A NEW MODEL FOR ESTIMATING CRITICAL GAP AND ITS DISTRIBUTION AT UNSIGNALIZED INTERSECTIONS BASED ON THE EQUILIBRIUM OF PROBABILITIES

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ABSTRACT

This paper presents a new model for estimation critical gaps at unsignalized intersection. The theoretical background of the new model is the probability equilibrium between the rejected and accepted gaps. The equilibrium is established macroscopically using the cumulative distribution of the rejected and accepted gaps. The model yields directly the probability distribution function of the critical gaps.

The new model has the following positive properties: a) solid theoretical background (equilibrium of probabilities), b) robust results, c) independent of any model assumptions, d) possibility of taking into account all relevant gaps (not only the maximum rejected gaps as is the case of the Troutbeck model (1992)), e) possibility of achieving the empirical probability distribution function of the critical gaps directly, and f) simple calculation procedure without iteration.

The implementation the new model is simple and robust. It can be carried out using spreadsheet programs (e.g. EXCEL, QuatroPro etc.). Thus, with the new model a useful and promising tool can be set up for professionals in traffic engineering.

1 INTRODUCTION

Critical gap is a major parameter for capacity analysis at unsignalized intersections. This parameter is a stochastically distributed value and it cannot be obtained directly by measurements. The estimation of critical gaps at unsignalized intersections from traffic observation is one of the most difficult tasks in traffic engineering science. For estimating the critical gaps, statistical models or procedures are required. There exist many different models for estimating critical gaps. Among them the models of Siegloch (1973), Raff et al. (1950), Aworth (1970), Harders (1968), Hewett (1983), and Troutbeck (1992) are the most important. Today we can find more than 20 or 30 models worldwide for estimating critical gaps. In practice - for unsaturated conditions - the most common models are that of Raff et al. (1950) and Troutbeck (1992).

Brilon and König (1997) gave an overview of the most important models. Using microscopic simulations, they also conducted an assessment of those models. It was found that the model of Troutbeck (1992) gives the best results. Thus, this model is recommended for estimating the critical gaps in many standard manuals for traffic engineering (e.g., HCM 2000, HBS 2001, etc.).

The model of Troutbeck (1992) is a microscopic model. That is, the single values of the measured gaps are used in the model. The model is based on the theory of Maximum Likelihood. In this model, only the maximum rejected gap and the accepted gap of single vehicles are treated pair wise. In this model, two assumptions are made: a) a log-normal distribution for the critical gaps and b) the driver behaviour is both homogeneous and consistent.

Such presumptions are disadvantages. Furthermore, the model of Troutbeck (1992) is very complicated to use and its results are not very robust. This model also requires a large sample size for establishing stable results.

To avoid such disadvantages, in this paper a totally new model for estimating the critical gaps is presented. The theoretical foundation of this new model is the probability equilibrium between the rejected and the accepted gaps. The equilibrium is established *macroscopically* from the cumulative distribution of the rejected and accepted gaps. It turns out that the model from the macroscopic equilibrium is more appropriate for estimating critical gaps. The new model yields similar results as that from Troutbeck's model. More importantly, the new model yields directly the empirical distribution of the critical gaps. The new model does not require any predefined assumptions and it is easy to use.

2 MODEL DESCRIPTION AN APPLICATIONS

2.1. The method of Raff and Troutbeck

Let $F_r(t)$ and $F_a(t)$ be the probability distribution functions (PDFs) of rejected and accepted gaps, respectively. Then $F_r(t)$ and $F_a(t)$ can be obtained empirically by *in situ* measurements. Thus, the observed probability that a gap of length t is rejected is $F_r(t)$, and that it is not rejected is $1\text{-}F_r(t)$.

More than forty years ago, Raff (1950) introduced a macroscopic model for estimating the critical gap. He defined the critical gap as the value of t where the functions $1\text{-}F_r(t)$ and $F_a(t)$ intercept. That is, the value t at which

$$F_a(t) = 1 - F_r(t) \tag{1}$$

is defined as the estimated critical gap t_c.

Raff's method was used in many countries in earlier times. Because of its simplicity, it is still being used today in some research projects.

