INJECTIVITY RADIUS AND DIAMETER OF THE MANIFOLDS OF FLAGS IN THE PROJECTIVE PLANES

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Abstract. The manifolds of flags in the projective planes $\mathbb{RP}^2$, $\mathbb{CP}^2$, $\mathbb{HP}^2$, and $\mathbb{OP}^2$ are among the very few compact manifolds that are known to admit metrics with positive sectional curvature. They also arise as isoparametric hypersurfaces in spheres. We show how their appearance in these two fields is related and study the global geodesic geometry of these spaces in detail.

1. Introduction

The manifolds $F_{\mathbb{R}}$, $F_{\mathbb{C}}$, $F_{\mathbb{H}}$, $F_{\mathbb{O}}$ of flags in the projective planes $\mathbb{RP}^2$, $\mathbb{CP}^2$, $\mathbb{HP}^2$, $\mathbb{OP}^2$ serve as important examples in two classical fields in Riemannian geometry: They are among the very few manifolds that are known to admit metrics with positive sectional curvature and they arise as isoparametric hypersurfaces in the spheres $\mathbb{S}^4$, $\mathbb{S}^7$, $\mathbb{S}^{13}$, and $\mathbb{S}^{25}$ (see e.g. [Ka]). Their appearance in these two fields seems to be rather unrelated since the standard metrics on the spheres $\mathbb{S}^4$, $\mathbb{S}^7$, $\mathbb{S}^{13}$, and $\mathbb{S}^{25}$ induce metrics with some negative sectional curvatures on all the hypersurfaces in the isoparametric foliations.

The isoparametric foliations of $\mathbb{S}^4$, $\mathbb{S}^7$, $\mathbb{S}^{13}$, and $\mathbb{S}^{25}$ are given by the isotropy representations of the rank 2 symmetric spaces

$$\text{(1)} \qquad \mathrm{SU}(3)/\mathrm{SO}(3),\ \mathrm{SU}(3)\times\mathrm{SU}(3)/\Delta\mathrm{SU}(3),\ \mathrm{SU}(6)/\mathrm{Sp}(3),\ \mathrm{E}_6/\mathrm{F}_4,$$

respectively. In this paper we study the geometry of the flag manifolds $F_{\mathbb{R}}$, $F_{\mathbb{C}}$, $F_{\mathbb{H}}$, $F_{\mathbb{O}}$ via the orbital geometry of the isotropy *actions* of the symmetric spaces in (1) rather than just via the infinitesimal picture provided by the isotropy *representations*. This approach in particular exhibits the following relation between the isoparametric foliations of the spheres and the metrics of positive sectional curvature on the flag manifolds:

Theorem 1.1. *Any of the flag manifolds* $F_{\mathbb{R}}$, $F_{\mathbb{C}}$, $F_{\mathbb{H}}$, $F_{\mathbb{O}}$ *with any homogeneous metric of positive (or even nonnegative) sectional curvature is isometric to a principal orbit of the isotropy action of the corresponding symmetric space in (1).*

In other words, when we consider sufficiently large distance spheres to a point in one of the symmetric spaces instead of the euclidian spheres in the tangent space then some of the hypersurfaces in the isoparametric foliation of the distance spheres inherit metrics with positive sectional curvature. All possible homogeneous metrics with positive sectional curvature on the hypersurfaces arise this way. This fact gives an answer in a particular case to one of the current general problems in the field of positive sectional curvature, namely, to the question of how one can construct positively curved metrics on spaces without using Riemannian submersions.

The isotropy actions of the symmetric spaces in (1) also allow us to determine the diameter of the normal homogeneous metrics on the flag manifolds:

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Theorem 1.2. *Let $\mathcal{O}$ be a principal orbit of the isotropy action of one of the symmetric spaces in (1) such that $\mathcal{O}$ is isometric to a flag manifold $\mathrm{F}_{\mathbb{R}}$, $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, or $\mathrm{F}_{\mathbb{O}}$ with a normal homogeneous metric. Then $\mathcal{O}$ passes through two points of maximal distance in the symmetric space and the intrinsic diameter of $\mathcal{O}$ is equal to the diameter of the symmetric space.*

Note that the diameter of a symmetric space is equal to the diameter of its maximal torus and can thus be determined in a rather simple way.

For the normal homogeneous metrics on the flag manifolds we know precisely which pairs of points have maximal distance. This knowledge enables us to give lower diameter bounds for all other homogeneous metrics with nonnegative curvature (these form a two dimensional family). It is of course interesting to compare these lower diameter bounds with the injectivity radius. Surprisingly, the injectivity radius can be determined precisely for all homogeneous metrics with nonnegative sectional curvature on the flag manifolds in the following way: For any positively curved homogeneous metric on the three flag manifolds $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, and $\mathrm{F}_{\mathbb{O}}$ the maximum of the sectional curvature was computed by Valiev [Va] in 1979. By Klingenberg's classical injectivity radius estimate the injectivity radius of any simply connected, even dimensional manifold with sectional curvature $0 < \sec \le \Lambda$ is bounded from below by $\frac{\pi}{\sqrt{\Lambda}}$. Inspecting the length of certain closed geodesics on the flag manifolds we observe:

Lemma 1.3. *Klingenberg's injectivity radius estimate is sharp for all homogeneous metrics with positive sectional curvature on the manifolds $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, and $\mathrm{F}_{\mathbb{O}}$.*

Klingenberg's estimate does not apply directly to the odd dimensional and non simply connected flag manifold $\mathrm{F}_{\mathbb{R}} = \mathrm{SO}(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. It turns out, however, that the injectivity radius of $\mathrm{F}_{\mathbb{R}}$ is determined by the totally geodesic inclusion $\mathrm{F}_{\mathbb{R}} \subset \mathrm{F}_{\mathbb{C}}$ for all positively curved homogeneous metrics on $\mathrm{F}_{\mathbb{C}}$.

Theorem 1.4. *For the normal homogeneous metrics on the four flag manifolds the diameter is $\frac{4}{3}$ times the injectivity radius. Moreover, we have*

$$\tfrac{\mathrm{diam}}{\mathrm{inj}} \;\ge\; \tfrac{3\sqrt{2}}{\pi}\arctan(\sqrt{2}) \approx 1.2901$$

for all homogeneous metrics with nonnegative sectional curvature.

