# THE HOPF CONJECTURE FOR MANIFOLDS WITH LOW COHOMOGENEITY OR HIGH SYMMETRY RANK

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Abstract. We prove that the Euler characteristic of an even-dimensional compact manifold with positive (nonnegative) sectional curvature is positive (nonnegative) provided that the manifold admits an isometric action of a compact Lie group $G$ with principal isotropy group $H$ and cohomogeneity $k$ such that $k - (\operatorname{rank} G - \operatorname{rank} H) \leq 5$. Moreover, we prove that the Euler characteristic of a compact Riemannian manifold $M^{4l+4}$ or $M^{4l+2}$ with positive sectional curvature is positive if $M$ admits an effective isometric action of a torus $T^l$, i.e., if the symmetry rank of $M$ is $\geq l$.

The Gauss-Bonnet theorem states that the Euler characteristic of a closed surface $M$ is determined by its total curvature: $\chi(M) = 2\pi \int_M K$. In particular, if the curvature is positive (nonnegative), the Euler characteristic of the surface is positive (nonnegative). H. Hopf [H] generalized in 1925 the Gauss-Bonnet theorem to even-dimensional hypersurfaces of Euclidean space and posed in the early 1930's (according to Berger [Be]) the question whether a compact even-dimensional manifold which admits a metric of positive (nonnegative) sectional curvature must have positive (nonnegative) Euler characteristic.

Indications that the Hopf conjecture should be true came from the generalizations of the Gauss-Bonnet theorem: Fenchel [F] and Allendoerfer [A] proved in 1940 independently a Gauss-Bonnet formula for submanifolds of Euclidean space with arbitrary codimension. Three years later Allendoerfer and Weil [AW] (using E. Cartan's result that any Riemannian manifold can locally be embedded into Euclidean space) established the theorem in its final intrinsic version: For any even-dimensional manifold the Euler characteristic can be obtained by integrating a function derived from the curvature tensor, the so called Gauss-Bonnet integrand. Chern [C1] gave the first intrinsic proof of this theorem in 1944.

After this, many attempts were made to settle the stronger algebraic Hopf conjecture: A curvature tensor with positive (nonnegative) sectional curvature yields a positive (nonnegative) Gauss-Bonnet integrand. Milnor (unpublished, see [C2]) actually proved the algebraic Hopf conjecture in dimension 4, but finally in 1976 Geroch [G] found curvature tensors with positive sectional curvature in all even dimensions $\geq 6$ that do not provide a positive Gauss-Bonnet integrand.

A different approach to the Hopf conjecture is to consider first Riemannian manifolds that have a certain amount of symmetry. Hopf himself and Samelson

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[HS] proved in 1941 that the Euler characteristic of every compact homogeneous space $G/H$ is nonnegative and positive if and only if $\operatorname{rank} G = \operatorname{rank} H$ holds. The key observations in their proof are that a regular element in the compact Lie group $G$ has at most finitely many fixed points in the homogeneous space $G/H$ and that each of these fixed points has fixed point index 1. In 1972 Wallach [W] then showed that for any even-dimensional homogeneous space of positive sectional curvature one actually has $\operatorname{rank} G = \operatorname{rank} H$. Therefore, the Hopf conjecture is true for homogeneous spaces. Recently, Podestà and Verdiani [PV] proved among other things that the Hopf conjecture also holds for cohomogeneity one manifolds. We show that much weaker symmetry assumptions are sufficient.

**Theorem 1.** *Let $M$ be a compact even-dimensional Riemannian manifold with positive (nonnegative) sectional curvature. Let $G \times M \to M$ be an isometric action of a compact Lie group $G$ with principal isotropy group $H$ and cohomogeneity $k$. If*

$$k - (\operatorname{rank} G - \operatorname{rank} H) \leq 5$$

*then $M$ has positive (nonnegative) Euler characteristic.*

*Proof.* If $M$ is nonorientable, then the action of $G$ can be lifted to an action by orientation preserving isometries (see [Br, Corollary I.9.4]) on the orientable double covering space of $M$. We can therefore assume that $M$ is orientable.

We consider the fixed point set $M^T = \{p \in M \mid \psi(p) = p \text{ for all } \psi \in T\}$ of a maximal torus $T$ of $G$. Note that $M^T$ is equal to the fixed point set of a generating element $\psi \in T$, i.e., of an element $\psi$ with $\overline{\{\psi^m \mid m \in \mathbb{Z}\}} = T\}$. If the fixed point set $M^T$ is empty then there exists a Killing field without zeros. This implies that $M$ cannot have positive sectional curvature (by Berger's theorem, see e.g. [W]) and that the Euler characteristic is zero. We can therefore assume that $M^T$ is nonempty. Now each of the finitely many components of $M^T$ is a totally geodesic submanifold of $M$ with even codimension and the Euler characteristic of $M$ is the sum of the Euler characteristics of the components (see [K, Chapter II]). By Theorem IV.5.3 of [Br] each component $N$ of $M^T$ satisfies

$$\dim N \leq k - (\operatorname{rank} G - \operatorname{rank} H) \leq 5.$$

Since $N$ is even-dimensional and the Hopf conjecture holds in dimensions 2 and 4 we are done. □

Note that $k - (\operatorname{rank} G - \operatorname{rank} H) \leq \dim M - 2 \operatorname{rank} G$ (see [Br, Corollary IV.5.4]) if the action of $G$ is effective. Hence we get as a special case of Theorem 1 that any compact even-dimensional Riemannian manifold $M^{2l+4}$ with positive (nonnegative) sectional curvature has positive (nonnegative) Euler characteristic if $M^{2l+4}$ admits an effective isometric torus action $T^l \times M \to M$. Using a result from [GS] we can improve this result in the case of positive sectional curvature.

**Theorem 2.** *Let $M^{4l+2}$ or $M^{4l+4}$ be a Riemannian manifold with positive sectional curvature that admits an almost effective isometric $T^l$-action. Then for any $T^1 \subset T^l$ the Euler characteristics of all the components of the fixed point set $\operatorname{Fix}(M; T^1)$ are positive. In particular, $\chi(M) > 0$.*

*Proof.* As above we can assume that $M$ is orientable in order to have even-dimensional fixed point sets. The proof is done by induction. For $l = 0$ note that the Hopf conjecture is true in dimensions 2 and 4. For the induction step consider $M^{4l+6}$ or $M^{4l+8}$ with an almost effective $T^{l+1}$-action. Consider any circle $T^1 \subset T^{l+1}$ and any component $N$ of its fixed point set in $M$. We will show that $\chi(N) > 0$. Choose an $\tilde{T}^1 \subset T^{l+1}$ such that $N \subset \mathrm{Fix}(M;\tilde{T}^1)$ and such that the component $\tilde{N}$ of $\mathrm{Fix}(M;\tilde{T}^1)$ that contains $N$ has maximal dimension. It follows from the slice theorem and from the representation theory of tori that $T^l = T^{l+1}/\tilde{T}^1$ acts almost effectively on $\tilde{N}$. If $\mathrm{codim}\,\tilde{N} \geq 4$ we know from the induction assumption that in particular $N$ as a component of $\mathrm{Fix}(\tilde{N};T^1)$ has positive Euler characteristic and hence we are done. In the case where $\mathrm{codim}\,\tilde{N} = 2$ we know from [GS] that $M$ is differentiably covered by a sphere or a complex projective space. From results of Bredon [Br, Chapter III and VII] it follows that all the components of the fixed point set of any circle action on $M$ have positive Euler characteristic. Thus in particular $N$ has positive Euler characteristic. $\square$

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