Troutbeck (1992) gave a procedure for estimating critical gaps based on the Maximum Likelihood techniques. This model is a microscopic model. In this model only the maximum rejected gaps are taken into account. Thus, for an accepted gap a_d, there is only a corresponding rejected gap r_d under consideration. The likelihood that a driver's critical gap is between a_d and r_d is given by $F_a(a_d)\text{-}F_r(r_d)$. The likelihood L^* with a sample of n observed drivers is

$$L^* = \prod_{i=1}^{n} \left[F_a(a_d) - F_r(r_d) \right] \tag{2}$$

If the PDF of the critical gaps, $F_{tc}(t)$, is given, the parameters of the PDF can be obtained by maximizing the likelihood L^*. In the practice, the log-normal distribution is often used as the PDF of the critical gaps. Furthermore, as model assumptions, the driver behaviour has to be both homogeneous and consistent. Normally, the maximization of the likelihood L^* can only be done using numerical and iteration techniques (cf., Troutbeck 1992).

2.2. The new model based on the macroscopic probability equilibrium

The macroscopic probability equilibrium of the accepted and rejected gaps can be established as follows.

According to the PDFs of the accepted ($F_a(t)$) and rejected ($F_r(t)$) gaps, the observed probability that a gap of length t is accepted is 1-$F_a(t)$ and that it is "not-accepted" is $F_a(t)$. And the observed probability that a gap of length t is rejected is $F_r(t)$ and that it is "not-rejected" is 1-$F_r(t)$. In general, we have $F_r(t) \neq 1 - F_a(t)$ and $1-F_r(t) \neq F_a(t)$ because an accepted gap in the major stream may not have the exact length of the actual critical gap. In fact, an accepted gap is always greater than the actual critical gap.

Denote the PDF of the critical gaps to be estimated by $F_{tc}(t)$, then the probability $P_{r,tc}(t)$ that a gap of length t in the major stream would be rejected is $F_{tc}(t)$, and the probability $P_{a,tc}(t)$ that it would be accepted is 1-$F_{tc}(t)$.

Considering the observed probability of both acceptance and rejection, we have the probability equilibrium

$$\begin{cases} P_{r,tc}(t) = F_r(t) \cdot P_{r,tc}(t) + F_a(t) \cdot P_{a,tc}(t) \\ P_{a,tc}(t) = \left(1 - F_a(t)\right) \cdot P_{r,tc}(t) + \left(1 - F_r(t)\right) \cdot P_{a,tc}(t) \end{cases} \tag{3}$$

The equation (3) can be rewritten in the following matrix form:

$$\begin{pmatrix} P_{r,tc}(t) \\ P_{a,tc}(t) \end{pmatrix} = \begin{pmatrix} F_r(t) & F_a(t) \\ 1 - F_r(t) & 1 - F_a(t) \end{pmatrix} \begin{pmatrix} P_{r,tc}(t) \\ P_{a,tc}(t) \end{pmatrix} \tag{4}$$

That is exactly the description of the equilibrium state of the probabilities $P_{a,tc}(t)$ and $P_{r,tc}(t)$ as a Markov Chain. In this formulation

$$\begin{pmatrix} P_{r,tc}(t) \\ P_{a,tc}(t) \end{pmatrix}$$

is the state vector and

$$\begin{pmatrix} F_r(t) & F_a(t) \\ 1 - F_r(t) & 1 - F_a(t) \end{pmatrix}$$

the transition matrix. The boundary condition $P_{a,tc}(t) + P_{r,tc}(t) = 1$ holds.

With $P_{r,tc}$(t)=$F_{tc}(t)$ and $P_{a,tc}$(t)=1-$F_{tc}(t)$, equation (4) yields

$$\begin{pmatrix} F_{tc}(t) \\ 1 - F_{tc}(t) \end{pmatrix} = \begin{pmatrix} F_r(t) & F_a(t) \\ 1 - F_r(t) & 1 - F_a(t) \end{pmatrix} \begin{pmatrix} F_{tc}(t) \\ 1 - F_{tc}(t) \end{pmatrix} \tag{5}$$

Solving equation (5) yields the PDF $F_{tc}(t)$ of the critical gaps:

$$F_{tc}(t) = \frac{F_a(t)}{F_a(t) + 1 - F_r(t)} = 1 - \frac{1 - F_r(t)}{F_a(t) + 1 - F_r(t)} \tag{6}$$

The PDF $F_{tc}(t)$ always lies between $F_r(t)$ and F_a(t) (see Figure 1).