Note that the normal homogeneous metric on the real flag manifold has constant positive sectional curvature while the normal homogeneous metrics on the other flag manifolds only have a nonnegative sectional curvature function, but are surrounded by homogeneous metrics with positive sectional curvature. From our lower diameter bound for all positively curved homogeneous metrics and the precise lower sectional curvature bound given by Valiev we deduce the following diameter pinching estimate:

Theorem 1.5. *The manifolds $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, and $\mathrm{F}_{\mathbb{O}}$ admit metrics such that*

$$(\min\sec)\cdot(\tfrac{\mathrm{diam}}{\pi})^2 \;\ge\; 0.0414.$$

For the normal homogeneous metric on the flag manifold $\mathrm{F}_{\mathbb{R}}$ we have

$$(\min\sec)\cdot(\tfrac{\mathrm{diam}}{\pi})^2 \;=\; \tfrac{1}{9}.$$

This result should be seen in complement to the diameter sphere theorem of Grove and Shiohama [GS] and the related rigidity theorem of Gromoll and Grove (see [GG] and [Wi1] for the result in its final form): Every Riemannian manifold with

$$(\min\sec)\cdot(\tfrac{\mathrm{diam}}{\pi})^2 \;>\; \tfrac{1}{4} \qquad (2)$$

is homeomorphic to the standard sphere, and if equality holds in (2) then the manifold is homeomorphic to the standard sphere or up to scaling isometric to one of the projective spaces $\mathbb{CP}^k$, $\mathbb{HP}^k$, or $\mathbb{OP}^2$.

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2. The manifolds of flags in the projective planes

A flag in a projective plane is a pair (p, l) consisting of a point and a line that are incident, i.e., the line passes through the point. The set of flags in $\mathbb{RP}^2$, $\mathbb{CP}^2$, $\mathbb{HP}^2$, and $\mathbb{OP}^2$ will be denoted by $\mathrm{F}_{\mathbb{R}}$, $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, and $\mathrm{F}_{\mathbb{O}}$, respectively. In each case the isometry group of the projective plane acts transitively on the set of flags. One therefore gets the following homogeneous representations

$$\begin{aligned} \mathrm{F}_{\mathbb{R}} &= \mathrm{SO}(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2), \\ \mathrm{F}_{\mathbb{C}} &= \mathrm{SU}(3)/\mathrm{T}^2, \\ \mathrm{F}_{\mathbb{H}} &= \mathrm{Sp}(3)/\mathrm{Sp}(1)^3, \\ \mathrm{F}_{\mathbb{O}} &= \mathrm{F}_4/\mathrm{Spin}(8), \end{aligned} \tag{3}$$

where in the first three cases the isotropy groups can be chosen to be the subgroups of all diagonal matrices. The dimensions of the flag manifolds are 3, 6, 12, and 24.

There are three natural projections from each of the flag manifolds to the corresponding projective plane: First, we can forget about the line in the pair. In this case the fiber over a point p consists of all lines passing through p. This set can be identified with the cut locus of the point p, i.e., with its dual projective line. Second, we can map each pair (p, l) to the point p_l dual to the line l. Here, the fiber obviously consists of all points in l. Finally, we can map (p, l) to the point $p_{l^\perp}$ dual to the unique line $l^\perp$ that also passes through p and is perpendicular to l.

In the homogeneous description of $\mathrm{F}_{\mathbb{C}}$, for example, these three fibrations are given as follows:

$$\mathbb{S}^2_j = \mathrm{U}(2)_j/\mathrm{T}^2 \;\to\; \mathrm{F}_{\mathbb{C}} = \mathrm{SU}(3)/\mathrm{T}^2 \;\to\; \mathrm{SU}(3)/\mathrm{U}(2)_j = \mathbb{CP}^2, \tag{4}$$

where $\mathrm{U}(2)_1$, $\mathrm{U}(2)_2$, $\mathrm{U}(2)_3$ denote the centralizers of the matrices

$$I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

in SU(3).

It can easily be seen from the isotropy representation of each of the flag manifolds that the three fibers $\mathbb{S}^m_1$, $\mathbb{S}^m_2$, and $\mathbb{S}^m_3$ (where m is 1, 2, 4, or 8 according to whether we are in the real, complex, quaternionic, or octonionic case) are mutually perpendicular for all homogeneous metrics on the flag manifold and that any homogeneous metric is determined by the size of the three fibers at one point, i.e., by three positive parameters. The fibers are connected components of fixed point sets of isometries given by the matrices I_j and hence totally geodesic. None of the three projection maps from a flag manifold to the corresponding projective plane is a Riemannian submersion until at least two of the fibers have the same size. Only in this case there is a free and isometric $\mathbb{Z}_2$-action on the flag manifold. The metrics for which all three fibers have the same size are precisely the normal homogeneous metrics, i.e., the metrics that are induced from the biinvariant metrics on the Lie groups SO(3), SU(3), Sp(3), and F_4, respectively. Only for the normal homogeneous metrics there is a free and isometric action of the symmetric group S_3 on three letters on the flag manifold. This free and isometric action is the action of the Weyl group on the flag manifold from the right (see Section 5).

There is the natural chain of inclusions

$$\mathrm{F}_{\mathbb{R}} \subset \mathrm{F}_{\mathbb{C}} \subset \mathrm{F}_{\mathbb{H}} \subset \mathrm{F}_{\mathbb{O}} \tag{6}$$

which identifies the cones of homogeneous metrics on the flag manifolds in the natural way, i.e., a homogeneous metric on $\mathrm{F}_{\mathbb{O}}$ induces metrics on $\mathrm{F}_{\mathbb{R}}$, $\mathrm{F}_{\mathbb{C}}$, and $\mathrm{F}_{\mathbb{H}}$ given by the same three parameters. Each of the inclusions in (6) is totally geodesic (it is easy to see that the submanifolds are fixed point sets of certain isometries). It is also not hard to see that given a point $p \in \mathrm{F}_{\mathbb{C}}$ and a tangent vector $v \in T_p\mathrm{F}_{\mathbb{O}}$ there is an element of the isotropy group of $p \in \mathrm{F}_{\mathbb{O}}$ that moves v into $T_p\mathrm{F}_{\mathbb{C}}$ (see [Wi2]). This shows that *the diameters of corresponding metrics on* $\mathrm{F}_{\mathbb{C}}$*,* $\mathrm{F}_{\mathbb{H}}$*, and* $\mathrm{F}_{\mathbb{O}}$ *are the same*, since a geodesic from p to a point of maximal distance to p in $\mathrm{F}_{\mathbb{O}}$ can be moved into the totally geodesic submanifold $\mathrm{F}_{\mathbb{C}}$.

