\rangle_{t,s}$ for each $t, s > 0$. Since $\vec{k} = (-2, 1, 1)^t$, we have $\hat{k}_0 = -\frac{2}{\sqrt{6}}$, $\hat{k}_1 = \hat{k}_2 = \frac{1}{\sqrt{6}}$. Therefore, the numbers

a_j, b_j, c_j, d_j, and ξ_j from Proposition 3.9 specialize as follows:

$$\begin{array}{llll} a_0 = \frac{1}{t}, & a_1 = a_2 = \frac{16s-3t}{4s^2}, & b_0 = -\frac{8s+3t}{4s^2}, & b_1 = b_2 = -\frac{3s-2t}{2s^2}, \\ c_0 = \frac{1}{t}, & c_1 = c_2 = \frac{t}{4s^2}, & d_0 = \frac{4s-3t}{4s^2}, & d_1 = d_2 = \frac{t}{4s^2} \\ \xi_0 = -\frac{2s-t}{2s^2}, & \xi_1 = \xi_2 = \frac{2s-t}{4s^2}. & & \end{array}$$

Lemma 5.1. *The minimal sectional curvature of the Riemannian metric $\langle \cdot, \cdot \rangle_{t,s}$ on the space $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ satisfies*

$$\min K \begin{cases} = \frac{t}{4s^2} & \text{if } t \leq \frac{2}{5}s, \\ = \frac{4}{s^2} \frac{(s-t)^3}{4s^2+16st-11t^2} & \text{if } \frac{2}{5}s \leq t \leq \frac{10}{7}s, \\ \leq \frac{4s-3t}{4s^2} & \text{if } \frac{10}{7}s \leq t. \end{cases}$$

In particular, the sectional curvature is positive if and only if $t < s$.

Proof. A direct computation yields

$$D_0 \; = \; \tfrac{3}{s^3}(10s - 7t), \quad D_1 = D_2 \; = \; \tfrac{3}{s^3 t}(2s^2 + 3st - 2t^2).$$

Since these numbers are positive for $t \leq \frac{10}{7}s$, we have

$$D_0 + D_1 + D_2 \; = \; \tfrac{3}{s^3 t}(4s^2 + 16st - 11t^2) > 0$$

and

$$a_0 + a_1 + a_2 - b_0 - b_1 - b_2 = \tfrac{1}{s^2 t}(4s^2 + 52st - 11t^2) > 0.$$

Therefore, by Proposition 4.11 we have $\min K = \min\{\bar\lambda, \hat\lambda\}$ for $t \leq \frac{10}{7}s$. Moreover,

$$\bar\lambda \; = \; \frac{\det A}{D_0+D_1+D_2} \; = \; \frac{4}{s^2}\,\frac{(s-t)^3}{4s^2+16st-11t^2}.$$

For the computation of $\hat\lambda$ note that $c_0 > d_0$ and $c_1 = d_1$ for all $t, s > 0$. Hence, d_0 and d_1 are the maxima of the hyperbolas $\lambda_0(\xi)$ and $\lambda_1(\xi)$, and the latter hyperbola degenerates to a cone. We have $d_0 \geq d_1$ for $t \leq s$. Moreover, the slope of the cone is identical with the slope of the asymptotes to the hyperbola. It is easy to verify that the maximum (ξ_0, d_0) of $\lambda_0(\xi)$ is contained in the convex set bounded by the cone if and only if $t \geq \frac{10}{7}s$. Therefore, $\hat\lambda = d_0$ for $t \geq \frac{10}{7}s$. On the other hand, the convex set bounded by the hyperbola $\lambda_0(\xi)$ is given by the inequalities $\lambda^2 - \lambda(c_0 + d_0) + c_0 d_0 - (\xi_0 - \xi)^2 \geq 0$ and $\lambda \leq d_0$. The maximum (ξ_1, d_1) of $\lambda_1(\xi)$ is contained in this set if and only if $(2s - 5t)(2s - t)(4s - t) \geq 0$, i.e., $t \leq \frac{2}{5}s$. Therefore, $\hat\lambda = d_1$ for $t \leq \frac{2}{5}s$. Since the slopes of the asymptotes and the cone are the same, the hyperbola and the cone intersect in exactly one point between the two maxima for $\frac{2}{5}s < t < \frac{10}{7}s$. The value at this intersection point is given by $\bar\lambda$, as can be verified by Lemma 4.8. Hence, $\hat\lambda = \bar\lambda$ for $\frac{2}{5}s < t < \frac{10}{7}s$. □

Lemma 5.2. *The maximal sectional curvature of the Riemannian metric $\langle \cdot, \cdot \rangle_{t,s}$ on the space $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ is given by*

$$\max K \; = \; \begin{cases} \frac{1}{t} & \text{if } t \leq \frac{8-2\sqrt{13}}{3}s, \\ \frac{16s-3t}{4s^2} & \text{if } \frac{8-2\sqrt{13}}{3}s \leq t \leq \frac{22}{7}s. \end{cases}$$

Proof. We apply Lemma 4.14. For the Riemannian metric $\langle\cdot,\cdot\rangle_{t,s}$ we have $2d_0-b_0=a_1=a_2$ and hence $\check{\eta}_{\mathrm{m}}=0$, $\Lambda_0=\max\{\frac{1}{t},\frac{16s-3t}{4s^2}\}$, and

$$\hat{R}(b_0,b_1,b_2,\xi_0)|_{\mathbb{C}_1^a\oplus\mathbb{C}_1^b} \;=\; \hat{R}(b_0,b_1,b_2,\xi_0)|_{\mathbb{C}_2^a\oplus\mathbb{C}_2^b} \;=\; \begin{pmatrix} \frac{t}{4s^2} & -3\,\frac{2s-t}{2\sqrt{2}s^2} \\ -3\,\frac{2s-t}{2\sqrt{2}s^2} & \frac{3s-t}{2s^2} \end{pmatrix}$$

with eigenvalues $\frac{12s-5t}{4s^2}$ and $-\frac{3s-2t}{2s^2}$. □

Theorem 5.3. *The pinching constant of the Riemannian metric $\langle\cdot,\cdot\rangle_{t,s}$ on the space $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ is given by*

$$\delta\big(\mathsf{SU}(3)/\mathsf{S}^1_{1,1},\langle\cdot,\cdot\rangle_{t,s}\big) \;=\; \begin{cases} \frac{t^2}{4s^2} & \text{if } t\le\frac{8-2\sqrt{13}}{3}s,\\ \frac{t}{16s-3t} & \text{if } \frac{8-2\sqrt{13}}{3}s\le t\le\frac{2}{5}s,\\ \frac{16(s-t)^3}{(16s-3t)(4s^2+16st-11t^2)} & \text{if } \frac{2}{5}s\le t<s. \end{cases}$$

In particular, the metric of optimal pinching is given by $t=\frac{2}{5}s$, and the corresponding pinching constant is $\frac{1}{37}$.

Proof. A direct combination of Lemma 5.1 and Lemma 5.2. □

Remark. The metric $\langle\cdot,\cdot\rangle_{2,5}$ of optimal pinching is Einstein (see Section 1.6). Moreover, this metric is induced by the Killing form on $\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1}=\mathsf{SU}(3)\times\mathsf{SO}(3)$ (see [Wi1]). The pinching constant $\frac{16}{29\cdot 37}$ of the metric $\langle\cdot,\cdot\rangle_{2,4}$ has been computed before by Huang (see [Hu]).

5.2. **The pinching constants of T^2-biinvariant metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$.** We first settle the question how the pinching constant $\delta\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\big)$ can be determined for a given pair (k,l). Then we ask how these numbers vary with the slope k/l, i.e., we investigate the continuous pinching function $\Delta:[0,1]\to[0,1]$ defined by $\delta(k/l)=\delta\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{k,l},\mathsf{T}^2\big)$. As in Section 3 we write $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$ for $\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$ with $\vec{k}=(k_0,k_1,k_2)=(-k-l,k,l)$ where appropriate.

For a fixed vector $\vec{k}$ we consider the function

$$\delta_{\vec{k}}:\mathcal{M}\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{\vec{k}},\mathsf{T}^2\big)\to\mathbb{R},\quad \langle\cdot,\cdot\rangle_{t,\vec{s}}\mapsto\delta\big(\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}},\langle\cdot,\cdot\rangle_{t,\vec{s}}\big).$$

We are interested in finding the supremum $\delta\big(\mathsf{SU}(3)\times\mathsf{T}^2/\mathsf{S}^1_{\vec{k}},\mathsf{T}^2\big)$ of $\delta_{\vec{k}}$. It suffices to consider only the metrics $\langle\cdot,\cdot\rangle_{t,\vec{s}}$ where $s_0+s_1+s_2=1$. We use this normalization throughout this section.

According to the definition in Section 4.4 Ω is the cone of tuples $(t,\vec{s})$ such that the metric $\langle\cdot,\cdot\rangle_{t,\vec{s}}$ has positive sectional curvature. Moreover, recall the inclusion $\Omega\subset\mathbb{R}_+\times\Omega'_d$. We set

$$\Sigma':=\{\vec{s}\in\Omega'_d\mid s_0+s_1+s_2=1\}\quad\text{and}\quad\Sigma:=]0,1[\,\times\Sigma'.$$

Lemma 5.4. *Σ contains up to scaling all T^2-biinvariant metrics of positive sectional curvature, i.e., $\{(t,\vec{s})\in\Omega\mid s_0+s_1+s_2=1\}\subset\Sigma$.*

Proof. By Lemma 4.17 we have $s_j<s_{j+1}+s_{j-1}$ on Ω'_d. Thus the normalization $s_0+s_1+s_2=1$ implies $s_j<\frac{1}{2}$. On the other hand, $\max\{\hat{k}_j^2\}\ge\frac{1}{2}$, as $|\hat{k}|=1$. Moreover, if $\langle\cdot,\cdot\rangle_{t,\vec{s}}$ has positive sectional curvature then in particular $a_0,a_1,a_2>0$. Hence, $t<\min\{\frac{8s_j}{9\hat{k}_j^2}\}\le\frac{8}{9}$. □

Let $\bar{\Sigma}$ denote the euclidean closure of Σ. The points in $\bar{\Sigma}$ which do not correspond to T^2-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$ are precisely the points in the set

$$\partial\Sigma \ := \ \{0\} \times \bar{\Sigma}' \ \cup \ [0,1] \times \big\{(0,\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2},0)\big\}.$$

For small $\vartheta > 0$ we define the compact subsets

$$\Sigma^{\vartheta}_{\vec{k}} := \Big\{(t,\vec{s}) \in \bar{\Sigma} \ \big| \ \hat{k}^2_{j+1}s^2_{j-1} \geq \vartheta\, \hat{k}^2_{j-1}s^2_{j+1}, \quad \tfrac{8\vartheta}{3(1+3\vartheta)\hat{k}^2_j} \leq t \text{ for all } j=0,1,2\Big\}$$

of $\bar{\Sigma}$. These subsets do not intersect $\partial\Sigma$.

Lemma 5.5. *Let $\bar{\delta}$ be the pinching constant of a positively curved T^2-biinvariant Riemannian metric on $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$. Then*

$$\big\{(t,\vec{s}) \ \big| \ \delta\big(\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}, \langle\cdot,\cdot\rangle_{t,\vec{s}}\big) \geq \bar{\delta}\big\} \ \subset \ \Sigma^{\bar{\delta}}_{\vec{k}}$$

Hence $\delta_{\vec{k}}$ attains all its maxima in the compact set $\Sigma^{\bar{\delta}}_{\vec{k}}$.

Proof. If a metric $\langle\cdot,\cdot\rangle_{t,\vec{s}}$ has a pinching constant $\geq \bar{\delta}$ then in particular $c_{j+1}/c_{j-1} \geq \bar{\delta}$ and $c_j/a_j \geq \bar{\delta}$ hold. These inequalities imply $(t,\vec{s}) \in \Sigma^{\bar{\delta}}_{\vec{k}}$. $\square$

Proposition 4.11 and Lemma 4.14 provide conditions that guarantee that $\min K = \min\{\hat{\lambda},\bar{\lambda}\}$ and $\max K = \Lambda_0$. However, these conditions are not a priori satisfied for all metrics $\langle\cdot,\cdot\rangle_{t,\vec{s}}$ with $(t,\vec{s}) \in \Sigma^{\bar{\delta}}_{\vec{k}}$. In order to determine the maximum of these pinching constants nevertheless, we define the function

$$\hat{\delta}_{\vec{k}} : \Sigma^{\bar{\delta}}_{\vec{k}} \to \mathbb{R}, \quad \langle\cdot,\cdot\rangle_{t,\vec{s}} \mapsto \tfrac{\min\{\bar{\lambda},\hat{\lambda}\}}{\max\{\Lambda_0,\Lambda_1,\Lambda_2\}},$$

where $\bar{\lambda}$, $\hat{\lambda}$, and $\Lambda_0,\Lambda_1,\Lambda_2$ are defined in Definition 4.1. The function $\hat{\delta}_{\vec{k}}$ is an upper bound for the function $\delta_{\vec{k}}$. The idea now is to maximize $\hat{\delta}_{\vec{k}}$ instead of $\delta_{\vec{k}}$ on $\Sigma^{\bar{\delta}}_{\vec{k}}$ and to verify by Proposition 4.11 and Lemma 4.14 that $\hat{\delta}_{\vec{k}}(t_{\max},\vec{s}_{\max}) = \delta_{\vec{k}}(t_{\max},\vec{s}_{\max})$ for a point $(t_{\max},\vec{s}_{\max})$ where $\hat{\delta}_{\vec{k}}$ attains its maximumal value. At this point also $\delta_{\vec{k}}$ attains its maximal value.