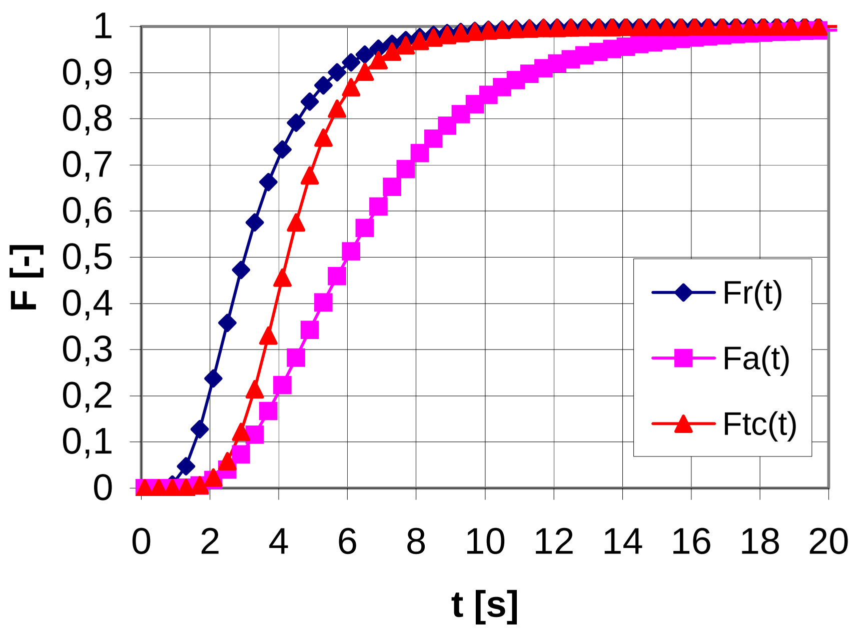

Figure 1 – Schematic relationship between the PDF's for the rejected gaps, the accepted gaps, and the estimated critical gaps from the new model

It should be noted that this distribution is only explicitly defined between, from the point of view of all vehicles, the minimum accepted gap $a_{\min}$ and the maximum rejected gap $r_{\max}$. For $t_c<a_{min}$ one has $F_{tc}(t) = 0$ and for $t_c>r_{max}$ one has $F_{tc}(t) = 1$.

According to Raff's definition for the critical gap (eq. (1)) we have

$$F_{tc}(t) = \frac{F_a(t)}{F_a(t)+1-F_r(t)} = \frac{F_a(t)}{F_a(t)+F_a(t)} = 0.5 \qquad (7)$$

This means that the critical gap estimated from Raff's method is the median value but not the mean value of the critical gap.

The new model has a solid theoretical foundation (in terms of the Markov Chain and equilibrium of probabilities) and robust results. It is also independent of any model assumptions. It requires neither predefined distribution function of the critical gaps nor the consistency nor the homogeneity of drivers. This model can take into account all relevant gaps (not only the maximum rejected gaps as is the case of the Troutbeck model (1992)) and yields the empirical PDF of the critical gaps directly. The calculation procedure of the model is simple and without iteration.

In particular, the property of the model that all the rejected, not only the maximum rejected, gaps can be taken into account makes the major difference between the present new model and the most used model of Troutbeck (1992). If only the maximum rejected gaps are used for estimating the critical gaps, the new model gives similar results (deviations smaller than 0.2s) for the mean critical gaps as that from Troutbeck (1992). If all rejected gaps are used, the estimated mean critical gaps must be shorter.

For implementing the proposed macroscopic model, a useful calculation procedure is recommended. This procedure can be easily implemented into a Spreadsheet (for example, EXCEL or QuatroPro). The procedure is described as follows:

1. insert all measured and relevant (according to whether all or only the maximum rejected gaps are taken into account) gaps t in the major stream into the column 1 of the spreadsheet
2. mark the accepted gaps with "a" and the rejected gaps with "r" in column 2 of the spreadsheet respectively
3. sort all gaps (together with their marks "a" and "r") in an ascending order
4. calculate the accumulate frequencies of the rejected gaps, n_{rj}, in column 3 of the spreadsheet (that is: for a given row j, if mark="r" then $n_{rj}=n_{rj}+1$ else $n_{rj}=n_{rj}$, with $n_{r0}=0$)
5. calculate the accumulate frequencies of the accepted gaps, n_{aj}, in column 4 of the spreadsheet (that is: for a given row j, if mark="a" then $n_{aj}=n_{aj}+1$ else $n_{aj}=n_{aj}$, with $n_{a0}=0$)
6. calculate the PDF of the rejected gaps, $F_j(r)$, in column 5 of the spreadsheet (that is: for a given raw j, $F_j(r)=n_{rj}/n_{r,max}$ with $n_{r,max}$=number of all rejected gaps)
7. calculate the PDF of the accepted gaps, $F_a(t_j)$, in column 6 of the spreadsheet (that is: for a given raw j, $F_a(t_j)=n_{aj}/n_{a,max}$ with $n_{a,max}$=number of all accepted gaps)
8. calculate (according to equation (6)) the PDF of the estimated critical gaps, $F_{tc}(t_j)$, in column 7 of the spreadsheet (that is: for a given raw j, $F_{tc}(t_j)=F_a(t_j)/[F_a(t_j)+1-F_r(t_j)]$
9. calculate the frequencies of the estimated critical gaps, $p_{tc}(t_j)$, between the raw j and j-1 in column 8 of the spreadsheet (that is: $p_{tc}(t_j)=F_{tc}(t_j)-F_{tc}(t_{j-1})$)
10. calculate the class mean, $t_{d,j}$, between the raw j and j-1 in column 9 of the spreadsheet (that is: $t_{d,j}=(t_j+t_{j-1})/2$)
11. calculate the average value and the variance of the estimated critical gaps (that is: ($t_{c,averge}=sum[p_{tc}(t_j)*t_{d,j}]$ and $\sigma^2=sum[p_{tc}(t_j)*t_{d,j}^2]-(sum[p_{tc}(t_j)*t_{d,j}])^2$)