The flag manifolds arise as isoparametric hypersurfaces of the spheres $\mathbb{S}^4$, $\mathbb{S}^7$, $\mathbb{S}^{13}$, and $\mathbb{S}^{25}$. The isoparametric foliations of these spheres are given by the isotropy representations of the rank 2 symmetric spaces in (1). This means that the isotropy groups of the symmetric spaces act on the euclidian spheres in the tangent space with hypersurface orbits that are diffeomorphic to the flag manifolds. As pointed out in the introduction we consider the isotropy actions of the symmetric spaces on themselves rather than the isotropy representations. Like the flag manifolds themselves the rank 2 symmetric spaces in (1) are nested totally geodesically. Using this fact it is easy to see that it suffices to investigate just one of the isotropy actions. Note that the symmetric space $\mathrm{SU}(3) \times \mathrm{SU}(3)/\Delta\mathrm{SU}(3)$ is the Lie group $\mathrm{SU}(3)$. In this case the isotropy action is just the action of $\mathrm{SU}(3)$ on itself by conjugation. We investigate this action and the geometry of its principal orbits in Section 4.

3. Sectional curvature and injectivity radius

As explained in the previous section the homogeneous metrics on the flag manifolds can be parametrized by three positive parameters s_1, s_2, s_3 such that $\frac{\pi}{2}\sqrt{s_j}$ is the diameter of the fiber $\mathbb{S}^m_j$ in the fibration (4) or its real, quaternionic, or octonionic analogues. Any such metric will be denoted by $\langle\cdot,\cdot\rangle_{\vec{s}}$. For our purposes, it is convenient to use the elementary symmetric polynomials

$$\sigma_1 = s_1 + s_2 + s_3, \qquad \sigma_2 = s_2s_3 + s_3s_1 + s_1s_2, \qquad \sigma_3 = s_1s_2s_3$$

and additionally the symmetric polynomial

$$\sigma \;:=\; 4\sigma_2 - \sigma_1^2 \;=\; 2s_2s_3 + 2s_3s_1 + 2s_1s_2 - s_1^2 - s_2^2 - s_3^2.$$

The three critical values of the sectional curvature of the metric $\langle\cdot,\cdot\rangle_{\vec{s}}$ on the real flag manifold $\mathrm{F}_{\mathbb{R}} = \mathrm{SO}(3)/\mathbb{Z}_2 \times \mathbb{Z}_2$ are given by the three numbers

$$d_1 = -\tfrac{\sigma}{4\sigma_3} + \tfrac{-s_1+s_2+s_3}{s_2s_3}, \quad d_2 = -\tfrac{\sigma}{4\sigma_3} + \tfrac{s_1-s_2+s_3}{s_3s_1}, \quad d_3 = -\tfrac{\sigma}{4\sigma_3} + \tfrac{s_1+s_2-s_3}{s_1s_2}$$

and the scalar curvature is given by

$$2(d_1 + d_2 + d_3) = \tfrac{\sigma}{2\sigma_3}.$$

Wallach [Wa] discovered in 1972 that the other three flag manifolds $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, and $\mathrm{F}_{\mathbb{O}}$ admit homogeneous metrics with positive sectional curvature as well. Since the real flag manifold $\mathrm{F}_{\mathbb{R}}$ is a totally geodesic submanifold of the other flag manifolds a metric $\langle\cdot,\cdot\rangle_{\vec{s}}$ on any of the four flag manifolds can only have positive sectional curvature if d_1, d_2, d_3 and hence also σ are positive.

Theorem 3.1 (Valiev, see [Va] or [Pü])**.** *For any homogeneous metric* $\langle\cdot,\cdot\rangle_{\vec{s}}$ *on the flag manifolds* $\mathrm{F}_{\mathbb{C}}$*,* $\mathrm{F}_{\mathbb{H}}$*, or* $\mathrm{F}_{\mathbb{O}}$ *we have*

$$\max\sec = \tfrac{4}{s_1} \quad \textit{and} \quad \min\sec = \min\left\{\tfrac{3\sigma_1\sigma_3 - \sigma_2\sigma}{2\sigma_2\sigma_3}, d_3\right\}$$

provided that $s_1 \le s_2 \le s_3$ *and* $d_3 \ge 0$.

It is important to note that $s_1 \leq s_2 \leq s_3$ and $d_3 \geq 0$ implies that $\frac{3\sigma_1\sigma_3-\sigma_2\sigma}{2\sigma_2\sigma_3} \geq 0$, i.e., the domains of homogeneous metrics with nonnegative sectional curvature on all four flag manifolds are identical. The domains of homogeneous metrics with positive sectional curvature differ only by the normal homogeneous metrics, which have positive sectional curvature in case of the real flag manifold $\mathrm{F}_{\mathbb{R}}$ and nonnegative sectional curvature in case of the other flag manifolds $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, and $\mathrm{F}_{\mathbb{O}}$.

We will now see that Klingenberg's injectivity radius estimate is sharp for all homogeneous metrics with positive sectional curvature on the flag manifolds $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, and $\mathrm{F}_{\mathbb{O}}$:

Corollary 3.2. *The injectivity radius of a metric $\langle\cdot,\cdot\rangle_{\vec{s}}$ with nonnegative sectional curvature on any of the four flag manifolds is equal to*

$$\tfrac{\pi}{2}\min\{\sqrt{s_1},\sqrt{s_2},\sqrt{s_3}\},$$

i.e., to the minimal diameter of the totally geodesic fibers $\mathbb{S}_1^m$, $\mathbb{S}_2^m$, $\mathbb{S}_3^m$ in (4).