We turn to the numerical investigation of the functions $\hat{\delta}_{\vec{k}}$. We consider 200 vectors $\vec{k}$ such that the corresponding values k_1/k_2 provide an equidistant partition of the interval $[0,1]$. It is important that each $\hat{\delta}_{\vec{k}}$ is Lipschitz. However, $\hat{\delta}_{\vec{k}}$ is not a concave function. In particular, we do not have any a priori argument for the uniqueness of the point where $\hat{\delta}_{\vec{k}}$ attains its maximum. We generate a fairly dense lattice of starting points and apply the simplex algorithm of Nelder and Mead [NM] for the maximization of $\hat{\delta}_{\vec{k}}$. Performing the computations we find:

Observation 5.6. *For each $\vec{k}$ that we considered in our numerical investigations the function $\hat{\delta}_{\vec{k}} : \Sigma^{\bar{\delta}}_{\vec{k}} \to \mathbb{R}$ attains its maximum at a unique point. This point satisfies the equations $\bar{\lambda} = \hat{\lambda} = \lambda_0(\xi) = \lambda_1(\xi) = \lambda_2(\xi)$ for some $\xi \in \mathbb{R}$.*

We introduce the function $\hat{\Delta} : [0,1] \to [0,1]$, which is defined by the property $\hat{\Delta}(k_1/k_2) = \sup \hat{\delta}_{\vec{k}}$. Note that $\hat{\delta}_{\vec{k}}(t,\vec{s})$ is Lipschitz in $(t,\vec{s})$ and in k_1/k_2. Hence also $\hat{\Delta}$ is Lipschitz. It is clear that $\hat{\Delta}$ is an upper bound for the pinching function Δ, since Δ is defined by the property $\Delta(k_1/k_2) = \sup \delta_{\vec{k}}$ and $\hat{\delta}_{\vec{k}}$ is an upper bound for $\delta_{\vec{k}}$.

Observation 5.7. *The functions* $\Delta : [0,1] \to [0,1]$ *and* $\hat{\Delta} : [0,1] \to [0,1]$ *coincide at the values* k_1/k_2 *of the considered vectors* $\vec{k}$.

In fact, we verify open conditions corresponding to the hypotheses of Proposition 4.11 and Lemma 4.14 in order to see that the equalities $\min\{\hat{\lambda}, \bar{\lambda}\} = \min K$ and $\Lambda_0 = \max K$ hold for the unique metric resulting from our maximization of $\hat{\delta}_{\vec{k}}$.

Observation 5.8. *The function* $\hat{\Delta} : [0,1] \to [0,1]$ *is strictly increasing (see Figure 1 on page 4). Hence, the maximum of* $\hat{\Delta}$ *is attained at* 1*, i.e.,* $\hat{\delta}\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{k,l}, \mathsf{T}^2\big)$ *is maximal for* $k = l = 1$.

Observation 5.9. *The* T^2*-biinvariant Riemannian metric on* $\big(\mathsf{SU}(3), \mathsf{S}^1_{1,1}\big)$ *realizing* $\delta\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{1,1}, \mathsf{T}^2\big)$ *is* $\mathsf{U}(2)$*-biinvariant. Hence,* $\delta\big(\mathsf{SU}(3) \times \mathsf{T}^2/\mathsf{S}^1_{1,1}, \mathsf{T}^2\big) = \delta\big(\mathsf{SU}(3) \times \mathsf{U}(2)/\mathsf{S}^1_{1,1}, \mathsf{U}(2)\big) = \frac{1}{37}$.

5.3. **The pinching constants of invariant metrics on** $\mathsf{SU}(3)/\mathsf{T}^2$**.** We compute the minimal and the maximal sectional curvatures of an invariant Riemannian metric on $\mathsf{SU}(3)/\mathsf{T}^2$ using the invariantly modified curvature operators of T^2-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}}$ and the fibration $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}} \to \mathsf{SU}(3)/\mathsf{T}^2$ with S^1-fibres.

The isotropy representation of $\mathsf{SU}(3)/\mathsf{T}^2$ decomposes into the three spaces $\mathbb{C}_j$ of (8). Therefore, we have a three parameter family of invariant Riemannian metrics $\langle\cdot,\cdot\rangle_{\vec{s}}$. Each metric $\langle\cdot,\cdot\rangle_{\vec{s}}$ is defined in the same way as $\langle\cdot,\cdot\rangle_{t,s}$, i.e., by scaling the restrictions of the $\mathrm{Ad}_{\mathsf{SU}(3)}$-invariant inner product $\langle\!\langle X, Y\rangle\!\rangle := -\frac{1}{2}\operatorname{Re}\operatorname{tr}(XY)$ to each $\mathbb{C}_j$ by s_j. For any $\vec{k}$ and any $t > 0$ the fibration $\mathsf{SU}(3)/\mathsf{S}^1_{\vec{k}} \to \mathsf{SU}(3)/\mathsf{T}^2$ of Proposition 3.3 is a Riemannian submersion with respect to the metrics $\langle\cdot,\cdot\rangle_{t,\vec{s}}$ and $\langle\cdot,\cdot\rangle_{\vec{s}}$. It follows from the O'Neill formulas that the curvature operator of $\langle\cdot,\cdot\rangle_{\vec{s}}$ can be obtained from that one of $\langle\cdot,\cdot\rangle_{t,\vec{s}}$ by restricting the latter one to

$$\Lambda_2(\mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2) \;=\; 3\bar{\varepsilon} \oplus V_0 \oplus V_1 \oplus V_2 \oplus \mathbb{C}_0^b \oplus \mathbb{C}_1^b \oplus \mathbb{C}_2^b$$

and setting $t = 0$.

The invariantly modified curvature operator $\hat{R}(\hat{\eta}_0, \hat{\eta}_1, \hat{\eta}_1, \xi)$ of $\langle\cdot,\cdot\rangle_{t,\vec{s}}$ given in Proposition 3.9 projects to an invariantly modified curvature operator of $\langle\cdot,\cdot\rangle_{\vec{s}}$, which we denote by $\hat{R}(\hat{\eta}_0, \hat{\eta}_1, \hat{\eta}_1)$. The entries of this operator are the numbers $\tilde{a}_j$, $\tilde{b}_j$, and $\tilde{d}_j = d_j$, which are obtained from a_j, b_j, and d_j by setting $t = 0$:

$$\tilde{a}_j = \tfrac{4}{s_j}\,, \quad \tilde{b}_j = -\tfrac{\sigma}{\sigma_3} + \tfrac{s_{j-1} - s_j + s_{j+1}}{s_{j-1}s_{j+1}}\,, \quad d_j = -\tfrac{\sigma}{4\sigma_3} + \tfrac{s_{j-1} - s_j + s_{j+1}}{s_{j-1}s_{j+1}}.$$

Since the d_j are sectional curvatures, we have

$$\sigma = 4\sigma_3(d_0 + d_1 + d_2) > 0 \quad \text{and} \quad s_{j-1} - s_j + s_{j+1} > 0$$

for a positively curved Riemannian metric $\langle\cdot,\cdot\rangle_{\vec{s}}$.

Lemma 5.10 (cf. [Va])**.** *For any positively curved invariant Riemannian metric* $\langle\cdot,\cdot\rangle_{\vec{s}}$ *on* $\mathsf{SU}(3)/\mathsf{T}^2$ *the minimum of the sectional curvature is given by* $\min K = \min\{\bar{\lambda}, d_0, d_1, d_2\}$*, where* $\bar{\lambda} := \frac{3\sigma_1\sigma_3 - \sigma_2\sigma}{2\sigma_2\sigma_3}$.

Proof. We apply Proposition 4.11. Notice that for $\mathsf{SU}(3)/\mathsf{T}^2$ the number $\hat{\lambda}$ is simply $\min\{d_0, d_1, d_2\}$. The lemma follows, since $\sum_{j=0}^{2}(\tilde{a}_j - \tilde{b}_j) = \frac{2\sigma + 4\sigma_2}{\sigma_3} > 0$ and $\hat{D}_j = 6\frac{\sigma}{\sigma_3}\, s_j(s_{j+1} + s_{j-1}) > 0$. ∎

Lemma 5.11 (cf. [Va]). *For any positively curved invariant Riemannian metric* $\langle\cdot,\cdot\rangle_{\vec{s}}$ *on* $\mathsf{SU}(3)/\mathsf{T}^2$ *the maximum of the sectional curvature is given by* $\max K = \max\{\tilde{a}_0, \tilde{a}_1, \tilde{a}_2\}$.

Proof. The Weyl group D_3 of $\mathsf{SU}(3)$ acts on $\mathsf{SU}(3)/\mathsf{T}^2$. Any permutation $\pi \in D_3$ induces an isometry between $\langle\cdot,\cdot\rangle_{\vec{s}}$ and $\langle\cdot,\cdot\rangle_{\pi(\vec{s})}$. Hence, we can assume $s_0 \leq s_1 \leq s_2$. The following inequalities are now apparent:

$$\tilde{a}_0 \geq \tilde{a}_1 \geq \tilde{a}_2, \quad \tilde{b}_0 \geq \tilde{b}_1 \geq \tilde{b}_2, \quad d_0 \geq d_1 \geq d_2.$$

Lemma 4.13 holds for $\mathsf{SU}(3)/\mathsf{T}^2$, too. Simply one has to ignore the c_j. A straightforward computation shows $2d_0 - \tilde{b}_0 < \tilde{a}_1$. Hence, $\Lambda_0 = \max\{\tilde{a}_0, \tilde{a}_1, \tilde{a}_2\} = \tilde{a}_0$. Moreover, it is easy to see that $\tilde{a}_0 > d_0 > \tilde{b}_0$. We can now apply Lemma 4.14. $\square$

At this point one has to solve a 2-dimensional maximization problem in order to compute $\delta\big(\mathsf{SU}(3), \mathsf{T}^2\big)$: One can for instance normalize the metric such that $s_2 = 1$, as Valiev did. Moreover, one can restrict to the domain where all $d_j > 0$ and $s_0 \leq s_1 \leq s_2$. On this domain one has to find the maximum of

$$\delta\big(\mathsf{SU}(3)/\mathsf{T}^2, \langle\cdot,\cdot\rangle_{\vec{s}}\big) = \min\{\tfrac{3\sigma_1\sigma_3 - \sigma_2\sigma}{2\sigma_2\sigma_3}, d_2\}/\tilde{a}_0.$$

Valiev solved this maximization problem in a straightforward manner. We do not reproduce his proof here, but just state the result. Note that the maximum is attained at the boundary of the domain.

Theorem 5.12 (see [Va]). *The pinching constant of* $\big(\mathsf{SU}(3), \mathsf{T}^2\big)$ *is* $\frac{1}{64}$. *It is attained, up to scaling, exactly for the three invariant Riemannian metrics* $\langle\cdot,\cdot\rangle_{1,2,2}$, $\langle\cdot,\cdot\rangle_{2,1,2}$, *and* $\langle\cdot,\cdot\rangle_{2,2,1}$ *on* $\mathsf{SU}(3)/\mathsf{T}^2$.

Valiev computed the pinching constants of three corresponding invariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{T}^2$, $\mathsf{Sp}(3)/\mathsf{Sp}(1)^3$, and $\mathsf{F}_4/\mathsf{Spin}(8)$ with a single computation. In fact, the coincidence of the pinching constants for corresponding invariant Riemannian metrics on these three spaces can be explained analogously to Section 7 using totally geodesic embeddings $\mathsf{SU}(3)/\mathsf{T}^2 \hookrightarrow \mathsf{Sp}(3)/\mathsf{Sp}(1)^3 \hookrightarrow \mathsf{F}_4/\mathsf{Spin}(8)$.