This calculation procedure ensures a monotonous ascending PDF for the critical gaps.

In Figure 2, an example of the procedure for estimating the critical gap with a spreadsheet is illustrated.

The new procedure still has a limitation: in the measured data, the minimum accepted gap a_{min} has to be smaller than the maximum rejected gap r_{max}. Otherwise the procedure yields no defined result because in this case the denominator of the equation (6) in the range $r_{max}<t<a_{min}$ is not defined. This case can occur if the sample size is very small.

(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
		accepted or rejected	if (2)="r", nr=nr+1	if (2)="a", na=na+1	(3)/nr,max	(4)/nr,max	(6)/[(6)+1-(5)]	(7)_j-(7)_j-1	[(1)_j-(1)_j-1]/2
index j	gap t		nr	na	Fr	Fa	Ftc	ftc	cl.m
1	5	r	1	0	0,00694444	0	0	0	2,5
2	7	r	2	0	0,01388889	0	0	0	6
3	7	r	3	0	0,02083333	0	0	0	7
4	7	r	4	0	0,02777778	0	0	0	7
5	7	r	5	0	0,03472222	0	0	0	7
6	8	r	6	0	0,04166667	0	0	0	7,5
7	9	r	7	0	0,04861111	0	0	0	8,5
8	10	r	8	0	0,05555556	0	0	0	9,5
9	10	r	9	0	0,0625	0	0	0	10
10	11	r	10	0	0,06944444	0	0	0	10,5
11	11	r	11	0	0,07638889	0	0	0	11
12	11	r	12	0	0,08333333	0	0	0	11
13	12	r	13	0	0,09027778	0	0	0	11,5
	...								
	...								
	...								
138	60	r	133	5	0,92361111	0,034722222	0,3125	0,018382353	60
139	61	r	134	5	0,93055556	0,034722222	0,333333333	0,020833333	60,5
140	62	r	135	5	0,9375	0,034722222	0,357142857	0,023809524	61,5
141	63	a	135	6	0,9375	0,041666667	0,4	0,042857143	62,5
142	63	a	135	7	0,9375	0,048611111	0,4375	0,0375	63
143	64	a	135	8	0,9375	0,055555556	0,470588235	0,033088235	63,5
144	64	a	135	9	0,9375	0,0625	0,5	0,029411765	64
145	64	a	135	10	0,9375	0,069444444	0,526315789	0,026315789	64
146	64	a	135	11	0,9375	0,076388889	0,55	0,023684211	64
147	65	r	136	11	0,94444444	0,076388889	0,578947368	0,028947368	64,5
148	66	r	137	11	0,95138889	0,076388889	0,611111111	0,032163743	65,5
149	67	a	137	12	0,95138889	0,083333333	0,631578947	0,020467836	66,5
150	67	a	137	13	0,95138889	0,090277778	0,65	0,018421053	67
151	68	a	137	14	0,95138889	0,097222222	0,666666667	0,016666667	67,5
152	69	r	138	14	0,95833333	0,097222222	0,7	0,033333333	68,5
	...								
	...								
	..								
279	328	a	144	135	1	0,9375	1	0	327
280	363	a	144	136	1	0,944444444	1	0	345,5
281	368	a	144	137	1	0,951388889	1	0	365,5
282	387	a	144	138	1	0,958333333	1	0	377,5
283	439	a	144	139	1	0,965277778	1	0	413
284	461	a	144	140	1	0,972222222	1	0	450
285	467	a	144	141	1	0,979166667	1	0	464
286	633	a	144	142	1	0,986111111	1	0	550
287	642	a	144	143	1	0,993055556	1	0	637,5
288	656	a	144	144	1	1	1	0	649
		summe	144	144				tc,mean	6,38
								sigma	1,11