Proof. The flag manifolds $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, and $\mathrm{F}_{\mathbb{O}}$ are simply connected and even dimensional. Klingenberg's estimate together with the previous theorem yields that $\frac{\pi}{2}\sqrt{s_1}$ is a lower bound for the injectivity radius if $s_1 \leq s_2 \leq s_3$ and $d_3 > 0$. This continues to hold if we weaken $d_3 > 0$ to $d_3 \geq 0$. By definition $\frac{\pi}{2}\sqrt{s_1}$ is the diameter of the totally geodesic fiber $\mathbb{S}_1^2$ and hence an upper bound for the injectivity radius of the flag manifold. Concerning the real flag manifold $\mathrm{F}_{\mathbb{R}}$ note that the totally geodesic inclusion $\mathrm{F}_{\mathbb{R}} \subset \mathrm{F}_{\mathbb{C}}$ implies that the injectivity radius of $\mathrm{F}_{\mathbb{R}}$ is not less than the injectivity radius of $\mathrm{F}_{\mathbb{C}}$. □

We would like to point out that Valiev did not consider the parameters s_1, s_2, s_3 as the size of fibers, so the corollary is not obvious from his paper. One actually has to check that his definition of the parameters is identical to our geometric one. This can be done by a straightforward computation.

4. Metrics on the complex flag manifold induced by the adjoint action of SU(3)

We will now be concerned with the fundamental action

$$\mathrm{SU}(3)\times\mathrm{SU}(3)\to\mathrm{SU}(3),\quad (A,B)\mapsto ABA^{-1} \tag{7}$$

of SU(3) on itself by conjugation. There are obviously three orbit types: If the matrix B is a scalar multiple of the identity (i.e., in the center of SU(3)) then B is a fixed point of the action. If only two eigenvalues of B are equal then its centralizer is isomorphic to U(2) and the orbit is diffeomorphic to a complex projective plane. If all three eigenvalues are distinct then the centralizer is a maximal torus of SU(3) and the orbit is diffeomorphic to the manifold $\mathrm{F}_{\mathbb{C}} = \mathrm{SU}(3)/\mathrm{T}^2$ of flags in the complex projective plane.

We characterize the metrics that the latter orbits inherit from SU(3) in an intrinsic way using the symmetric polynomials from the previous section.

Lemma 4.1. *The metrics on the complex flag manifold $\mathrm{F}_{\mathbb{C}}$ induced by conjugation on the Lie group SU(3) equipped with the biinvariant metric $-\frac{\kappa}{2}\operatorname{tr}(XY)$ are precisely all metrics $\langle\cdot,\cdot\rangle_{\vec{s}}$ for which $\kappa\sigma = \sigma_3$. The metrics on the flag manifold $\mathrm{F}_{\mathbb{C}}$ induced by the adjoint action of SU(3) on its Lie algebra are precisely all metrics for which $\sigma = 0$.*

Note that any metric with $\sigma > 0$ can be rescaled such that $\kappa\sigma = \sigma_3$. Hence, we see from the previous section that all homogeneous metrics with positive sectional curvature on the flag manifold $\mathrm{F}_{\mathbb{C}}$ are induced by the action of SU(3) on itself by conjugation. As one expects the condition $\sigma = 0$ for the metrics that are induced by the adjoint action of SU(3) on its Lie algebra $\mathfrak{su}(3)$ arises when taking the limit $\kappa \to \infty$. It is easy to see that at least one of sectional curvatures d_j is negative in the case $\sigma = 0$. Hence, if we consider the flag manifolds as isoparametric submanifolds of the euclidean spheres $\mathbb{S}^4$, $\mathbb{S}^7$, $\mathbb{S}^{13}$, $\mathbb{S}^{25}$ then they always inherit metrics with some negative sectional curvatures. In order to put metrics on the spheres $\mathbb{S}^4$, $\mathbb{S}^7$, $\mathbb{S}^{13}$, $\mathbb{S}^{25}$ that induce metrics with positive sectional curvature on the flag manifolds one can pass to sufficiently large distance spheres to any base point in the rank 2 symmetric spaces given in (1). Such a distance sphere is the union of adjoint orbits that pass through a circle around the origin in E that intersects the domain F' in Figure 2.

Proof of Lemma 4.1. We will only prove the first part of the lemma. The proof of the second part is similar and simpler. Let T^2 be the fixed maximal torus of SU(3) that consists of all diagonal matrices. This torus meets all orbits and is perpendicular to all of them. We consider the universal covering map

$$\begin{aligned} E = \{\vec{\alpha} \in \mathbb{R}^3 \mid \alpha_1 + \alpha_2 + \alpha_3 = 0\} \ &\to \ \mathrm{T}^2, \\ \vec{\alpha} \ &\mapsto \ \mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}). \end{aligned}$$

The maximal torus T^2 corresponds to the lattice in E generated by the vectors

$$\vec{v}_1 = 2\pi \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = 2\pi \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

while the elements in the center of SU(3) correspond to the lattice in E generated by the vectors

$$\vec{w}_1 = \tfrac{2\pi}{3} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{w}_2 = \tfrac{2\pi}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