6. Extremal sectional curvatures on $\mathsf{Sp}(2)/(\mathsf{Sp}(1) \times \mathsf{S}^1) \approx \mathbb{CP}^3$

In this section we compute the extremal values of the sectional curvature for the two parameter family of homogeneous Riemannian metrics on $\mathbb{CP}^3$ coming from the fibration $\mathbb{CP}^3 \to \mathbb{HP}^1$ with fibers $\mathbb{S}^2$. We use the corresponding quotient description $\mathsf{Sp}(2)/(\mathsf{Sp}(1) \times \mathsf{S}^1)$ for $\mathbb{CP}^3$. Note that by Wallach's classification [Wa] any homogeneous Riemannian metric of positive curvature on $\mathbb{CP}^3$ is known to be isometric to such an element of $\mathcal{M}\big(\mathsf{Sp}(2), \mathsf{Sp}(1) \times \mathsf{S}^1\big)$. Although the optimal pinched metric on $\mathbb{CP}^3$ is known to be the symmetric Fubini-Study metric by Berger's rigidity theorem, our computations are nevertheless useful, since $\mathsf{Sp}(2)/(\mathsf{Sp}(1) \times \mathsf{S}^1)$ is the universal covering space of $\mathsf{SO}(5)/\hat{\mathsf{U}}(2)$. The latter space is a totally geodesic submanifold of the Berger space $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$, and we shall use this fact in the next section to determine the pinching constant of $\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big)$. The computations themselves demonstrate in an easier setting how to apply the techniques developed in Section 2.

We fix our conventions for the quaternions $\mathbb{H} = \operatorname{span}_{\mathbb{R}}\{1, I, J, K\}$. Since we consider the space $\mathbb{H}^n$ of column vectors to be a right vector space, i.e., scalars operate by multiplication on the right, multiplication by quaternionic matrices on the left yields $\mathbb{H}$-linear maps of $\mathbb{H}^n$. The embedding $\mathbb{C} \hookrightarrow \mathbb{H}$, $a + ib \mapsto a + Ib$ induces an isomorphism $\mathbb{C}^{2n} \to \mathbb{H}^n$, $(z_1, z_2)^t \mapsto z_1 + Jz_2$ of complex vector spaces and hence an algebra homomorphism

$$\mathrm{Mat}(n, \mathbb{H}) \to \mathrm{Mat}(2n, \mathbb{C}), \quad A_1 + JA_2 \mapsto \left(\begin{smallmatrix} A_1 & -\bar{A}_2 \\ A_2 & \bar{A}_1 \end{smallmatrix}\right).$$

This homomorphism identifies the group $\mathsf{Sp}(n)$ of quaternionic unitary matrices with the subgroup of $\mathsf{SU}(2n)$ consisting of all matrices preserving the complex bilinear form $\mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$, $(z_1, z_2) \mapsto z_1^t \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right) z_2$.

We consider the embedding

$$\mathsf{Sp}(1) \times \mathsf{S}^1 \to \mathsf{Sp}(2), \quad (q, \lambda) \mapsto \left(\begin{smallmatrix} q & 0 \\ 0 & \bar{\lambda} \end{smallmatrix}\right). \tag{16}$$

The quotient $\mathsf{Sp}(2)/(\mathsf{Sp}(1) \times \mathsf{S}^1)$ is diffeomorphic to $\mathbb{CP}^3$, but the biinvariant Riemannian metric $\langle\!\langle X, Y \rangle\!\rangle = -\operatorname{Re}\operatorname{tr}(XY)$ on $\mathsf{Sp}(2)$ (viewed as a group of quaternionic matrices) does not induce the symmetric Fubini-Study metric on $\mathbb{CP}^3$. The involution $\theta : \mathsf{Sp}(2) \to \mathsf{Sp}(2)$, $A \mapsto -JAJ$, which is the Cartan involution of the symmetric pair $(\mathsf{Sp}(2), \mathsf{U}(2))$, maps the subgroup $\mathsf{Sp}(1) \times \mathsf{S}^1$ to itself and hence defines an involution on $\mathsf{Sp}(2)/(\mathsf{Sp}(1) \times \mathsf{S}^1)$, which we also denote by θ. More precisely, θ induces on $\mathsf{Sp}(1)$ the Cartan involution of the symmetric pair $(\mathsf{Sp}(1), \mathsf{U}(1))$ and acts on S^1 by $\lambda \mapsto \lambda^{-1}$. Note that the map θ is nothing but conjugation by the element $\left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right) \in \mathsf{Sp}(2)$. Therefore, we can extend the group $\mathsf{Sp}(1) \times \mathsf{S}^1$ by $\left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$ to a subgroup $(\mathsf{Sp}(1) \times \mathsf{S}^1)^\wedge \subset \mathsf{Sp}(2)$. In this way we get a quotient $\mathsf{Sp}(2)/(\mathsf{Sp}(1) \times \mathsf{S}^1)^\wedge \approx \mathbb{CP}^3/\mathbb{Z}_2$. Note also that θ is given by the complex conjugation on $\mathsf{SU}(4)$ if $\mathsf{Sp}(2)$ is embedded into $\mathsf{SU}(4)$ as above.

Lemma 6.1. *The isotropy representation of* $\mathsf{Sp}(2)/(\mathsf{Sp}(1) \times \mathsf{S}^1)$ *decomposes into two inequivalent irreducible* $\mathsf{Sp}(1) \times \mathsf{S}^1$*-representations, which can be identified with* $\mathbb{C}$ *and* $\mathbb{H}$ *in such a way that* $\mathrm{Ad}_{(q,\lambda)}\, w = \lambda^2 w$ *for* $w \in \mathbb{C}$ *and* $\mathrm{Ad}_{(q,\lambda)}\, u = qu\lambda$ *for* $u \in \mathbb{H}$. *Hence, the space* $\mathcal{M}\big(\mathsf{Sp}(2), \mathsf{Sp}(1) \times \mathsf{S}^1\big)$ *of invariant Riemannian metrics* $\langle \cdot, \cdot \rangle_{t,s}$ *on* $\mathsf{Sp}(2)/(\mathsf{Sp}(1) \times \mathsf{S}^1)$ *can be parametrized by* $t, s > 0$. *The involution* θ *acts isometrically for each* $\langle \cdot, \cdot \rangle_{t,s}$.

Proof. The images of the $\mathbb{R}$-linear maps

$$\begin{aligned} \mathbb{C} &\to \mathfrak{sp}(2), \quad w \mapsto \tfrac{1}{2\sqrt{t}}\left(\begin{smallmatrix} 0 & 0 \\ 0 & Jw \end{smallmatrix}\right), \\ \mathbb{H} &\to \mathfrak{sp}(2), \quad u \mapsto \tfrac{1}{2\sqrt{s}}\left(\begin{smallmatrix} 0 & u \\ -\bar{u} & 0 \end{smallmatrix}\right). \end{aligned} \tag{17}$$

are invariant under the adjoint action of $\mathsf{Sp}(1) \times \mathsf{S}^1$ and their sum is a subspace of $\mathfrak{sp}(2)$ complementary to the Lie algebra $\mathfrak{sp}(1) \oplus \mathbb{R}\left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$ of $\mathsf{Sp}(1) \times \mathsf{S}^1$. These maps induce $\mathrm{Ad}_{(q,\lambda)}\, w = \lambda^2 w$ for $w \in \mathbb{C}$ and $\mathrm{Ad}_{(q,\lambda)}\, u = qu\lambda$ for $u \in \mathbb{H}$.

We restrict the biinvariant metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ to the subspaces of $\mathfrak{su}(2)$ corresponding to $\mathbb{C}$ and $\mathbb{H}$, and define the invariant Riemannian metric $\langle \cdot, \cdot \rangle_{t,s}$ on $\mathsf{Sp}(2)/(\mathsf{Sp}(1) \times \mathsf{S}^1)$ by scaling these restrictions by $4t$ and $2s$, respectively. In this way our identifications (17) induce the standard metrics on $\mathbb{C}$ and $\mathbb{H}$.

The involution θ is an isometry of the biinvariant metric on $\mathsf{Sp}(2)$ that preserves the decomposition. It acts on $\mathbb{C}$ by complex conjugation and on $\mathbb{H}$ by $\theta(u) = -JuJ$, i.e., as the complex conjugation of $\mathbb{C}^2$. Hence θ is an isometry for each $\langle \cdot, \cdot \rangle_{t,s}$. $\square$

Lemma 6.2. *The decomposition of* $\Lambda_2(\mathbb{C} \oplus \mathbb{H})$ *into irreducible* $(\mathsf{Sp}(1) \times \mathsf{S}^1)^\wedge$*-representations is given by*

$$\Lambda_2(\mathbb{C} \oplus \mathbb{H}) \;\approx\; 2\bar{\varepsilon} \oplus \mathbb{C} \oplus \mathbb{R}^3 \oplus \mathbb{H} \oplus \tilde{\mathbb{H}},$$

where $2\bar{\varepsilon}$ is a trivial 2-dimensional $\mathsf{Sp}(1)\times\mathsf{S}^1$-representation, on which θ acts as $-\mathrm{id}$, where $\mathbb{R}^3$ is equivalent to $\mathfrak{sp}(1)$ viewed as a representation space, and where $\tilde{\mathbb{H}}$ is inequivalent to $\mathbb{H}$.

Proof. The space $\Lambda_2(\mathbb{C}\oplus\mathbb{H})$ splits naturally into $\Lambda_2\mathbb{C}$, $\Lambda_2\mathbb{H}$, and $\mathbb{C}\otimes\mathbb{H}$. The space $\Lambda_2\mathbb{C}$ is spanned by the vector $\epsilon_0 = 1\wedge i$. In order to decompose $\Lambda_2\mathbb{H}$ we restrict the Lie bracket of $\mathfrak{sp}(2)$ to this space, which gives us an equivariant linear map $\Lambda_2\mathbb{H}\to\mathfrak{sp}(2)\approx\mathfrak{sp}(1)\oplus\mathbb{R}\left(\begin{smallmatrix}0&0\\0&I\end{smallmatrix}\right)\oplus\mathbb{C}\oplus\mathbb{H}$. For $u_1,u_2\in\mathbb{H}$ we have

$$[u_1,u_2] \;=\; \tfrac{1}{4s}\left(\begin{smallmatrix}-u_1\bar{u}_2+u_2\bar{u}_1 & 0\\ 0 & -\bar{u}_1u_2+\bar{u}_2u_1\end{smallmatrix}\right)\;\in\mathbb{H}^\perp\subset\mathfrak{sp}(2).$$

The images of the three unit vectors

$$\begin{aligned}
\epsilon_1 &:= \tfrac{1}{\sqrt{2}}(1\wedge I - J\wedge K), & [\epsilon_1] &= -\tfrac{1}{\sqrt{2}s}\left(\begin{smallmatrix}0&0\\0&I\end{smallmatrix}\right),\\
v &:= \tfrac{1}{\sqrt{2}}(1\wedge J + I\wedge K), & [v] &= -\tfrac{1}{\sqrt{2}s}\left(\begin{smallmatrix}0&0\\0&J\end{smallmatrix}\right),\\
x &:= \tfrac{1}{\sqrt{2}}(1\wedge I + J\wedge K), & [x] &= \tfrac{1}{\sqrt{2}s}\left(\begin{smallmatrix}I&0\\0&0\end{smallmatrix}\right),
\end{aligned}$$

are contained in the three inequivalent representation spaces in $\subset\mathfrak{sp}(2)$ that are perpendicular to $\mathbb{H}$. Schur's Lemma implies that $\dim[\mathbb{H},\mathbb{H}] = 6 = \dim\Lambda_2\mathbb{H}$, and therefore $\Lambda_2\mathbb{H}$ decomposes into $\mathbb{R}\epsilon_1$, $\mathbb{C}$, and $\mathbb{R}^3\approx\mathfrak{sp}(1)$.

For the decomposition of $\mathbb{C}\otimes\mathbb{H}$ we identify $\mathbb{C}\otimes\mathbb{H}$ with two copies $\hat{\mathbb{H}}$ and $\tilde{\mathbb{H}}$ of the quaternions by the two $\mathbb{R}$-linear maps

$$\begin{aligned}
\hat{\mathbb{H}}\to\mathbb{C}\otimes\mathbb{H}, &\quad \hat{u}\mapsto\; 1\otimes\hat{u}J - i\otimes\hat{u}K,\\
\tilde{\mathbb{H}}\to\mathbb{C}\otimes\mathbb{H}, &\quad \tilde{u}\mapsto\; 1\otimes\tilde{u}J + i\otimes\tilde{u}K.
\end{aligned}$$

The pair $(q,\lambda)\in\mathsf{Sp}(1)\times\mathsf{S}^1$ acts on $\hat{\mathbb{H}}$ by $\hat{u}\mapsto q\hat{u}\lambda$ and on $\tilde{\mathbb{H}}$ by $\tilde{u}\mapsto q\tilde{u}\bar{\lambda}^3$. The involution θ acts on both spaces in the standard way. $\square$

Lemma 6.3. *The $(\mathsf{Sp}(1)\times\mathsf{S}^1)^\wedge$-invariant elements $T\in\Lambda^4(\mathbb{C}\oplus\mathbb{H})$ are precisely the duals of*

$$\eta\cdot 1_{\mathbb{H}}\wedge I\wedge J\wedge K + \xi\cdot 1_{\mathbb{C}}\wedge i\wedge(1_{\mathbb{H}}\wedge I - J\wedge K),$$

where $\eta,\xi\in\mathbb{R}$.