Figure 2 - Example of a spreadsheet for estimating the critical gap

In Figure 3 and Figure 4, the results of two examples are presented (data: Weinert, 2001). In these calculations, only the maximum rejected gaps are used for reason of comparability to the model of Troutbeck (1992). It can be recognised that the mean values of the critical gaps are similar for both models. Also, the PDF estimated from the new model are comparable to the predefined PDF (log-normal) from Troutbeck's model. This indicates that the predefined log-normal distribution in Troutbeck's model is suitable for describing the distribution of critical gaps.

In Figure 5 and Figure 6, results for the same examples but using all rejected gaps are presented. It can be seen that the mean values of the critical gaps are shorter compared to the results in Figure 3 and Figure 4. The average difference is about 15%. To demonstrate this effect clearly, the resulted PDF for both cases are illustrated together in Figure 7 and Figure 8.

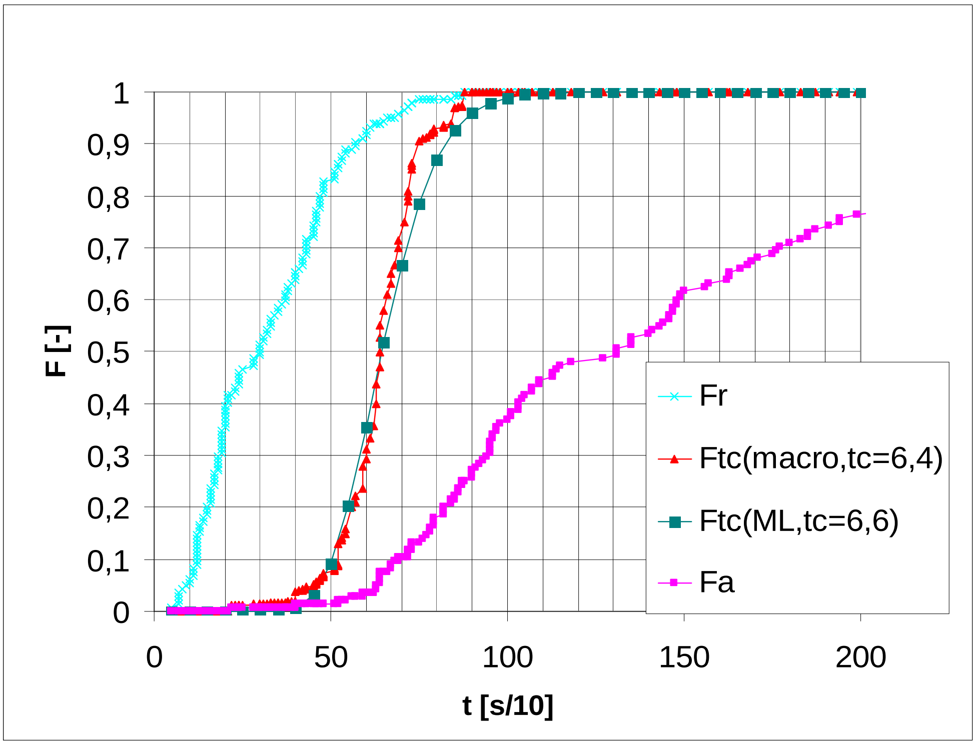

Figure 3 - Example for critical gap estimation. F_r=PDF of the <u>maximum</u> rejected gaps, F_a=PDF of the accepted gaps, F_{tc}(ML)=PDF of the estimated critical gaps from Maximum Likelihood model of Troutbeck, F_{tc}(macro)=PDF of the estimated critical gaps from the new model for macroscopic equilibrium (Data: Weinert, 2001, Bad Nauheim 3, minor right-turn).