These two lattices are illustrated in Figure 1. The solid lines represent all singular matrices in the torus T^2, i.e., all matrices whose centralizer is larger than T^2. Let E' be the plane E without the solid lines, so that the orbits through matrices corresponding to points in E' are diffeomorphic to $\mathrm{F}_{\mathbb{C}} = \mathrm{SU}(3)/\mathrm{T}^2$. We parametrize the plane E' by

$$\tfrac{\beta_1}{2\pi}\, \vec{w}_2 - \tfrac{\beta_2}{2\pi}\, \vec{w}_1 \ = \ \tfrac{1}{3} \begin{pmatrix} \beta_1 + 2\beta_2 \\ -2\beta_1 - \beta_2 \\ \beta_1 - \beta_2 \end{pmatrix}$$

with $\beta_1, \beta_2 \in \mathbb{R}$. A straightforward computation shows that the orbit through the matrix determined by β_1 and β_2 inherits the metric given by the parameters

$$s_1 = 2\kappa(1 - \cos\beta_1), \quad s_2 = 2\kappa(1 - \cos\beta_2), \quad s_3 = 2\kappa(1 - \cos\beta_3) \tag{8}$$

where $\beta_3 = -(\beta_1 + \beta_2)$. From these expressions for s_1, s_2, s_3 we get immediately

$$\sigma_3 - \kappa\sigma \ = \ 4\kappa^3 \,(-1 + \cos^2\beta_1 + \cos^2\beta_2 + \cos^2\beta_3 - 2\cos\beta_1 \cos\beta_2 \cos\beta_3).$$

In order to see that this expression is actually zero we substitute the right hand side of the identity

$$\cos\beta_1 \ = \ \cos(\beta_2 + \beta_3) \ = \ \cos\beta_2 \cos\beta_3 - \sin\beta_2 \sin\beta_3$$

for one of the two factors $\cos\beta_1$ in the square $\cos^2\beta_1$ and do the same with the $\cos^2\beta_2$ and $\cos^2\beta_3$. This yields

$$\begin{aligned} \sigma_3 - \kappa\sigma \ &= 4\kappa^3 \,(-1 + \cos\beta_1 \cos\beta_2 \cos\beta_3 - \cos\beta_1 \sin\beta_2 \sin\beta_3 \\ &\qquad\qquad - \sin\beta_1 \cos\beta_2 \sin\beta_3 - \sin\beta_1 \sin\beta_2 \cos\beta_3) \\ &= 4\kappa^3 \,(-1 + \cos(\beta_1 + \beta_2 + \beta_3)) \ = \ 0. \end{aligned}$$

We will now investigate which points in E' will provide isometric metrics. If a matrix C is in the center of SU(3) then the orbit through BC is evidently an isometric copy of the orbit through B. Moreover, the reflection in E' that fixes $\mathbb{R}\cdot\vec{w}_1$ maps the parameter tripel (s_1, s_2, s_3) to the tripel (s_1, s_3, s_2), and the reflection in

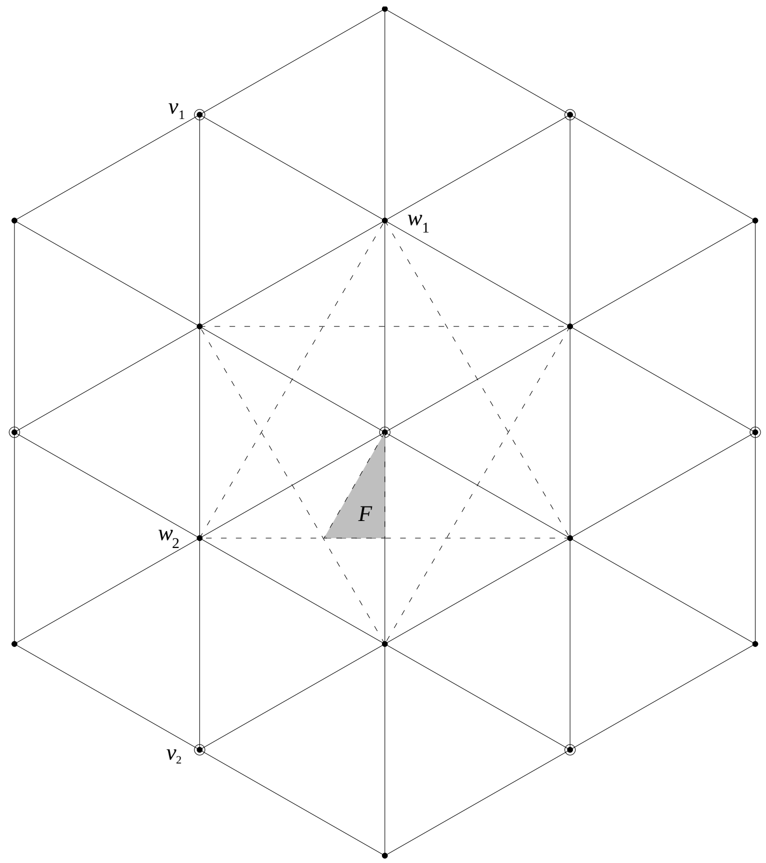

FIGURE 1. The lattices in E that correspond to the maximal tori in SU(3) and PSU(3).

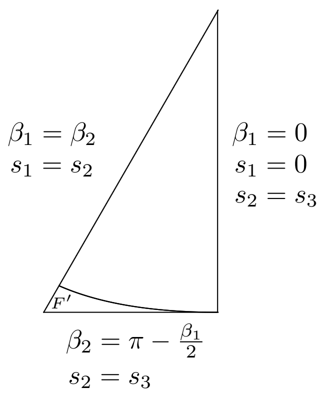

FIGURE 2. The fundamental domain F in the maximal torus of SU(3) and the subdomain F' that corresponds to homogeneous metrics of nonnegative sectional curvature on the flag manifolds.

E' that fixes $\mathbb{R} \cdot \vec{v}_2$ maps the tripel (s_1, s_2, s_3) to the tripel (s_3, s_2, s_1). These two reflections generate the dihedral group D_6. A fundamental domain F for the action on E' generated by the two reflections and the translations by central elements is given by

$$0 < \beta_1 \leq \tfrac{2\pi}{3} \quad \text{and} \quad \beta_1 \leq \beta_2 \leq \pi - \tfrac{\beta_1}{2}$$

and depicted in Figure 1 by the shaded triangle (see also Figure 2). It is easy to see that the parameters s_1, s_2, s_3 induced by points in F satisfy $s_1 \leq s_2 \leq s_3$.