Proof. This follows from the standard decomposition formula and Lemma 6.2. $\square$

Lemma 6.4. *The modified curvature operator $\hat{R}_T$ of the invariant Riemannian metric $\langle\cdot,\cdot\rangle_{t,s}$ on $\mathsf{Sp}(2)/(\mathsf{Sp}(1)\times\mathsf{S}^1)$ is given by*

$$\begin{aligned}
\hat{R}_T|_{\mathbb{C}} &= \tfrac{s-t}{s^2}-\eta, & \hat{R}_T|_{\mathbb{R}^3} &= \tfrac{2s-t}{2s^2}+\eta,\\
\hat{R}_T|_{\mathbb{H}} &= \tfrac{1}{2s}-\xi, & \hat{R}_T|_{\tilde{\mathbb{H}}} &= -\tfrac{s-t}{2s^2}+\xi,
\end{aligned}$$

$$\hat{R}_T|_{2\bar{\varepsilon}} = \begin{pmatrix} \frac{1}{t} & \frac{2s-t}{\sqrt{2}s^2}+\sqrt{2}\xi\\ \frac{2s-t}{\sqrt{2}s^2}+\sqrt{2}\xi & \frac{2s+t}{2s^2}-\eta\end{pmatrix}.$$

Proof. The formulas are obtained by straightforward computations using Lemma 2.7 and the unit vectors from the proof of Lemma 6.2. $\square$

We use Lemma 6.4 to derive the extremal sectional curvatures of $\langle\cdot,\cdot\rangle_{t,s}$. These were already known to Ziller (see [Zi]).

Proposition 6.5. *The extremal sectional curvatures of the invariant Riemannian metric* $\langle\cdot,\cdot\rangle_{t,s}$ *on* $\mathsf{Sp}(2)/(\mathsf{Sp}(1)\times\mathsf{S}^1)$ *are given by*

$$\min K \;=\; \begin{cases} \frac{t}{4s^2} & \text{if } t\le s,\\ \frac{4s-3t}{4s^2} & \text{if } s\le t,\end{cases} \qquad \max K \;=\; \begin{cases} \frac{1}{t} & \text{if } t\le s,\\ \frac{1}{s} & \text{if } s\le t\le \frac{5}{2}s.\end{cases}$$

In particular, the sectional curvature is strictly positive if and only if $4s>3t$.

Proof. Inspecting the modified curvature operator in Lemma 6.4, it is easy to see that the numbers $\frac{1}{t}$, $\frac{1}{s}$, $\frac{t}{4s^2}$, and $\frac{4s-3t}{4s^2}$ are the sectional curvatures of the planes $1_{\mathbb{C}}\wedge i$, $1_{\mathbb{H}}\wedge I$, $1_{\mathbb{C}}\wedge 1_{\mathbb{H}}$, and $1_{\mathbb{H}}\wedge J$, respectively. The proposition is proved by applying Proposition 2.5 and checking a few inequalities. $\square$

The planes $1_{\mathbb{C}}\wedge 1_{\mathbb{H}}$ and $1_{\mathbb{H}}\wedge J$ are the extremal planes in the tangent space of the totally geodesic $\mathbb{RP}^3=\mathrm{Fix}\,\theta$ with tangent space $\mathrm{span}\{1_{\mathbb{C}},1_{\mathbb{H}},J\}$. The Riemannian covering space of this $\mathbb{RP}^3$ is a Berger sphere with extremal sectional curvatures $\frac{t}{4s^2}$ and $\frac{4s-3t}{4s^2}$. We interpret the other planes in terms of the fibration

$$\mathbb{S}^2\to\mathbb{CP}^3\approx\mathsf{Sp}(2)/(\mathsf{Sp}(1)\times\mathsf{S}^1)\;\to\;\mathsf{Sp}(2)/(\mathsf{Sp}(1)\times\mathsf{Sp}(1))\approx\mathbb{HP}^1. \tag{18}$$

The plane $1_{\mathbb{C}}\wedge i$ is the tangent space of the totally geodesic $\mathbb{S}^2$-fiber, while the planes given by $1_{\mathbb{H}}\wedge I$ and $J\wedge K$ are horizontal lifts of the tangent spaces of the two $\mathbb{CP}^1\subset\mathbb{HP}^1$. Note that these $\mathbb{CP}^1$ admit totally geodesic lifts, while the entire base space $\mathbb{HP}^1$ can not be lifted totally geodesically. In fact, there doesn't exist any section of the bundle (18).

Corollary 6.6. *The pinching constant* $\delta\big(\mathsf{Sp}(2)/(\mathsf{Sp}(1)\times\mathsf{S}^1),\langle\cdot,\cdot\rangle_{t,s}\big)$ *of the invariant Riemannian metric* $\langle\cdot,\cdot\rangle_{t,s}$ *on* $\mathsf{Sp}(2)/(\mathsf{Sp}(1)\times\mathsf{S}^1)$ *satisfies*

$$\delta\big(\mathsf{Sp}(2)/(\mathsf{Sp}(1)\times\mathsf{S}^1),\langle\cdot,\cdot\rangle_{t,s}\big) \;=\; \begin{cases} \frac{t^2}{4s^2} & \text{if } t\le s,\\ \frac{4s-3t}{4s} & \text{if } s\le t\le\frac{4}{3}s.\end{cases}$$

Proof. This is a direct corollary of Proposition 6.5. $\square$

Now we turn to the quotient spaces $\mathsf{SO}(5)/\mathsf{U}(2)$ and $\mathsf{SO}(5)/\hat{\mathsf{U}}(2)$, where the subgroups $\mathsf{U}(2)$ and $\hat{\mathsf{U}}(2)$ are embedded into $\mathsf{SO}(5)$ by composing the homomorphism

$$\begin{gathered}\hat{\mathsf{U}}(2)=\mathsf{U}(2)\rtimes\mathbb{Z}_2\;\hookrightarrow\;\mathsf{SO}(4),\\ (A+iB,\mathrm{id})\;\mapsto\;\left(\begin{smallmatrix}A & -B\\ B & A\end{smallmatrix}\right),\quad(\mathbb{1},\theta)\;\mapsto\;\left(\begin{smallmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{smallmatrix}\right)\end{gathered} \tag{19}$$

with the standard embedding $\mathsf{SO}(4)\hookrightarrow\mathsf{SO}(5)$, $A\mapsto\left(\begin{smallmatrix}1 & 0\\ 0 & A\end{smallmatrix}\right)$.

Lemma 6.7. *The adjoint action of* $\mathsf{Sp}(2)\subset\mathsf{SU}(4)$ *on* $\mathfrak{sp}(2)^\perp\subset\mathfrak{su}(4)$ *induces a double covering map* $\rho:\mathsf{Sp}(2)\to\mathsf{SO}(5)$*, which maps* $\mathsf{Sp}(1)\times\mathsf{Sp}(1)\subset\mathsf{Sp}(2)$ *onto the subgroup* $\mathsf{SO}(4)\subset\mathsf{SO}(5)$*, and so there is the following commutative diagram:*

$$\begin{tikzcd}
\mathbb{CP}^3 \arrow[r, equal] \arrow[d, "2:1"'] & \mathsf{Sp}(2)/(\mathsf{Sp}(1)\times\mathsf{S}^1) \arrow[r, "\approx", "\rho"'] \arrow[d, "2:1"'] & \mathsf{SO}(5)/\mathsf{U}(2) \arrow[d, "2:1"'] & \\
\mathbb{CP}^3/\mathbb{Z}_2 \arrow[r, equal] \arrow[d] & \mathsf{Sp}(2)/(\mathsf{Sp}(1)\times\mathsf{S}^1)^\wedge \arrow[r, "\approx", "\rho"'] \arrow[d] & \mathsf{SO}(5)/\hat{\mathsf{U}}(2) \arrow[d] & \\
\mathbb{HP}^1 \arrow[r, equal] & \mathsf{Sp}(2)/(\mathsf{Sp}(1)\times\mathsf{Sp}(1)) \arrow[r, "\approx", "\rho"'] & \mathsf{SO}(5)/\mathsf{SO}(4) \arrow[r, equal] & \mathbb{S}^4
\end{tikzcd}$$

Proof. The orthogonal complement of $\mathfrak{sp}(2)$ in $\mathfrak{su}(4)$ with respect to the $\mathrm{Ad}_{\mathsf{SU}(4)}$-invariant inner product $-\frac{1}{4}\operatorname{Re}\operatorname{tr}(XY)$ on $\mathfrak{su}(4)$ is a 5-dimensional euclidean vector space spanned by

$$\tilde{b}_0 := \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \tilde{b}_1 := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \tilde{b}_2 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$i\tilde{b}_1 := \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad i\tilde{b}_2 := \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

The adjoint action of $\mathsf{Sp}(2)$ on this vector space provides a nontrivial homomorphism $\rho : \mathsf{Sp}(2) \to \mathsf{SO}(5)$. Since both groups are 10-dimensional and simple, ρ is a covering map. In fact, it is the map that is customarily used in order to identify $\mathsf{Spin}(5)$ with $\mathsf{Sp}(2)$.

By our choice of the orthogonal basis in $\mathfrak{sp}(2)^\perp \subset \mathfrak{su}(4)$, it is clear that the subgroup $\mathsf{Sp}(1) \times \mathsf{Sp}(1) \subset \mathsf{Sp}(2)$ given by diagonal quaternionic matrices stabilizes the vector $\tilde{b}_0$, and thus its image under ρ is contained in the standard embedded $\mathsf{SO}(4) \subset \mathsf{SO}(5)$ as considered before. Therefore, ρ induces a double covering map $\mathsf{Sp}(1) \times \mathsf{Sp}(1) \to \mathsf{SO}(4) \subset \mathsf{SO}(5)$.

Restricting ρ to $\mathsf{Sp}(1) \times \mathsf{S}^1$ we obtain the 4-dimensional subgroup discribed in (19). By direct computations one sees that $\mathsf{Sp}(1) \times \mathsf{S}^1$ is a double covering space of $\mathsf{U}(2) \subset \mathsf{SO}(4)$ and that the elements $\pm\left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$ corresponding to the involution θ of $\mathsf{Sp}(2)$ are mapped onto $\left(\begin{smallmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{smallmatrix}\right) \in \mathsf{SO}(4)$ corresponding to the involution $-\theta$ of $\hat{\mathsf{U}}(2)$. □

Let us finally identify $\mathcal{M}\big(\mathsf{Sp}(2), \mathsf{Sp}(1) \times \mathsf{S}^1\big)$ with $\mathcal{M}\big(\mathsf{SO}(5), \hat{\mathsf{U}}(2)\big)$ by the diffeomorphism $\mathsf{Sp}(2)/(\mathsf{Sp}(1) \times \mathsf{S}^1) \to \mathsf{SO}(5)/\mathsf{U}(2)$. We denote the unique invariant Riemannian metric on $\mathsf{SO}(5)/\mathsf{U}(2)$ which pulls back to the Riemannian metric $\langle\cdot,\cdot\rangle_{t,s}$ also by $\langle\cdot,\cdot\rangle_{t,s}$.

Since the diffeomorphism is equivariant the isotropy representation of $\mathsf{SO}(5)/\mathsf{U}(2)$ decomposes also into two irreducible representations. These correspond to the spaces $\mathfrak{u}(2)^\perp \subset \mathfrak{so}(4)$ and $\mathfrak{so}(4)^\perp \subset \mathfrak{so}(5)$, where the orthogonal complement is taken with respect to the $\mathrm{Ad}_{\mathsf{SO}(5)}$-invariant inner product $\langle\!\langle X, Y\rangle\!\rangle = -\frac{1}{2}\operatorname{Re}\operatorname{tr}(XY)$ on $\mathfrak{so}(5)$. In order to see when the diffeomorphism $\mathsf{Sp}(2)/(\mathsf{Sp}(1)\times\mathsf{S}^1) \to \mathsf{SO}(5)/\mathsf{U}(2)$ is an isometry, it suffices to compute its differential for the two vectors

$$1_{\mathbb{C}} \;\cong\; \tfrac{1}{2\sqrt{t}}\left(\begin{smallmatrix} 0 & 0 \\ 0 & J \end{smallmatrix}\right) \;\cong\; \frac{1}{2\sqrt{t}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$1_{\mathbb{H}} \;\cong\; \tfrac{1}{2\sqrt{s}}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \;\cong\; \frac{1}{2\sqrt{s}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

By direct computations one obtains

$$\mathrm{ad}_{1_{\mathbb{C}}} \;\cong\; \frac{1}{2\sqrt{t}}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \in \mathfrak{u}(2)^\perp \subset \mathfrak{so}(4),$$
$$\mathrm{ad}_{1_{\mathbb{H}}} \;\cong\; \frac{1}{\sqrt{s}}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(4)^\perp \subset \mathfrak{so}(5).$$

Therefore, the diffeomorphism $\mathsf{Sp}(2)/(\mathsf{Sp}(1) \times \mathsf{S}^1) \to \mathsf{SO}(5)/\mathsf{U}(2)$ is isometric if the restrictions of $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ to $\mathfrak{u}(2)^\perp \subset \mathfrak{so}(4)$ and $\mathfrak{so}(4)^\perp \subset \mathfrak{so}(5)$ are scaled by $2t$ and s, respectively.