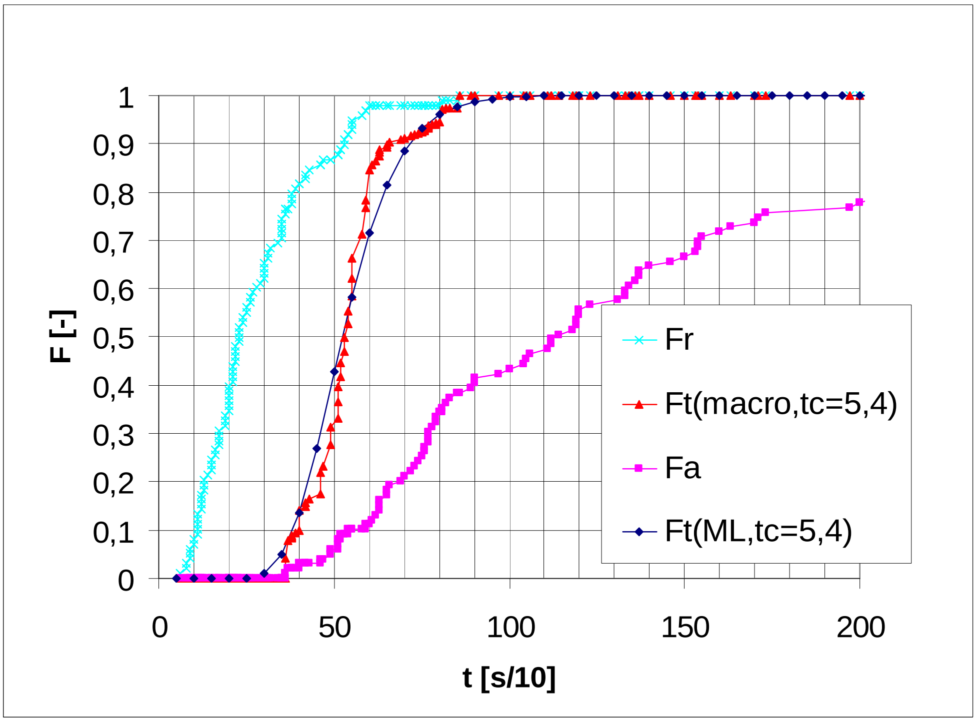

Figure 4 - Example for critical gap estimation. F_r=PDF of the <u>maximum</u> rejected gaps, F_a=PDF of the accepted gaps, F_{tc}(ML)=PDF of the estimated critical gaps from Maximum Likelihood model of Troutbeck, F_{tc}(macro)=PDF of the estimated critical gaps from the new model for macroscopic equilibrium (Data: Weinert, 2001, Köln 1, major left-turn).

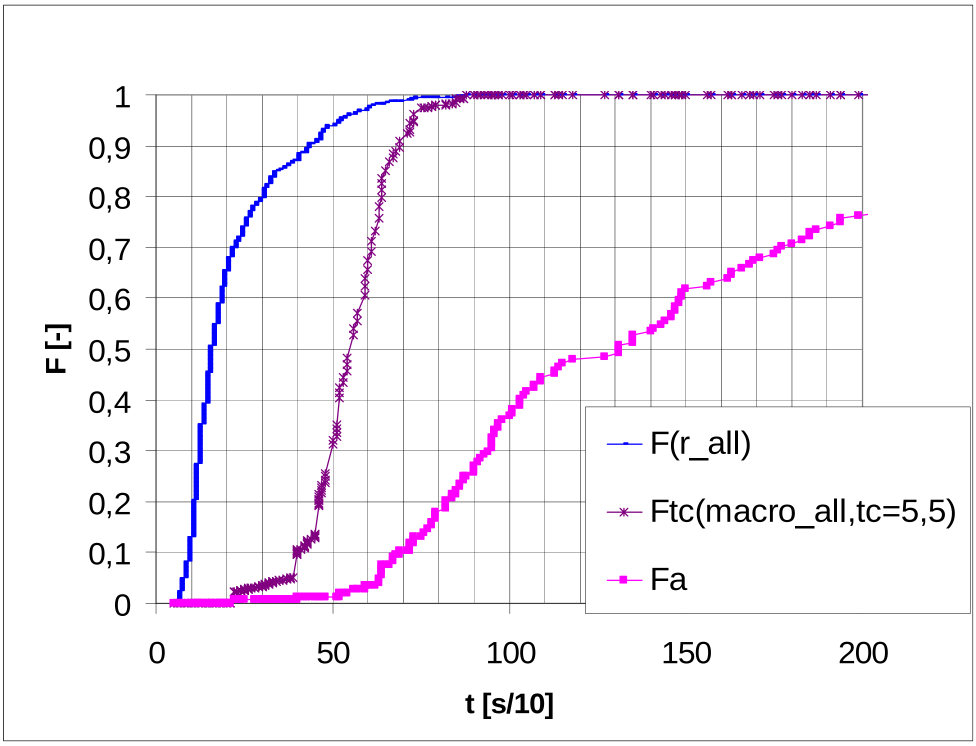

Figure 5 - Example for critical gap estimation. $F_{r,all}$=PDF of <u>all</u> gaps, F_a=PDF of the accepted gaps, F_{tc}(macro)=PDF of the estimated critical gaps from the new model for macroscopic equilibrium (Data: Weinert, 2001, Bad Nauheim 3, minor right-turn).