It can be shown by a sequence of elementary arguments that $0 < s_1 \leq s_2 \leq s_3$ and $\kappa\sigma = \sigma_3$ implies that $s_3 \leq 4\kappa$ and that s_3 is uniquely determined by s_1 and s_2 by the formula

$$s_3 = s_1 + s_2 - \tfrac{s_1 s_2}{2\kappa} + \sqrt{(s_1 + s_2 - \tfrac{s_1 s_2}{2\kappa})^2 - (s_1 - s_2)^2}.$$

From this fact it follows that the assignment (8) maps the fundamental domain F injectively into the set S of parameters with $0 < s_1 \leq s_2 \leq s_3$ and $\kappa\sigma = \sigma_3$. It is not difficult to see that this assignment maps F also surjectively onto the connected set S (first inspect the assignment along the boundary and then apply a connectedness argument). □

Remark 4.2. The principal orbits of the adjoint action of SU(3) that inherit normal homogeneous metrics correspond to the lower left corner of the domain F in Figure 1, i.e., to the midpoints of the equilateral triangles bounded by the solid lines. If one considers the adjoint action of PSU(3) instead of the adjoint action of SU(3) then these orbits become isolated exceptional orbits diffeomorphic to $\mathrm{F}_{\mathbb{C}}/\mathbb{Z}_3$.

5. The diameter of the flag manifolds

In order to describe the points of maximal distance in the flag manifolds it is useful to consider the Weyl groups of the flag manifolds first. The Weyl group of a homogeneous space G/H is by definition the quotient $N(H)/H$ where $N(H)$ denotes the normalizer of H in G. The Weyl group is obviously a homogeneous subspace of G/H, namely, the fixed point set of the isotropy group H. (The Weyl group also acts freely from the right on G/H by diffeomorphisms. This latter action induces an action on the cone of left invariant metrics on G/H. In our case of the flag manifolds the three parameters of a left invariant metric are permuted.) For the homogeneous representations of the flag manifolds given in (3) the Weyl groups consist of the following six points:

$$\begin{aligned} p_{\mathrm{id}} &= \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad p_{\to} = \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}, \quad p_{\leftarrow} = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix}, \\ p_{23} &= \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}, \quad p_{31} = \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}, \quad p_{12} = \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}, \end{aligned} \tag{9}$$

where the asterisks $*$ can be filled in any way that is consistent with the group G. This makes immediately sense for the matrix groups SO(3), SU(3), and Sp(3) in the case of the flag manifolds $\mathrm{F}_{\mathbb{R}}$, $\mathrm{F}_{\mathbb{C}}$, and $\mathrm{F}_{\mathbb{H}}$, and it is not difficult to give some meaning to it in the case of the octonionic flag manifold $\mathrm{F}_{\mathbb{O}} = \mathrm{F}_4/\mathrm{Spin}(8)$. The Weyl groups are, as indicated by the notation, isomorphic to the symmetric group on three letters. The three points in the first line of (9) form the alternating group on three letters, i.e., the cyclic group of order three. The points in the second line form the other coset of this group. At each point p of the Weyl group there are the three fibers $\mathbb{S}^m$ of the fibrations to the projective space. The antipodes of p in these three fibers are precisely the points in the coset of the alternating group to which the point p does not belong. Hence, there are in total nine spheres that meet perpendicularly at the points of the Weyl group. The entire configuration

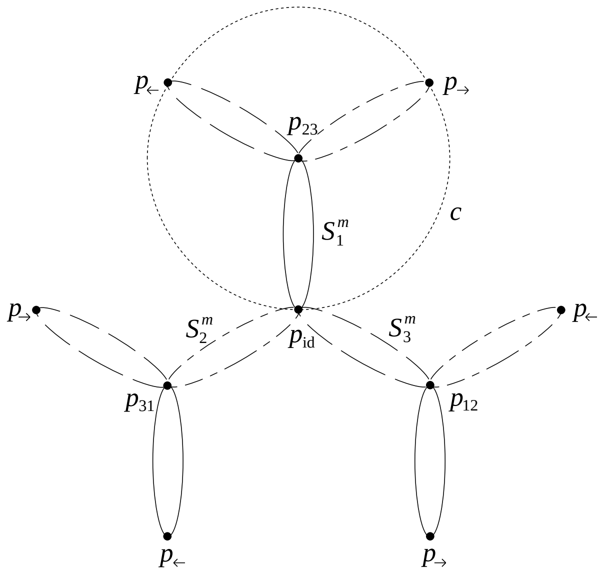

FIGURE 3. The nine spheres that join the six points of the Weyl group.

is illustrated in Figure 3. The Normalizer $N(H)$ acts transitively on the set of unordered pairs of points in $N(H)/H$ which belong to a common coset of the alternating group:

$$\begin{aligned} &\{p_{\mathrm{id}}, p_{\to}\}, \quad \{p_{\to}, p_{\leftarrow}\}, \quad \{p_{\leftarrow}, p_{\mathrm{id}}\}, \\ &\{p_{12}, p_{31}\}, \quad \{p_{31}, p_{23}\}, \quad \{p_{23}, p_{12}\}. \end{aligned} \tag{10}$$

Therefore, the distance between the two points in one of these pairs is independent of the pair. It will turn out that the diameter of the flag manifolds is given by this distance (at least for the normal homogeneous metrics).

Lemma 5.1. *Let $\langle\cdot,\cdot\rangle_{\vec{s}}$ be a metric on one of the flag manifolds that corresponds to a diagonal matrix D in the fundamental domain F of the previous section given by the parameters β_1 and β_2. Then the distance between the two points p_{id} and $p_{\to}$ with respect to the metric $\langle\cdot,\cdot\rangle_{\vec{s}}$ is bounded from below as follows:*

$$\mathrm{dist}(p_{\mathrm{id}}, p_{\to}) \ \geq \ \sqrt{\beta_1^2 + \beta_2^2 + \beta_1\beta_2}\,.$$