7. Extremal sectional curvatures on $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$

We determine the extremal values of the sectional curvature for all invariant Riemannian metrics of positive sectional curvature on $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$ by means of the Taimanov embedding and the totally geodesic embedding $\mathsf{SO}(5)/\hat{\mathsf{U}}(2)\hookrightarrow \mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$. In particular, we compute the pinching constants for these metrics and prove that $\delta\big(\mathsf{SU}(5),\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1\big)=\frac{1}{37}$.

In order to define the subgroup $\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1\subset\mathsf{SU}(5)$, we identify $\mathsf{Sp}(2)$ with the subgroup of matrices in $\mathsf{SU}(4)$ preserving the complex bilinear form $\mathbb{C}^4\times\mathbb{C}^4\to\mathbb{C}$, $(z_1,z_2)\mapsto z_1^t\left(\begin{smallmatrix}0&-1\!\!1\\1\!\!1&0\end{smallmatrix}\right)z_2$, just as in the preceding section. The image of $\mathsf{Sp}(2)\times\mathsf{S}^1$ under the homomorphism

$$\iota:\mathsf{SU}(4)\times\mathsf{S}^1\to\mathsf{U}(4),\quad (A,\lambda)\mapsto\lambda A,$$

contains $\mathsf{Sp}(2)$ and S^1 as subgroups and is covered twice by $\mathsf{Sp}(2)\times\mathsf{S}^1$. The subgroup $\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1\subset\mathsf{SU}(5)$ is the image of $\iota(\mathsf{Sp}(2)\times\mathsf{S}^1)$ under the embedding

$$\mathsf{U}(4)\hookrightarrow\mathsf{SU}(5),\quad A\mapsto\left(\begin{smallmatrix}\det A^{-1}&0\\0&A\end{smallmatrix}\right).$$

The involution $\theta:\mathsf{SU}(5)\to\mathsf{SU}(5)$, $A\mapsto\bar{A}$ maps $\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1$ to itself. This involution is the Cartan involution of the symmetric pair $(\mathsf{SU}(5),\mathsf{SO}(5))$. Moreover, θ induces on $\mathsf{Sp}(2)$ the Cartan involution of the symmetric pair $(\mathsf{Sp}(2),\mathsf{U}(2))$ and acts on S^1 by $\lambda\mapsto\lambda^{-1}$. Hence, θ decends to an involution on the quotient $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$.

Lemma 7.1. *The isotropy representation of $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$ decomposes into two inequivalent irreducible $(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$-representations. Therefore, the space $\mathcal{M}\big(\mathsf{SU}(5),\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1\big)$ of invariant Riemannian metrics $\langle\cdot,\cdot\rangle_{t,s}$ can be parametrized by $t,s>0$. The involution θ acts isometrically for the each $\langle\cdot,\cdot\rangle_{t,s}$.*

Proof. The Lie subalgebras $\mathbb{R}\left(\begin{smallmatrix}-4i&0\\0&i1\!\!1\end{smallmatrix}\right)$, $\mathfrak{sp}(2)$, $\mathfrak{su}(4)\subset\mathfrak{su}(5)$ are clearly invariant under the adjoint action of $\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1$. We consider the $\mathrm{Ad}_{\mathsf{SU}(5)}$-invariant inner product $\langle\!\langle X,Y\rangle\!\rangle=-\frac{1}{2}\operatorname{Re}\operatorname{tr}(XY)$ on $\mathfrak{su}(5)$ and identify the orthogonal complement of $\mathbb{R}\left(\begin{smallmatrix}-4i&0\\0&i1\!\!1\end{smallmatrix}\right)\oplus\mathfrak{su}(4)$ in $\mathfrak{su}(5)$ with $\mathbb{C}^4$ via the map $z\mapsto\frac{1}{\sqrt{s}}\left(\begin{smallmatrix}0&-\bar{z}^t\\z&0\end{smallmatrix}\right)$. Then $\mathrm{Ad}_{(A,\lambda)}\,z=\lambda^5Az$, and $\mathbb{C}^4$ is the irreducible standard representation of $\mathsf{Sp}(2)$. We denote the standard base of $\mathbb{C}^4$ by $e_1,\dots,e_4$.

The action of S^1 on $\mathbb{R}\left(\begin{smallmatrix}-4i&0\\0&i1\!\!1\end{smallmatrix}\right)\oplus\mathfrak{su}(4)$ is trivial, and $\mathsf{Sp}(2)=\mathsf{Spin}(5)$ acts as $\mathsf{SO}(5)$ on the orthogonal complement of $\mathfrak{sp}(2)$ in $\mathfrak{su}(4)$ (see Lemma 6.7). We denote the standard base of $\mathbb{R}^5$ by b_0,b_1,ib_1,b_2,ib_2 and identify $\mathfrak{sp}(2)^\perp$ with $\mathbb{R}^5$ by assigning to each b_j the vector $\frac{1}{2\sqrt{t}}\tilde{b}_j$ from formula (20) and to each ib_j the vector $\frac{1}{2\sqrt{t}}i\tilde{b}_j$.

We restrict $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ to the images of $\mathbb{R}^5$ and $\mathbb{C}^4$, and define the homogeneous metric $\langle\cdot,\cdot\rangle_{t,s}$ on $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$ by scaling these restrictions by $2t$ and s, respectively. In this way our identifications induce the standard metrics on $\mathbb{R}^5$ and $\mathbb{C}^4$.

The involution θ is an isometry of the biinvariant Riemannian metric on $\mathsf{SU}(5)$, acts on $\mathbb{C}^4$ by complex conjugation and preserves $\mathbb{R}^5$. Hence it is an isometry for the entire family of homogeneous metrics on $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$. ∎

Lemma 7.2. *The Berger space $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$ is an $\mathbb{RP}^5$-bundle over $\mathbb{CP}^4$ with totally geodesic fibers of constant curvature $\frac{1}{t}$.*

Proof. Let $S := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -\mathbb{1}_{4\times 4}\end{smallmatrix}\right) \in \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1$. The isometry of $\mathsf{SU}(5)$ given by $A \mapsto SAS^{-1}$ fixes $\mathsf{U}(4) \supset \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1$. Hence by Lemma A.3 the fibration

$$\mathbb{RP}^5 = \mathsf{U}(4)/\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1 \to \mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1) \to \mathsf{SU}(5)/\mathsf{U}(4) = \mathbb{CP}^4$$

has totally geodesic fibers. The sectional curvature is constant on each of these fibers, since $\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1$ acts as $\mathsf{SO}(5)$ on the tangent space $\mathbb{R}^5$. The value $\frac{1}{t}$ can be read off from the totally geodesic embeddings given below. $\square$

The sectional curvature of the planes $x \wedge z$, where $x \in \mathbb{R}^5$ and $z \in \mathbb{C}^4$, is also constant and equals $\frac{t}{4s^2}$. Since $\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1$ acts as $\mathsf{SO}(5)$ on $\mathbb{R}^5$, we can assume $x = b_0$. The Lie bracket of x and z is simply given by the multiplication matrix by vector, and the matrix b_0 preserves the norm of the column vector z. Now Lemma 2.7 proves the assertion.

Proposition 7.3. *The fixed point set of θ is a connected 6-dimensional submanifold of $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$ diffeomorphic to $\mathsf{SO}(5)/\hat{\mathsf{U}}(2)$. The submanifold is totally geodesic for any of the metrics $\langle\cdot,\cdot\rangle_{t,s}$ on $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$, and each $\langle\cdot,\cdot\rangle_{t,s}$ induces the metric $\langle\cdot,\cdot\rangle_{t,s}$ on $\mathsf{SO}(5)/\hat{\mathsf{U}}(2)$ defined in the preceding section, i.e., there is an identification of the spaces $\mathcal{M}\big(\mathsf{SU}(5), \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1\big)$ and $\mathcal{M}\big(\mathsf{SO}(5), \hat{\mathsf{U}}(2)\big)$.*

Proof. For any metric $\langle\cdot,\cdot\rangle_{t,s}$, θ is an isometry. Hence, every component of $\mathrm{Fix}\,\theta$ is a totally geodesic submanifold of $(\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1), \langle\cdot,\cdot\rangle_{t,s})$. Let us first determine the connected component Q of $\mathrm{Fix}\,\theta$ containing $[\mathbb{1}]$ and then prove that $\mathrm{Fix}\,\theta$ is already connected, i.e., $Q = \mathrm{Fix}\,\theta$.

The tangent space of $\mathrm{Fix}\,\theta$ in $[\mathbb{1}]$ is spanned by $b_1, b_2, e_1, e_2, e_3, e_4$. Hence, Q is 6-dimensional. The involution θ fixes the subgroup $\mathsf{SO}(5)$ of $\mathsf{SU}(5)$, and Lemma A.3 implies that Q is diffeomorphic to $\mathsf{SO}(5)/(\mathsf{SO}(5) \cap \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$. It remains to prove that $\mathsf{SO}(5) \cap \mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1 = \hat{\mathsf{U}}(2)$. This can be verified by direct computations, but we prefer to give the following argument: The subgroup $\hat{\mathsf{U}}(2) \subset \mathsf{SO}(4) \subset \mathsf{SO}(5)$ of formula (19) is clearly contained in $\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1$. Since $\mathsf{SO}(5)/\hat{\mathsf{U}}(2)$ is 6-dimensional, Lemma A.3 implies that $\mathsf{SO}(5)/\hat{\mathsf{U}}(2) \to Q$ is a covering map. But by Synge's Theorem there can be no quotient of $\mathsf{SO}(5)/\hat{\mathsf{U}}(2) = \mathbb{CP}^3/\mathbb{Z}_2$. Hence, the map $\mathsf{SO}(5)/\hat{\mathsf{U}}(2) \to Q$ is a diffeomorphism.

By Lemma A.4 any component of $\mathrm{Fix}\,\theta$ is even dimensional, since Q is even dimensional. Moreover, the components are homogeneous spaces. If we choose a metric $\langle\cdot,\cdot\rangle_{t,s}$ on $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$ of positive sectional curvature, the components inherit metrics of positive curvature. Hence by Lemma A.2 their Euler characteristics are all positive. Lemma A.4 yields $\Lambda(\theta) \geq \chi(Q) = \chi(\mathbb{CP}^3/\mathbb{Z}_2) = 2$ and $\Lambda(\theta) > 2$, if there exists more than one component.

Let us now determine the Lefschetz number $\Lambda(\theta)$. The cohomology ring of $M^{13} = \mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$ is given in Table 3. The involution θ of $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$H^k(M^{13};\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}_5$	0	$\mathbb{Z}_5$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
generator	1		a_2		a_2^2		a_2^3		a_2^4	b_9		a_2b_9		$a_2^2b_9$

TABLE 3. Cohomology of $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$ due to Heintze [He2].

induces the map θ^* on cohomology, which maps a_2 onto $\pm a_2$ and b_9 onto $\pm b_9$. Let us denote these two signs by η_2 and η_9. Then the Lefschetz number is $\Lambda(\theta) = (2+\eta_2)(1-\eta_9)$. We shall show that $\eta_2 = -1$, so $\Lambda(\theta) \in \{0, 2\}$. Hence, by the inequalites above, we can without computing η_9 conclude that $\Lambda(\theta) = 2$ and that $\operatorname{Fix}\theta$ is connected.