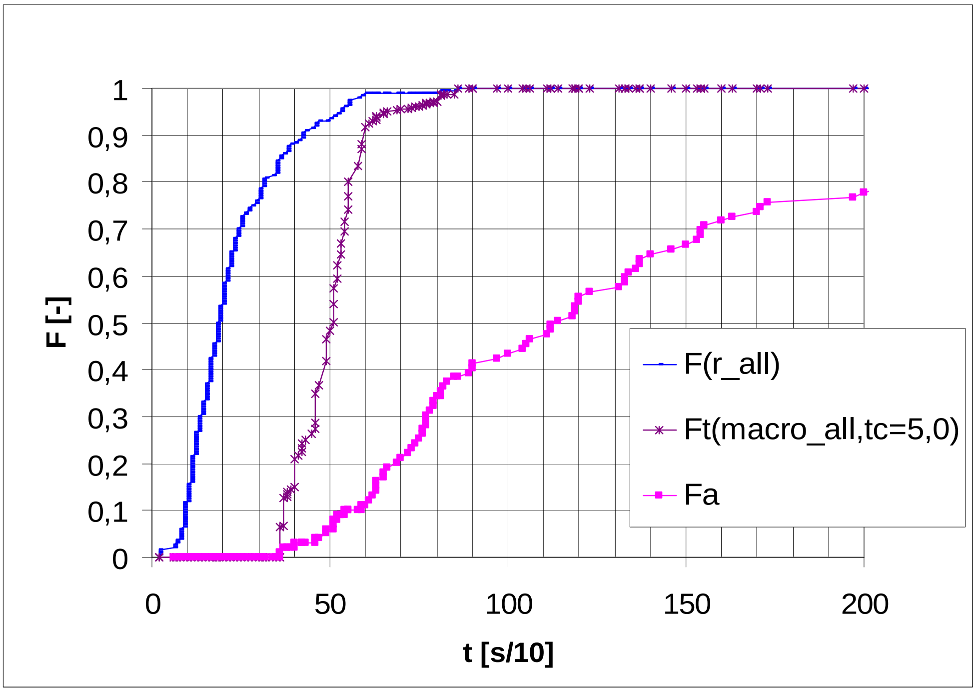

Figure 6 - Example for critical gap estimation. F_r=PDF of <u>all</u> rejected gaps, F_a=PDF of the accepted gaps, F_{tc}(macro)=PDF of the estimated critical gaps from the new model for macroscopic equilibrium (Data: Weinert, 2001, Köln 1, major left-turn).