Proof. The distances between the two points p_{id} and $p_{\to}$ are the same in the three flag manifolds $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, and $\mathrm{F}_{\mathbb{O}}$ since any geodesic in $\mathrm{F}_{\mathbb{O}}$ between the two points can be moved isometrically into the totally geodesic submanifold $\mathrm{F}_{\mathbb{C}}$. In $\mathrm{F}_{\mathbb{R}}$ the distance between the two points cannot be smaller than the distance in $\mathrm{F}_{\mathbb{C}}$. Hence, it suffices to consider $\mathrm{F}_{\mathbb{C}}$. The orbit of the diagonal matrix D with respect to the action by conjugation on $\mathrm{SU}(3)$ is isometric to $\mathrm{F}_{\mathbb{C}}$ with the metric $\langle\cdot,\cdot\rangle_{\vec{s}}$. The isometry is given by the embedding

$$\mathrm{SU}(3)/\mathrm{T}^2 \to \mathrm{SU}(3), \quad A\cdot\mathrm{T}^2 \mapsto ADA^{-1}.$$

The point p_{id} is mapped to the point $\mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$ in $T \subset \mathrm{SU}(3)$, and the point $p_{\to}$ is mapped to the point $\mathrm{diag}(e^{i\alpha_3}, e^{i\alpha_1}, e^{i\alpha_2})$. It is easy to compute that the distance between these two points with respect to the biinvariant metric $-\frac{1}{2}\,\mathrm{tr}(XY)$ on $\mathrm{SU}(3)$ is given by $\sqrt{\beta_1^2+\beta_2^2+\beta_1\beta_2}$. □

Theorem 5.2. *The diameter of each of the flag manifolds with the normal homogeneous metric* $\langle\cdot,\cdot\rangle_{(3,3,3)}$ *is equal to the diameter of the Lie group* SU(3) *with the biinvariant metric* $-\frac{1}{2}\operatorname{tr}(XY)$*, i.e., equal to* $\frac{2\pi}{\sqrt{3}}$.

Proof. The metric induced on the orbit of the adjoint action on SU(3) through the point in the fundamental domain F given by $\beta_1 = \beta_2 = \frac{2\pi}{3}$ is the metric $\langle\cdot,\cdot\rangle_{(3,3,3)}$. Hence, the previous lemma shows that the diameter is $\geq \frac{2\pi}{\sqrt{3}}$. In order to see the converse inequality for $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, and $\mathrm{F}_{\mathbb{O}}$ note that the diameter of PSU(3) equipped with the biinvariant metric $-\frac{1}{2}\operatorname{tr} XY$ equals $\frac{2\pi}{3}$ (see Figure 1) and that there is a Riemannian submersion from PSU(3) equipped with the metric $-\frac{3}{2}\operatorname{tr} XY$ to the flag manifold $\mathrm{F}_{\mathbb{C}} = \mathrm{SU}(3)/\mathrm{T}^2$ equipped with the metric $\langle\cdot,\cdot\rangle_{(3,3,3)}$. Since Riemannian submersions are distance nonincreasing, the diameter of $\mathrm{F}_{\mathbb{C}}$, $\mathrm{F}_{\mathbb{H}}$, and $\mathrm{F}_{\mathbb{O}}$ is $\leq \frac{2\pi}{\sqrt{3}}$. In the case of $\mathrm{F}_{\mathbb{R}}$ we refer to the comments on the cut locus of $\mathrm{F}_{\mathbb{R}}$ at the end of this section. □

For the normal homogeneous metrics a minimal geodesic segment from p_{id} to $p_{\rightarrow}$ is explicitely given as follows: Consider the geodesic

$$c(t) = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{3}\cos t \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \frac{1}{\sqrt{3}}\sin t \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

in SO(3) with $c(0) = \mathbb{1}$ and $c(\frac{2\pi}{3}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. The projection of the segment $c([0, \frac{2\pi}{3}])$ to the real flag manifold $\mathrm{F}_{\mathbb{R}} = \mathrm{SO}(3)/\mathbb{Z}_2 \times \mathbb{Z}_2$ is a bijection and joins the points p_{id} and $p_{\rightarrow}$ (see Figure 3).

Theorem 5.3. *On the four flag manifolds the inequality*

$$\frac{\mathrm{diam}}{\mathrm{inj}} \geq \frac{2\sqrt{3}}{\pi} \approx 1.1026$$

holds for all homogeneous metrics with $\sigma \geq 0$ *and the inequality*

$$\frac{\mathrm{diam}}{\mathrm{inj}} \geq \frac{3\sqrt{2}}{\pi}\arctan(\sqrt{2}) \approx 1.2901$$

holds for all homogeneous metrics with nonnegative sectional curvature.

Proof. Every homogeneous metric with $\sigma > 0$ can be rescaled such that $\sigma = \sigma_3$. The latter metrics correspond to the fundamental domain F of the previous section that is depicted in Figure 1 and Figure 2. From Lemma 5.1 and Corollary 3.2 we see that

$$f(\beta_1, \beta_2) := \frac{\beta_1^2 + \beta_2^2 + \beta_1\beta_2}{2(1-\cos\beta_1)} \leq \left(\frac{\pi}{2}\,\frac{\mathrm{diam}}{\mathrm{inj}}\right)^2.$$

Since $\frac{\partial f}{\partial \beta_2} > 0$ holds on the interior of the fundamental domain F, the infimum of f on F is approched when $0 \in F$ is approached along the left edge. Along this edge we have $\beta_1 = \beta_2$ and

$$f(\beta, \beta) = \frac{3\beta^2}{2(1-\cos\beta)} = 3\left(\frac{\beta/2}{\sin(\beta/2)}\right)^2$$

which is increasing in β and tends to 3 if $\beta \to 0$. This proves the first part of the proposition.

The subdomain F' of F that corresponds to all metrics with nonnegative sectional curvature on the flag manifolds is in the notation of Section 3 given by $d_3 \geq 0$. This is equivalent to $\tan(\beta_1/2)\tan(\beta_2/2) \geq 2$ (note that $\sigma = \sigma_3$). The subdomain F' is given in Figure 2. The upper boundary curve is given by the function

$$\beta_2(\beta_1) = 2\arctan\frac{2}{\tan(\beta_1/2)}.$$

with $\beta_2(0) = \pi$. The derivative of this function is

$$\beta_2'(\beta_1) = -2 + \frac{6}{4+\tan(\beta_1/2)^2} \leq -\frac{1}{2}.$$

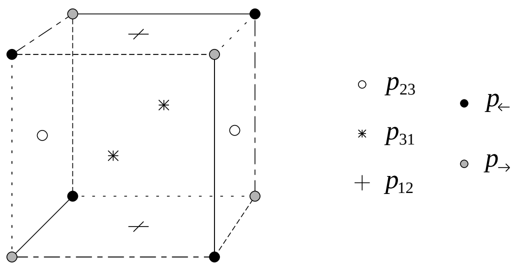

FIGURE 4. The identifications in the tangential cut locus of $F_{\mathbb{R}}$ at p_{id}.