It remains to show that $\eta_2 = -1$. Note that a_2 is the image of the generator of $H^2(\mathbb{CP}^4;\mathbb{Z})$ under the map induced by the projection $M^{13} \to \mathbb{CP}^4$ of Lemma 7.2. Moreover, θ projects to the complex conjugation on $\mathbb{CP}^4$. Evidently, There exists subspaces $\mathbb{CP}^1 \subset \mathbb{CP}^4$, that are mapped into theirselves (reversing the orientation of these $\mathbb{CP}^1$). Thus, we get a commutative diagram

$$\begin{array}{ccccc} \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^4 & \longleftarrow & M^{13} \\ {\scriptstyle\theta}\big\downarrow & & {\scriptstyle\theta}\big\downarrow & & {\scriptstyle\theta}\big\downarrow \\ \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^4 & \longleftarrow & M^{13} \end{array},$$

where all horizontal maps induce isomorphisms on the second cohomology groups. Therefore, $\theta^*(a_2) = -a_2$. □

Lemma 7.4. *There is an isometric copy $\tilde{Q}$ of $Q = \operatorname{Fix}\theta$ through $[\mathbb{1}]$ whose tangent space is given by* $\operatorname{span}\{ib_1, b_2, ie_1, e_2, e_3, ie_4\}$.

Proof. Apply the isometry of $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$ given by the matrix

$$\iota\big(\operatorname{diag}(e^{i\pi/4}, e^{-i\pi/4}, e^{-i\pi/4}, e^{i\pi/4}), e^{i\pi/20}\big)$$

to the fixed point set $Q = \operatorname{Fix}\theta$. □

Note that $\mathsf{SU}(5)$ doesn't act effectively on $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$ and that the isometry given by the matrix above is of order 4.

For any $p \in \mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$ there are actually at least eight isometric copies $Q_0, Q_1, \dots, Q_7$ of $Q = \operatorname{Fix}\theta$ which intersect orthogonally at p. These can be easily produced using elements of the standard maximal torus of $\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1$. If numbered in the right way, the connected components containing p of the orthogonal intersections $Q_l \cap Q_{l+4}$, $Q_l \cap Q_{l+2}$, $Q_l \cap Q_{l+1}$, and $Q_l \cap Q_{l+3}$ are totally geodesic submanifolds isometric to an $\mathbb{RP}^2$ with constant curvature $\frac{1}{t}$, to an $\mathbb{RP}^2$ with constant curvature $\frac{1}{s}$, and, in the last two cases, to lense spaces $\mathsf{SO}(3)/\mathbb{Z}_2$ covered by Berger spheres.

Let $S := \left(\begin{smallmatrix} \mathbb{1}_{3\times 3} & 0 \\ 0 & -\mathbb{1}_{2\times 2}\end{smallmatrix}\right) \in \mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1 \subset \mathsf{SU}(5)$ and $\sigma : \mathsf{SU}(5) \to \mathsf{SU}(5)$, $A \mapsto SAS^{-1}$. The map σ induces a map on $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$, which is an isometry for any metric $\langle\cdot,\cdot\rangle_{t,s}$. We denote this map also by σ. The following lemma is a refinement of a result of I.A. Taimanov (cf. [Ta]).

Proposition 7.5. *The connected component of $\operatorname{Fix}\sigma$ containing $[\mathbb{1}]$ is diffeomorphic to the Aloff–Wallach space $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$. Each metric $\langle\cdot,\cdot\rangle_{t,s}$ on the space $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$ induces the metric $\langle\cdot,\cdot\rangle_{t,s}$ on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$, i.e., there is an identification of $\mathcal{M}\big(\mathsf{SU}(5), \mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1\big)$ with the space $\mathcal{M}\big(\mathsf{SU}(3)\times\mathsf{U}(2)/\mathsf{S}^1_{1,1}, \mathsf{U}(2)\big)$ of $\mathsf{U}(2)$-biinvariant Riemannian metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$.*

Proof. The map σ fixes $\mathsf{S}(\mathsf{U}(3)\times\mathsf{U}(2))$ in $\mathsf{SU}(5)$ and $\iota(\mathsf{U}(2)^\bullet\times_{\mathbb{Z}_2}\mathsf{S}^1)$ in $\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1$, where

$$\mathsf{U}(2)^\bullet = \left\{\left(\begin{smallmatrix} B & 0 \\ 0 & \bar{B}\end{smallmatrix}\right) \,\middle|\, B \in \mathsf{U}(2)\right\} \subset \mathsf{Sp}(2).$$

By Lemma A.3 the quotient $\mathsf{S}(\mathsf{U}(3)\times\mathsf{U}(2))/\iota(\mathsf{U}(2)^\bullet\times_{\mathbb{Z}_2}\mathsf{S}^1)$ is the connected component of $\operatorname{Fix}\sigma$ containing $[\mathbb{1}]$. The embedding $\mathsf{SU}(3)\to\mathsf{S}(\mathsf{U}(3)\times\mathsf{U}(2))$, $A\mapsto\left(\begin{smallmatrix}A&0\\0&\mathbb{1}\end{smallmatrix}\right)$ yields a diffeomorphism

$$\mathsf{SU}(3)/\mathsf{S}^1_{1,1}\to\mathsf{S}(\mathsf{U}(3)\times\mathsf{U}(2))/\iota(\mathsf{U}(2)^\bullet\times_{\mathbb{Z}_2}\mathsf{S}^1)$$

and hence an embedding $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}\hookrightarrow\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$, which can easily be seen to be isometric with respect to the metrics $\langle\cdot,\cdot\rangle_{t,s}$. $\square$

It follows directly from Lemma A.4 and Lemma A.6 that any other component of $\operatorname{Fix}\sigma$ has dimension ≤ 5, provided such a component exists at all.

Lemma 7.6. *There are at least four isometric copies W_0,W_1,W_2,W_3 of $W=\operatorname{Fix}\sigma$ which intersect pairwise orthogonally at $[\mathbb{1}]$. The tangent spaces are given by*

$$\begin{aligned}
T_{[\mathbb{1}]}W_0 &= \operatorname{span}\{b_0,b_1,ib_1,e_1,ie_1,e_2,ie_2\},\\
T_{[\mathbb{1}]}W_1 &= \operatorname{span}\{b_0,b_2,ib_2,e_2,ie_2,e_3,ie_3\},\\
T_{[\mathbb{1}]}W_2 &= \operatorname{span}\{b_0,b_1,ib_1,e_3,ie_3,e_4,ie_4\},\\
T_{[\mathbb{1}]}W_3 &= \operatorname{span}\{b_0,b_2,ib_2,e_1,ie_1,e_4,ie_4\}.
\end{aligned}$$

Proof. The preimage of $S_0:=S$ in $\mathsf{Sp}(1)\times\mathsf{S}^1$ contains $\big(\operatorname{diag}(i,i,-i,-i),-i\big)$. Applying the Weyl group of $\mathsf{Sp}(2)$ to the first part and projecting down to $\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1$ yields the three matrices

$$S_1:=\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&-1\end{pmatrix},\quad S_2:=\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&-1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\quad S_3:=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&-1&0&0\\0&0&0&-1&0\\0&0&0&0&1\end{pmatrix},$$

which are conjugate to S by elements of $\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1$. By Lemma A.5 the fixed point sets of the corresponding pairwise commuting involutive isometries $\sigma_j:A\mapsto S_jAS_j^{-1}$ are isometric copies W_j of the fixed point set W of σ, which intersect orthogonally at $[\mathbb{1}]$. The tangent spaces $T_{[\mathbb{1}]}W_j$ are given by the fixed point sets of Ad_{S_j} in $\mathbb{R}^5\oplus\mathbb{C}^4$. $\square$

The connected components containing $[\mathbb{1}]$ of the orthogonal intersections of $W_j\cap W_{j+1}$ and $W_j\cap W_{j+2}$ are totally geodesic Berger spheres $\mathbb{S}^3$ with extremal curvatures $\frac{t}{4s^2}$ and $\frac{16s-3t}{4s^2}$, and totally geodesic $\mathbb{RP}^3$ of constant curvature $\frac{1}{t}$, respectively. This can be deduced directly from the tangent spaces and the discussion of totally geodesic submanifolds of $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ in Section 3.

Lemma 7.7. *Given any tangent vector $X\in T_{[\mathbb{1}]}\mathsf{SU}(5)/(\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1)$, there is a matrix $A\in\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1$ such that $\operatorname{Ad}_A X\in T_{[\mathbb{1}]}W_0$.*

Proof. Since $\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1$ acts as $\mathsf{SO}(5)$ on $\mathbb{R}^5$, we can assume that $pr_{\mathbb{R}^5}X=b_0$. The subgroup $\mathsf{Sp}(1)\times\mathsf{Sp}(1)\subset\mathsf{Sp}(2)$ of Lemma 6.7 stabilizes b_0. The first $\mathsf{Sp}(1)=\mathsf{SU}(2)$-factor acts transitively on the unit sphere in $\operatorname{span}\{e_1,ie_1,e_3,ie_3\}$ and the second factor transitively on the unit sphere in $\operatorname{span}\{e_2,ie_2,e_4,ie_4\}$. Hence we can move X into $\operatorname{span}\{b_0,e_1,e_2\}$. $\square$

Lemma 7.8. *The $\mathsf{Sp}(2)\times_{\mathbb{Z}_2}\mathsf{S}^1$-representations $\mathbb{R}^5$ and $\mathbb{C}^4$ decompose into the irreducible $\iota(\mathsf{U}(2)^\bullet\times_{\mathbb{Z}_2}\mathsf{S}^1)$-representations $\mathbb{R}^3$, $\mathbb{C}$, and $\mathbb{C}^2$, $\bar{\mathbb{C}}^2$, respectively. The space $\mathbb{R}^3\oplus\mathbb{C}^2$ corresponds to the tangent space $T_{[\mathbb{1}]}W_0$.*

Proof. Let (B,λ) be an abbreviation for the element $\left(\left(\begin{smallmatrix} B & 0 \\ 0 & \bar{B}\end{smallmatrix}\right),\lambda\right) \in \mathsf{U}(2)^\bullet \times \mathsf{S}^1$. In view of the embeddings

$$\mathfrak{su}(2) \to \mathfrak{su}(4),\ \ C \mapsto \tfrac{1}{2\sqrt{t}}\left(\begin{smallmatrix} C & 0 \\ 0 & -\bar{C}\end{smallmatrix}\right), \qquad \mathbb{C} \to \mathfrak{su}(4),\ \ w \mapsto \tfrac{1}{2\sqrt{t}}\left(\begin{smallmatrix} 0 & 0 & 0 & w \\ 0 & 0 & -w & 0 \\ 0 & \bar{w} & 0 & 0 \\ -\bar{w} & 0 & 0 & 0\end{smallmatrix}\right)$$

we have $\mathrm{Ad}_{\iota(B,\lambda)}\, C = BCB^{-1}$ for $C \in \mathfrak{su}(2)$ and $\mathrm{Ad}_{\iota(B,\lambda)}\, w = (\det B)w$ for $w \in \mathbb{C}$. Furthermore, we have

$$\mathrm{Ad}_{\iota(B,\lambda)}\, z \;=\; \lambda^5 \begin{pmatrix} B\left(\begin{smallmatrix} z_1 \\ z_2\end{smallmatrix}\right) \\ \bar{B}\left(\begin{smallmatrix} z_3 \\ z_4\end{smallmatrix}\right)\end{pmatrix} \qquad \text{for } z \in \mathbb{C}^4.$$

□

Lemma 7.9. *The extremal sectional curvatures of* $\left(\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1), \langle\cdot,\cdot\rangle_{t,s}\right)$ *are attained at planes tangent to* W_0, W_1, Q, *or* $\tilde{Q}$, *i.e., tangent to a totally geodesically embedded Aloff–Wallach space* $\left(\mathsf{SU}(3)/\mathsf{S}^1_{1,1}, \langle\cdot,\cdot\rangle_{t,s}\right)$ *or tangent to a totally geodesically embedded* $\left(\mathsf{SO}(5)/\hat{\mathsf{U}}(2), \langle\cdot,\cdot\rangle_{t,s}\right)$.

Proof. Suppose the contrary. Consider a plane $X \wedge Y$ of extremal sectional curvature. By Lemma 7.7 we can assume $X \in T_{[1\!\!1]}W_0$. Since the sectional curvature is extremal, Y is an eigenvector of the Jacobi field operator $R(\,\cdot\,, X)X$. The Codazzi equation implies that this operator is block diagonal, i.e., it maps $T_{[1\!\!1]}W_0$ into itself. We therefore have $Y \in T_{[1\!\!1]}W_0^\perp \approx \mathbb{C} \oplus \bar{\mathbb{C}}^2$ because of our assumption. By the action of $\iota(\mathsf{U}(2)^\bullet \times_{\mathbb{Z}_2} \mathsf{S}^1)$ we can achieve $Y \in \mathrm{span}\{b_2, e_3\}$, while X remains in $T_{[1\!\!1]}W_0$. Now we have $Y \in T_{[1\!\!1]}W_1 \cap T_{[1\!\!1]}Q \cap T_{[1\!\!1]}\tilde{Q}$, and by the same reasoning as above X must be perpendicular to each of these three tangent spaces. But their direct sum is the whole tangent space of $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$. □

Theorem 7.10. *The sectional curvature of the invariant Riemannian metric* $\langle\cdot,\cdot\rangle_{t,s}$ *on* $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$ *is strictly positive if and only if* $t < s$. *In this case, the pinching constant of the metric* $\langle\cdot,\cdot\rangle_{t,s}$ *coincides with the pinching constant of the Aloff–Wallach space* $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ *endowed with the metric* $\langle\cdot,\cdot\rangle_{t,s}$. *Hence, we have optimal pinching precisely for* $t = \frac{2}{5}s$, *where the pinching constant is* $\frac{1}{37}$.