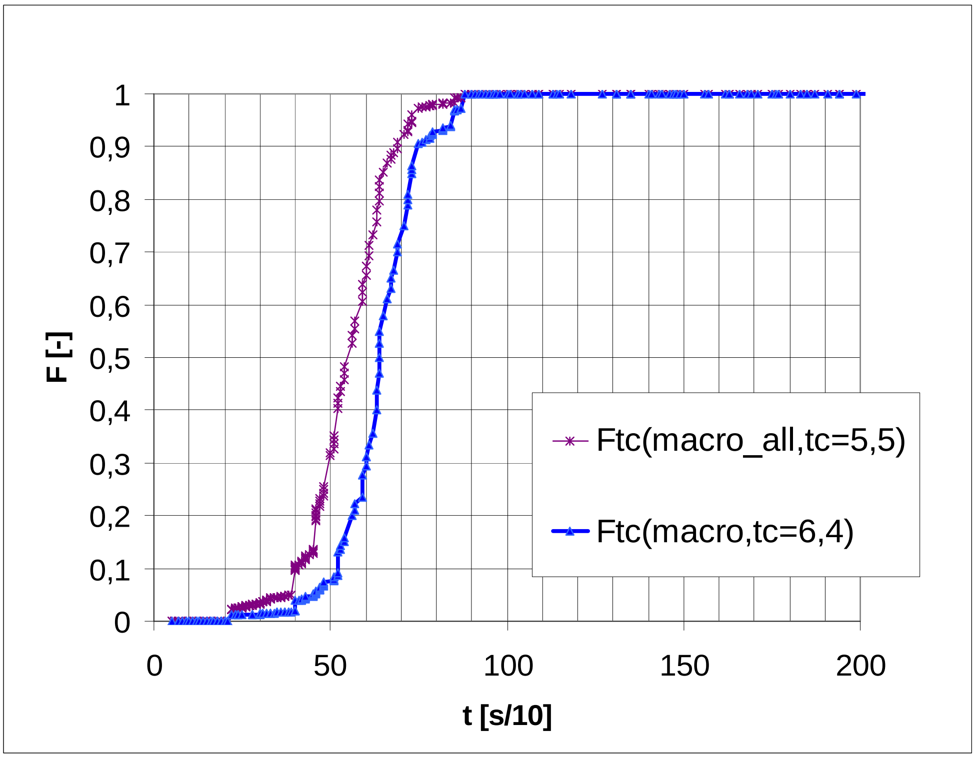

Figure 7 – Comparison of the estimated distributions of critical gaps. F_{tc}(macro)=PDF of the estimated critical gaps from the new model with only the maximum rejected gaps, F_{tc}(macro_all)=PDF of the estimated critical gaps from the new model with all rejected gaps (Data: Weinert, 2001, Bad Nauheim 3, minor right-turn).

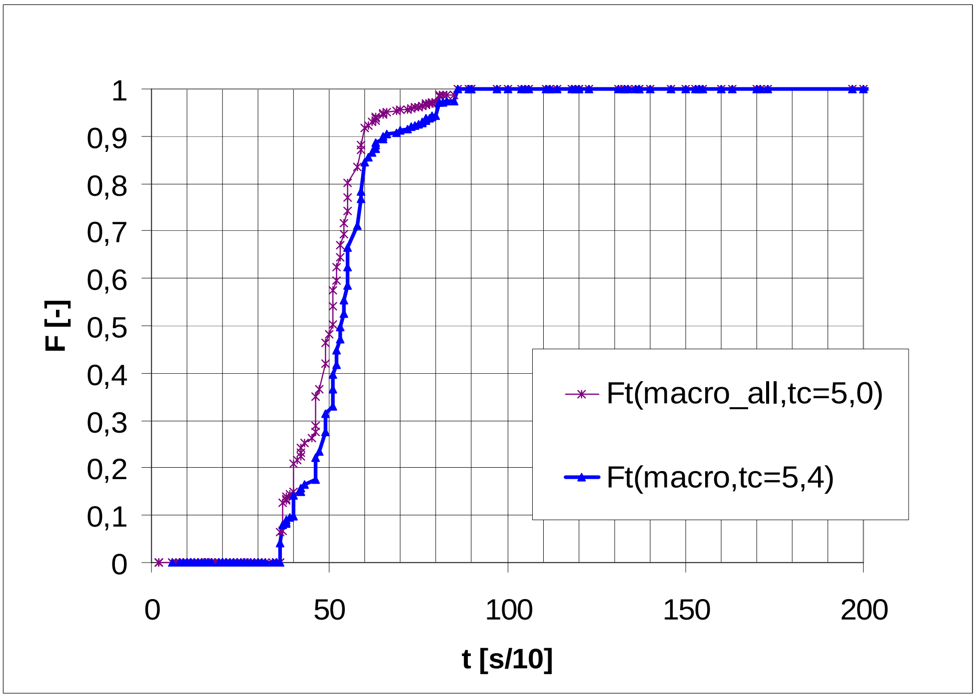

Figure 8 - Comparison of the estimated distributions of critical gaps. F_{tc}(macro)=PDF of the estimated critical gaps from the new model with only the maximum rejected gaps, F_{tc}(macro_all)=PDF of the estimated critical gaps from the new model with all rejected gaps (Data: Weinert, 2001, Köln 1, major left-turn).

3 SUMMARY AND CONCLUSIONS

Using the equilibrium of probabilities for rejected and accepted gaps, a new model for estimating the critical gap and its distribution can be established. The new model does not require any *a priori* assumptions and the results are accurate.

The proposed macroscopic model (equation (6)) gives a generalised procedure for estimating critical gaps. With this procedure, the PDF of the critical gaps can be estimated empirically.

The procedure for implementing the new model is simple and robust. It can be carried out using spreadsheet programs (e.g., EXCEL, QuatroPro etc.) without iteration. Thus, with the new model, a useful and promising tool can be set up for professionals of traffic engineering. For practical applications, an implemented EXCEL-spreadsheet can be obtained from the author.

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