Therefore $\beta_2(\beta_1) \leq \pi - \frac{\beta_1}{2}$ and the point given by β_1 and $\beta_2(\beta_1)$ belongs to F until $\beta_2(\beta_1) = \beta_1$, i.e., if and only if $0 < \beta_1 \leq 2\arctan\sqrt{2}$. The arguments that were used to determine the infimum of f on the whole fundamental domain F show that the minimum of the function f on F' is attained along the upper boundary curve of F'. Therefore, we consider the derivative of the function $g(\beta_1) := f(\beta_1, \beta_2(\beta_1))$:

$$g'(\beta_1) := \frac{(2\beta_1+2\beta_2\beta_2'+\beta_2+\beta_1\beta_2')(1-\cos\beta_1)-(\beta_1^2+\beta_2^2+\beta_1\beta_2)\sin\beta_1}{2(1-\cos\beta_1)^2}$$

If $2\beta_1 + 2\beta_2\beta_2' + \beta_2 + \beta_1\beta_2' \leq 0$ then evidently $g'(\beta_1) < 0$ for $0 < \beta_1 \leq 2\arctan\sqrt{2}$. Otherwise, the elementary inequalities

$$1 - \cos\beta \leq \frac{\beta_1^2}{2} \quad \text{and} \quad \sin\beta_1 \geq \frac{\beta_1}{3} \quad \text{for } 0 < \beta_1 \leq \frac{2\pi}{3}$$

together with $\beta_1 \leq \beta_2$ and $\beta_2' \leq -\frac{1}{2}$ show that $g'(\beta_1) < 0$ for $0 < \beta_1 \leq 2\arctan\sqrt{2}$. Therefore, the minimum of f on F' is attained for $\beta_1 = \beta_2 = 2\arctan\sqrt{2}$. □

Proof of Theorem 1.5. It follows from Lemma 4.1 with $\kappa = 1$ and Theorem 3.1 that the minimal sectional curvature of a metric corresponding to a point in F' is given by

$$\min\left\{\frac{3\sigma_1}{2\sigma_2} - \frac{1}{2}, \frac{s_1+s_2-s_3}{s_1s_2} - \frac{1}{4}\right\} \tag{11}$$

where the parameters $s_1 \leq s_2 \leq s_3$ are determined by (8). Hence, Lemma 5.1 implies

$$(\beta_1^2 + \beta_2^2 + \beta_1\beta_2) \cdot \min\left\{\frac{3\sigma_1}{2\sigma_2} - \frac{1}{2}, \frac{s_1+s_2-s_3}{s_1s_2} - \frac{1}{4}\right\} \leq \operatorname{diam}^2 \cdot(\min\sec).$$

It can be seen that the left hand side attains its maximum at the point in F' given by

$$\beta_1 = \arccos\frac{1}{8}, \quad \beta_2 = \arccos(-\tfrac{3}{4}),$$

which corresponds to the metric $\langle\cdot,\cdot\rangle_{\vec{s}}$ with the parameters

$$s_1 = \frac{7}{4}, \quad s_2 = s_3 = \frac{7}{2}. \tag{12}$$

This metric provides the lower bound for the diameter pinching constant that is stated in Theorem 1.5 in the introduction. □

Note that the metric given by the parameters in (12) is also the metric of optimal sectional curvature pinching $\frac{1}{64}$. A reason for this coincidence is that the minimum of the sectional curvature in (11) is given by the minimum of two functions, which are equal for these specific parameters.

Finally, we describe the cut locus of the real flag manifold $F_{\mathbb{R}}$ with the normal homogeneous metric $\langle\cdot,\cdot\rangle_{(1,1,1)}$. Since $F_{\mathbb{R}}$ can be regarded equivalently as the quotient of the sphere $\mathbb{S}^3$ with radius 2 by the quaternion group the cut locus can be determined by explicit computations, which we omit. The result is as follows:

Given a unit initial vector v in the tangent space at p_{id} the geodesic defined by v is minimal precisely on the interval from 0 to $2\operatorname{arccot}(|v|_\infty)$. Here, the ∞-norm is taken with respect to the coordiante system given by the three irreducible subspaces of the isotropy representation. It follows that the diameter of $\mathrm{F}_{\mathbb{R}}$ is $\frac{2\pi}{3}$. The identifications in the tangential cut locus are illustrated in Figure 4. For the sake of simplicity we have projected the tangential cut locus radially to a cube. Each edge is pointwise identified with two other edges as indicated. Each point in the interior of a face is identified with just one other point on the opposite face in a way that is determined by the identification on the edges.

An important subset of the cut locus of a point p in a Riemannian manifold is the set of critical points for the distance function to the point p in the sense of Grove and Shiohama (see [GS]). This is the set of points q for which the initial vectors of minimal geodesic segments from q to p are not contained in an open halfspace. A point of maximal distance to p is necessarily a critical point for the distance function to p. For the normal homogeneous metric on $\mathrm{F}_{\mathbb{R}}$ one can see from Figure 4 that only the points in the Weyl group are critical points for the distance function to the point p_{id}. The points in the Weyl group are critical points of the distance function to the point p_{id} for all homogeneous metrics on all four flag manifolds. This follows immediately from the isotropy representations. A larger set of points that is contained in the cut locus of p_{id} for all homogeneous metrics is given by the union of the six spheres in Figure 3 that do not contain p_{id}, since these spheres are connected components of the fixed point sets of the isometries I_1, I_2, or I_3 given in (5) that all fix p_{id} as well. In case of the normal homogeneous metric on $\mathrm{F}_{\mathbb{R}}$ these six spheres can be found as the diagonals of the faces in Figure 4.

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