Proof. The theorem is a direct consequence of Lemma 7.9, Lemma 5.1, Lemma 5.2, and Proposition 6.5. □

Appendix A. Totally geodesic submanifolds

In this appendix we summarize some facts about totally geodesic submanifolds. First, we consider the homogeneous situation. A complete totally geodesic submanifold of a homogeneous space is itself intrinsically homogeneous (see [KN], chapter VII, Corollary 8.10), since the tangential part of a Killing field is itself a Killing field. We restate this lemma in the case of positive sectional curvature. Note that a totally geodesic submanifold inherits the sectional curvature of its ambient manifold because of the Gauss equation.

Lemma A.1. *A complete, totally geodesic submanifold of dimension* ≥ 2 *in a homogeneous space of positive curvature is a homogeneous space of positive curvature and is therefore covered by a space in Wallach's and Bérard Bergery's classification lists (see Table 1 and Table 2).*

A direct topological consequence of Wallach's classification is the following fact, which is important in view of Lemma A.3 and Lemma A.4 below.

Lemma A.2. *The Euler characteristic of any even dimensional homogeneous space of positive curvature is strictly positive.*

Let ψ be an isometry of a complete Riemannian manifold (M, g). It is well known that each connected component of $\operatorname{Fix}\psi$ is a complete, totally geodesic submanifold. In this particular case the embedding is equivariant:

Lemma A.3. *Let G be a Lie group of isometries acting transitively on a complete Riemannian manifold (M, g), $\psi \in \mathsf{G}$, and $p \in \operatorname{Fix}\psi$. Let G' be the connected component of the centralizer $\{\varphi \in \mathsf{G} \mid \varphi \circ \psi = \psi \circ \varphi\}$ containing the identity. Then G' acts naturally on $\operatorname{Fix}\psi$, and the orbit $\mathsf{G}' \cdot p$ is a connected component of $\operatorname{Fix}\psi$, i.e., the connected component of $\operatorname{Fix}\psi$ containing p is diffeomorphic to $\mathsf{G}'/\mathsf{G}'_p$.*

Proof. The orbit $\mathsf{G}' \cdot p$ is a homogeneous space and hence a complete submanifold of M, that is clearly contained in $\operatorname{Fix}\psi$. Therefore, it suffices to prove that the tangent space to the orbit $\mathsf{G}' \cdot p$ at p is the full tangent space of $\operatorname{Fix}\psi$ at p. The action of G' on M induces the embedding $\mathsf{G}'/\mathsf{G}'_p \hookrightarrow \mathsf{G}/\mathsf{G}_p$. Now the result can easily be seen on Lie algebra level. □

Note that the normalizer of $\langle\psi\rangle$ also acts on $\operatorname{Fix}\psi$ and that the centralizer of ψ may be disconnected. In general, this may provide some isometric copies among the connected components.

There are some topological restrictions for the connected components of the fixed point set of an isometry. Furthermore, the existence of more than one component provides some information on the cut locus. The following lemma extends a result of Kobayashi (see [Ko]).

Lemma A.4. *Let ψ be an isometry of a compact orientable Riemannian manifold M with fixed point set $N := \operatorname{Fix}\psi \neq \emptyset$. Denote by $N_1, \dots, N_m$ the connected components of N. Then:*

1. *For fixed $l \in \{1, \dots, m\}$ each N_j, $j \neq l$, is contained in the cut locus of any point in N_l. In particular, $N_j \subset \operatorname{cut}(N_l)$.*
2. *The codimensions of the N_j are either all even or all odd dimensional, depending on whether ψ preserves or reverses the orientation of M.*
3. *The Lefschetz number $\Lambda(\psi)$ of $\psi : M \to M$ equals to the sum of the Euler characteristics of the fixed point sets N_j, i.e.,*

$$\Lambda(\psi) = \sum_{j=1}^{m} \chi(N_j).$$

Proof. 1. Let $p_l \in N_l$, $p_j \in N_j$ and consider a minimal geodesic from p_l to p_j. If ψ fixed this geodesic, $d\psi_{|p_l}$ would fix its tangent vector v in p_l. Since N_l is totally geodesic, this would imply $p_j \in N_l$. Contradiction.

2. Since 1 is not contained in the spectrum of $d\psi_{|p}|_{(T_p \operatorname{Fix}\psi)^\perp}$, the claim follows directly from the Jordan canonical form theorem for orthogonal matrices applied to $d\psi_{|p}$ for $p \in N$.

3. We use the notations and conventions of [Do]. Note that by Poincaré duality and by the universal coefficient theorem, we can compute the Lefschetz number $\Lambda(\psi)$ in homology or cohomology with integral or rational coefficients. Moreover,

since M is a compact euclidean neighborhood retract, $\Lambda(\psi)$ coincides with the fixed point index I_ψ. This fixed point index has the following fundamental property: If M is covered by a finite union of open sets U_j such that $U_j \cap U_l \cap \mathrm{Fix}(\psi) = \emptyset$ for $j \neq l$, then $I_\psi = \sum_j I_{\psi|_{U_j}}$. We take $U_0 = M \setminus \Big(\bigcup_{j=1}^m \overline{B(N_j, \varepsilon/2)}\Big)$ and $U_j = B(N_j, \varepsilon)$ for $j = 1, \dots, m$, where $B(N_j, \varepsilon)$ denotes the image of the union of balls with radius ε in the normal bundle of N_j under the exponential map, i.e., a solid Fermi tube around N_j. The number $\varepsilon > 0$ is chosen, such that the union of balls is mapped injectively. Since ψ has no fixed point in U_0 and hence $I_{\psi|_{U_0}} = 0$, we find that

$$I_\psi \;=\; \sum_{j=0}^{m} (I_{\psi|_{U_j}}) \;=\; \sum_{j=1}^{m} (I_{\psi|_{U_j}}).$$

Moreover, for $j \in \{1, \dots, m\}$ we have $I_{\psi|_{U_j}} = I_{\psi|_{\bar U_j}}$, since U_j is an open set in the compact euclidean neighborhood retract $\bar U_j$. The N_j are strong deformation retracts of the $\bar U_j$, and we get commutative diagrams

$$\begin{array}{ccc} N_j & \xrightarrow{\mathrm{id}} & N_j \\ \big\downarrow & & \big\downarrow \\ \bar U_j & \xrightarrow{\psi} & \bar U_j \end{array} \qquad \text{and} \qquad \begin{array}{ccc} H(N_j) & \xrightarrow{\mathrm{id}} & H(N_j) \\ \approx\big\downarrow & & \approx\big\downarrow \\ H(\bar U_j) & \xrightarrow{\psi_*} & H(\bar U_j) \end{array}.$$

Hence $\psi_* = \mathrm{id}$ and $I_{\psi|_{\bar U_j}} = \Lambda(\psi|_{\bar U_j}) = \chi(N_j)$. $\square$

Given a point $p \in M$, we can ask for isometric copies of the fixed point set of an isometry intersecting pairwise orthogonally at p. We use the knowledge of such configurations in Section 7 to understand the sectional curvature of the 13-dimensional Berger space $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times_{\mathbb{Z}_2} \mathsf{S}^1)$ to some amount. In the case of an involution, we have the following result.

Lemma A.5. *Let θ be an involutive isometry of a complete Riemannian manifold (M, g) and $p \in \mathrm{Fix}\,\theta$. Then pairwise orthogonally intersecting isometric copies of $\mathrm{Fix}\,\theta$ correspond to pairwise commuting involutive isometries of (M, g) which fix p and are conjugate to θ in $\mathrm{Isom}(M, g)$.*

Proof. Recall that two involutive orthogonal maps of an euclidean vector space commute if and only if their fixed point sets intersect orthogonally. In the following, we shall apply this implicitly to the differential of involutive isometries of (M, g) at some fixed point. An isometry of (M, g) is determined by the image of a point and its differential at this point. This implies that two involutive isometries which have the same fixed point set are identical and that two fixed point sets of involutive isometries containing a common point p intersect orthogonally if and only if the two involutions commute.

Now, if ψ is any isometry of (M, g) then $\psi(\mathrm{Fix}\,\theta) = \mathrm{Fix}(\psi \circ \theta \circ \psi^{-1})$, i.e., any isometric copy of $\mathrm{Fix}\,\theta$ is itself the fixed point set of a uniquely determined involutive isometry. On the other hand, the fixed point sets of two involutive isometries are isometric if the isometries are conjugate by an isometry. $\square$

Example. $\mathbb{CP}^2 = \mathbb{S}^5/\mathsf{S}^1$, where $\mathbb{S}^5 \subset \mathbb{C}^3$. Let $\pi : \mathbb{S}^5 \to \mathbb{CP}^2$ denote the canonical projection, and let σ be the orientation preserving isometry given by the matrix $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$. Then σ fixes the point $p = \pi\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ and its cut locus $\mathbb{CP}^1 = \pi\big(\big\{\left(\begin{smallmatrix} 0 \\ z \end{smallmatrix}\right) \mid z \in$

$\mathbb{S}^3 \subset \mathbb{C}^2\}\big)$. We have $\Lambda(\sigma) = \chi(\mathbb{CP}^2) = 3 = \chi(\{p\}) + \chi(\mathbb{CP}^1)$. There are exactly three isometric copies of $\operatorname{Fix}\sigma$ intersecting orthogonally at p. The matrices of the two other involutions can be obtained from σ by applying the Weyl group of $\mathsf{SU}(3)$.

If (M, g) admits a metric of positive sectional curvature, then the following lemma limits the number of "high dimensional" connected components. There are further applications of this lemma in Section 7. The lemma is due to Frankel [Fr].

Lemma A.6. *Let M be a complete Riemannian manifold of positive sectional curvature and N_1, N_2 two compact totally geodesic submanifolds. If* $\dim N_1 + \dim N_2 \geq \dim M$ *then $N_1 \cap N_2 \neq \emptyset$.*

Proof. Suppose $N_1 \cap N_2 = \emptyset$. Because of compactness, the restriction of the distance function of M to $N_1 \times N_2$ attains a positive minimum at some point (p_1, p_2). The distance is realized by some geodesic $c : [a, b] \to M$ with $c(a) = p_1$, $c(b) = p_2$. The first variation formula implies $\dot c(a) \perp T_{p_1}N_1$ and $\dot c(b) \perp T_{p_2}N_2$. We use parallel transport along c to move $T_{p_1}N_1$ into the orthogonal complement of $\dot c(b)$ in $T_{p_2}M$. The resulting subspace of $T_{p_2}M$ intersects $T_{p_2}N_2$ at least in a line, since $\dim N_1 + \dim N_2 \geq \dim M$. Therefore, we get a parallel unit vector field X along c with $X(a) \in T_{p_1}N_1$ and $X(b) \in T_{p_2}N_2$. Since N_1 and N_2 are totally geodesic, the variation $c_s(t) = \exp_{c(t)}(sX(t))$ of $c = c_0$ consists of curves c_s with $c_s(a) \in N_1$ and $c_s(b) \in N_2$. But M has positive sectional curvature and hence the second variation formula yields $\frac{d^2}{ds^2}L(c_s) < 0$, which is a contradiction to the fact that (p_1, p_2) is a minimum for the restricted distance function. ☐

Appendix B. The curvature tensor of a compact Lie group

We give a formula for the curvature tensor R of a left invariant Riemannian metric $g = \langle\cdot,\cdot\rangle$ on a compact Lie group G. We denote by X, Y, Z, W simultaneously elements in $\mathfrak{g}$ and the corresponding left invariant vector fields. Let $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ be any biinvariant Riemannian metric on G. The covariant derivative ∇ with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ at e is given by $\nabla_X Y = \frac{1}{2}[X, Y]$ and the curvature tensor R_0 by

$$R_0(X, Y; Z, W) \;=\; -\tfrac{1}{4}\langle\!\langle [X, Y], [Z, W]\rangle\!\rangle.$$

Since these two objects are easy to compute, we use of the following general identity (see [AS]):

$$\begin{aligned}
&R(X, Y; Z, W) \;= \\
&\tfrac{1}{4}\big(R_0(G\,X, Y; Z, W) + R_0(X, G\,Y; Z, W) \\
&\qquad\qquad\qquad + R_0(X, Y; G\,Z, W) + R_0(X, Y; Z, G\,W)\big) \\
&-\tfrac{1}{2}\big(\nabla^2_{\{X,W\}}g(Y, Z) - \nabla^2_{\{Y,W\}}g(X, Z) - \nabla^2_{\{X,Z\}}g(Y, W) + \nabla^2_{\{Y,Z\}}g(X, W)\big) \\
&-\;\;\big(\langle\!\langle B(X, W), G^{-1}B(Y, Z)\rangle\!\rangle - \langle\!\langle B(X, Z), G^{-1}B(Y, W)\rangle\!\rangle\big).
\end{aligned}$$

Here, $\nabla^2_{\{X,W\}}g := \frac{1}{2}\big(\nabla^2_{X,W}g + \nabla^2_{W,X}g\big)$ denotes the symmetrized second covariant derivative, G is the field of symmetric automorphisms which describes $\langle\cdot,\cdot\rangle$ with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle$, i.e., $\langle X, Y\rangle = \langle\!\langle GX, Y\rangle\!\rangle$, and $G^{-1}\,B = \Gamma$ is the difference between the covariant derivative of $\langle\cdot,\cdot\rangle$ and the covariant derivative ∇ of $\langle\!\langle\cdot,\cdot\rangle\!\rangle$, which means B is the tensor defined by the equation

$$2\langle\!\langle B(X,Y),Z\rangle\!\rangle \;=\; \nabla_X g(Y,Z)+\nabla_Y g(X,Z)-\nabla_Z g(X,Y).$$

First, we compute B and $\nabla^2_{\{\cdot,\cdot\}}g$. Since $\langle Y,Z\rangle$ is a constant function on G for arbitrary left invariant vector fields Y and Z, we have

$$\begin{aligned}\nabla_X g(Y,Z) \;&=\; d_X\big(\langle Y,Z\rangle\big)-\langle\nabla_X Y,Z\rangle-\langle Y,\nabla_X Z\rangle\\ &=\; \tfrac{1}{2}\,\langle[Y,X],Z\rangle+\tfrac{1}{2}\,\langle Y,[Z,X]\rangle.\end{aligned} \tag{21}$$

This yields

$$2\,\langle\!\langle B(X,Y),Z\rangle\!\rangle \;=\; \langle[Z,X],Y\rangle+\langle[Z,Y],X\rangle \;=\; \langle\!\langle Z,[X,G\,Y]\rangle\!\rangle+\langle\!\langle Z,[Y,G\,X]\rangle\!\rangle.$$

We have used that ad_X is skew-symmetric, i.e., $\langle\!\langle[X,Y],Z\rangle\!\rangle=-\langle\!\langle Y,[X,Z]\rangle\!\rangle$. Now we get

$$B(X,Y) \;=\; \tfrac{1}{2}\big([X,G\,Y]+[Y,G\,X]\big).$$

Differentiating (21) once more yields

$$\begin{aligned}\nabla^2_{W,X}g(Y,Z) \;&=\tfrac{1}{2}\nabla_W g([Y,X],Z)+\tfrac{1}{2}\nabla_W g(Y,[Z,X])\\ &=\tfrac{1}{4}\Big(\,\langle\big[[Y,X],W\big],Z\rangle+\langle[Y,X],[Z,W]\rangle\\ &\qquad+\langle[Y,W],[Z,X]\rangle+\langle\big[[Z,X],W\big],Y\rangle\Big)\end{aligned}$$

and

$$\begin{aligned}\nabla^2_{\{X,W\}}g(Y,Z) \;&=\; -\tfrac{1}{4}\big(\langle[X,Y],[Z,W]\rangle+\langle[X,Z],[Y,W]\rangle\big)\\ &\quad-\tfrac{1}{8}\Big(\,\langle\big[[Y,X],W\big],Z\rangle+\langle\big[[Y,W],X\big],Z\rangle\\ &\qquad+\langle\big[[Z,X],W\big],Y\rangle+\langle\big[[Z,W],X\big],Y\rangle\Big).\end{aligned}$$

We apply the Jacobi-identity to the half of the second and third expression in the last two rows.

$$\begin{aligned}\nabla^2_{\{X,W\}}g(Y,Z) \;&=\; -\tfrac{1}{4}\big(\langle[X,Y],[Z,W]\rangle+\langle[X,Z],[Y,W]\rangle\big)\\ &\quad+\tfrac{3}{16}\big(\langle\big[[Y,X],W\big],Z\rangle+\langle\big[[Z,W],X\big],Y\rangle\big)\\ &\quad+\tfrac{1}{16}\big(\langle\big[[X,W],Y\big],Z\rangle+\langle\big[[Y,W],X\big],Z\rangle\big)\\ &\quad+\tfrac{1}{16}\big(\langle\big[[Z,X],W\big],Y\rangle+\langle\big[[W,X],Z\big],Y\rangle\big).\end{aligned}$$

Thus, the last two rows are symmetric in X and Y resp. in Z and W. Now it is easy to compute

$$\begin{aligned}R(X,Y;Z,W)=&-\tfrac{1}{16}\big(\,\langle\!\langle[G\,X,Y],[Z,W]\rangle\!\rangle+\langle\!\langle[X,G\,Y],[Z,W]\rangle\!\rangle\\ &\qquad+\langle\!\langle[X,Y],[G\,Z,W]\rangle\!\rangle+\langle\!\langle[X,Y],[Z,G\,W]\rangle\!\rangle\big)\\ &-\tfrac{3}{16}\Big(\,\langle\big[[X,Y],Z\big],W\rangle-\langle\big[[X,Y],W\big],Z\rangle\\ &\qquad+\langle\big[[Z,W],X\big],Y\rangle-\langle\big[[Z,W],Y\big],X\rangle\Big)\\ &-\tfrac{1}{4}\big(\langle[X,W],[Y,Z]\rangle-\langle[X,Z],[Y,W]\rangle-2\,\langle[X,Y],[Z,W]\rangle\big)\\ &-\;\;\big(\langle\!\langle B(X,W),G^{-1}B(Y,Z)\rangle\!\rangle-\langle\!\langle B(X,Z),G^{-1}B(Y,W)\rangle\!\rangle\big).\end{aligned}$$

Finally we get

Lemma B.1. *The curvature tensor R of a compact Lie group G with left invariant metric $\langle\cdot,\cdot\rangle$ is given by*

$$\begin{aligned} R(X,Y;Z,W) = &-\tfrac{1}{4}\big(\langle\!\langle [G\,X,Y],[Z,W]\rangle\!\rangle + \langle\!\langle [X,G\,Y],[Z,W]\rangle\!\rangle \\ &\qquad + \langle\!\langle [X,Y],[G\,Z,W]\rangle\!\rangle + \langle\!\langle [X,Y],[Z,G\,W]\rangle\!\rangle\big) \\ &- \tfrac{1}{4}\big(\langle [X,W],[Y,Z]\rangle - \langle [X,Z],[Y,W]\rangle - 2\,\langle [X,Y],[Z,W]\rangle\big) \\ &- \big(\langle\!\langle B(X,W),G^{-1}B(Y,Z)\rangle\!\rangle - \langle\!\langle B(X,Z),G^{-1}B(Y,W)\rangle\!\rangle\big), \end{aligned}$$

where B is defined by $B(X,Y) = \frac{1}{2}([X,G\,Y]+[Y,G\,X])$.

References

[AM] U. Abresch, W.T. Meyer, "Injectivity radius estimates and sphere theorems," pp. 1–47 in *Comparison geometry*, edited by K. Grove et al., Cambridge 1997.

[AS] U. Abresch, V. Schroeder, "Analytic manifolds of nonpositive curvature," pp. 1–67 in *Actes de la table ronde de géométrie différentielle en l'honneur de Marcel Berger* (Marseille 1992), edited by Arthur L. Besse, Séminaires & Congrès, Collection SMF, Paris 1996.

[AW] S. Aloff, N.R. Wallach, *An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures*, Bull. Amer. Math. Soc. **81** (1975), 93–97.

[Ba] Ya. V. Bazaikin, *On a certain family of closed 13-dimensional Riemannian manifolds of positive curvature*, Sib. Math. J. **37** (1996), 1068–1085.

[BB] L. Bérard Bergery, *Les variétés riemanniennes homogènes simplement connexes de dimension impaire à courbure strictement positive*, J. Math. pure et appl. **55** (1976), 47–68.

[Be] M. Berger, *Les variétés riemanniennes homogènes normales à courbure strictement positive*, Ann. Scuola. Norm. Sup. Pisa. **15** (1961), 179–246.

[Bs] A. Besse, *Einstein Manifolds*, Springer, Berlin-Heidelberg-New York, 1987.

[Do] A. Dold, *Lectures on Algebraic Topology*, Springer, Berlin-Heidelberg-New York, 1972.

[El] H.I. Eliasson, *Die Krümmung des Raumes* $\mathsf{Sp}(2)/\mathsf{SU}(2)$ *von Berger*, Math. Ann. **164** (1966), 317–327.

[Esch] J.-H. Eschenburg, *New examples of manifolds with strictly positive curvature*, Invent. math. **66** (1982), 469–480.

[Fr] T. Frankel, *Manifolds with positive curvature*, Pac. J. Math. **11** (1961), 165–174.

[Gr] K. Grove, *The even dimensional pinching problem and* $\mathsf{SU}(3)/\mathsf{T}^2$, Geom. Dedicata **29** (1989), 327–334.

[He1] E. Heintze, *The curvature of* $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times\mathsf{S}^1)$, Invent. Math. **13** (1971), 205–212.

[He2] E. Heintze, *Krümmung und Topologie des Raumes* $\mathsf{SU}(5)/(\mathsf{Sp}(2)\times\mathsf{S}^1)$, thesis, University of Bonn, 1970.

[Hl] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[Hu] H.-M. Huang, *Some remarks on the pinching problems*, Bull. Inst. Math. Acad. Sin. **9** (1981), 321–340.

[Ko] S. Kobayashi, *Fixed points of isometries*, Nogoya Math. J. **13** (1958), 63–68.

[KN] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry II*, Wiley Interscience, New York, 1969.

[KV] O. Kowalski, Z. Vlasek, *Homogeneous Einstein metrics on Aloff–Wallach spaces*, Diff. Geom. Appl. **3** (1993), 157–167.

[KS1] M. Kreck, S. Stolz, *Some nondiffeomorphic homeomorphic homogeneous 7-manifolds with positive sectional curvature*, J. Diff. Geom. **33** (1991), 465–486.

[KS2] M. Kreck, S. Stolz, *Nonconnected moduli spaces of positive sectional curvature*, J. Amer. Math. Soc. **6** (1993), 825–850

[Kr] B. Kruggel, *A homotopy classification of certain 7-manifolds*, Trans. Amer. Math. Soc. **349** (1997), 2827–2843

[NM] J.A. Nelder, R. Mead, *A simplex method for function minimization*, Computer Journal **7** (1965), 308–313.

[Ro] E.D. Rodionov, *Some examples of homogeneous Einstein manifolds*, preprint, Altai State Univ. 1995.

[Ta] I.A. Taimanov, *On totally geodesic embeddings of 7-dimensional manifolds into 13-dimensional manifolds of positive sectional curvature*, Sb. Math. **187** (1996), 1853–1867.

[Th1] J.A. Thorpe, *The zeros of nonnegative curvature operators*, J. Diff. Geom. **5** (1971), 113–125.

[Th2] J.A. Thorpe, *Erratum to: The zeros of nonnegative curvature operators*, J. Diff. Geom. **11** (1976), 315.

[Va] F.M. Valiev, *Precise estimates for the sectional curvatures of homogeneous Riemannian metrics on Wallach spaces*, Sib. Mat. Zhurn. **20** (1979), 248–262.

[Wa] N.R. Wallach, *Compact homogeneous Riemannian manifolds with strictly positive sectional curvature*, Ann. of Math. **96** (1972), 277–295.

[Wg] M. Wang, *Some examples of homogeneous Einstein manifolds in dimension seven*, Duke Math. J. **49** (1982), 23–28.

[Wi1] B. Wilking, *The normal homogeneous space* $\mathsf{SU}(3) \times \mathsf{SO}(3)/\mathsf{U}(2)$ *has positive sectional curvature*, preprint, Univ. Münster 1996 (to appear in Proc. Amer. Soc.)

[Wi2] B. Wilking, *Riemannsche Submersionen und positive Krümmung*, notes for a seminar, Univ. Münster 1997

[Zi] W. Ziller, *Homogeneous Einstein metrics on Spheres and projective spaces*, Math. Ann. **259** (1982), 351–358

[Zo] S.M. Zoltek, *Nonnegative curvature operators: Some nontrivial examples*, J. Diff. Geom. **14** (1979), 303–315.

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