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Workshop: Gemischte und nicht-standard Finite-Elemente-Methoden mit Anwendungen

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Advantage and Disadvantage of the Method of Enhanced Assumed Strains

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Abstract. Finite elements with enhanced assumed strains are equivalent to mixed methods associated to the Hellinger–Reissner principle. The softening of the energy which avoids locking phenomena and the role of the hidden LBB condition are elucidated for nearly incompressible material.

1. The Need of Softening

In structural mechanics often *problems with a small parameter* are encountered which give rise to locking phenomena:

- nearly incompressible material (volume locking),
- plates, beams and cantilevers (shear locking),
- shells (membrane locking),
- 3-dim. models of plates and shells (thickness locking).

Locking means that a portion of the stored energy is too stiff and must be softened. Here appropriate variational formulations and finite elements are required which are able to cope with these situations. In particular, the stiff part of the energy has to be softened.

If we exclude shells, the energy that is to be minimized is of the following form

$$\Pi(u) = \frac{1}{2}a_0(u,u) + \frac{1}{2t^2}\int_\Omega (Bu)^2 dx - (f,u).$$

Here a_0 is a (nice) quadratic form that can be treated by standard methods, t is the small parameter, B a mapping into $L_2(\Omega)$, and the last term represents the external load.

We consider a nearly incompressible material, then $a_0(u,u) = \mu \int \varepsilon(u)^2 dx$, $t^2 = \frac{\mu}{\lambda}$ and

$$Bu = \operatorname{div} u.$$

The kernel of the operator, $\ker u = \{v \in H_0^1(\Omega); \operatorname{div} v = 0 \ a.e.\}$, is a very rich (thick) space. – This is quite different in the finite element space, in particular when the quadrilateral Q_1 element (with bilinear functions) is chosen. The kernel consists of the rigid body motions and a linear one direction shear only. The kernel is a low dimensional space, and the dimension is not increased by refinements of the finite element mesh.

[Similarly, when the Mindlin plate is considered, we have $u = (\theta, w)$ with θ being the rotation, w the vertical displacement and

$$Bu := \theta - \nabla w$$

is the shear term that has to be softened.]

The canonical way of softening via a mixed method with penalty [2, p. 301] is not popular.

2. Softening Behavior of the Hellinger–Reissner Formulation

Another way of softening is the use of finite elements with enhanced assumed strains [4]. It is equivalent to the mixed method associated to the Hellinger–Reissner principle. For convenience, we describe it for the mixed formulation of the Poisson equation $\sigma - \nabla u = 0,\ \operatorname{div}\sigma = -f$, since the analogous treatment of the Lamé equations is less transparent. Here we consider the pairing with the spaces $L_2(\Omega)$-$H^1(\Omega)$ and not $H(\operatorname{div},\Omega)$-$L_2(\Omega)$. Let $S_h \subset L_2(\Omega)$ and $V_h \subset H^1(\Omega)$ be the finite element spaces for σ and u, resp. Then

$$\begin{aligned} (\sigma_h,\tau) - (\tau,\nabla u_h) &= 0 && \forall \tau \in S_h, \\ -(\sigma_h,\nabla v) &= -(f,v) && \forall v \in V_h. \end{aligned} \tag{1}$$

Let P_{S_h} be the L_2-orthogonal projector onto S_h. The first equation says $\sigma_h = P_{S_h}(\nabla u_h)$, and the second equation that we are computing the solution of the minimum problem

$$\frac{1}{2}\int_\Omega \left(P_{S_h}\nabla u_h\right)^2 dx - \int_\Omega f u_h dx \to \min!$$

Since the projection reduces the energy, we have got a softening effect.

Instead of defining the target space S_h of the projection, we can specify the orthogonal complement. Note that $\sigma_h = P_{S_h}(\nabla u_h)$ implies (in the framework of the Lamé equation the elasticity matrix is found as a factor)

$$\sigma_h = \nabla u_h + \tilde{\varepsilon}_h \quad \text{with } \tilde{\varepsilon}_h \in \tilde{E}_h \perp S_h.$$

Here $\tilde{E}_h$ is the space of enhanced strains.

Lemma 1. [5] *The mixed method (1) is equivalent to the variational formulation*

$$\begin{aligned} (\nabla u_h,\nabla v)_{0,\Omega} + (\tilde{\varepsilon}_h,\nabla v)_{0,\Omega} &= (f,v)_{0,\Omega} && \text{for } v \in V_h, \\ (\nabla u_h,\tilde{\eta})_{0,\Omega} + (\tilde{\varepsilon}_h,\eta)_{0,\Omega} &= 0 && \text{for } \eta \in \tilde{E}_h, \end{aligned} \tag{2}$$

if the space $\tilde{E}_h$ of enhanced gradients satisfies the decomposition rule $\nabla V_h \subset S_h \oplus \tilde{E}_h$.

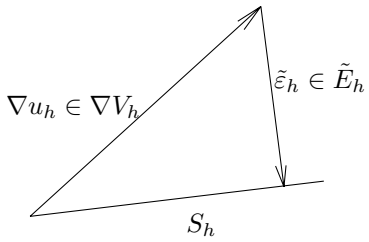

Figure 1. Projection of ∇u_h to S_h and $\tilde{E}_h$ in the EAS method.

Now it is clear that one cannot combine arbitrary pairs of spaces V_h and $\tilde{E}_h$ since one has to guarantee the LBB condition. This is often forgotten. Fortunately it can be expressed in terms of displacements and enhanced strains.

Lemma 2. [2, p.153] *The spaces V_h and S_h satisfy the inf-sup condition with a constant $\beta > 0$ if and only if a strengthened Cauchy inequality holds,*

$$(\nabla v_h, \eta_h)_{0,\Omega} \leq \sqrt{1-\beta^2}\ \|\nabla v_h\|_0\, \|\eta_h\|_0 \quad \textit{for } v_h \in V_h,\ \eta_h \in \tilde{E}_h.$$

3. Interpolation

In particular, volume locking disappears if the divergence is replaced by its mean value on each element. The volume term reads

$$\frac{\lambda}{2} \sum_{T\subset\Omega} \int_T (\overline{\operatorname{div}}\ v_h)^2 dx \tag{3}$$

This projection is also achieved with enhanced strains by Simo and Rifai [4]. The projection is related to the Stokes problem and the Q_1-P_0 element. Unfortunately the Q_1-P_0 element is unstable. This induces complications also in 7 (or more) almost equivalent remedies of locking known in engineering literature. Fortunately, a stable variant is obtained by a filter, and the filtered element is sufficient for the EAS method and linear problems for the following reason.

The efficient divergence of a finite element pair $(v_h, \tilde{\varepsilon}_h)$ is obviously $\operatorname{div} v_h + \operatorname{trace} \tilde{\varepsilon}_h$. There will be no locking if we find an interpolation operator $I_h : v \mapsto (v_h, \tilde{\varepsilon}_h)$ with the usual approximation properties and the additional prperty

$$\operatorname{div} v_h + \operatorname{trace} \tilde{\varepsilon}_h = 0 \quad \text{if } \operatorname{div} v = 0. \tag{4}$$

An interpolation with these properties is indeed given by the Stokes problem with the filtered Q_1-P_0 element [3] and related to (3). Thus the EAS method provides a stable locking-free element without the need of implementing the filter. In particular, the regularity result in [1] admits a robust approximation with a constant that is independent of the large parameter λ [3],

$$\lambda \,\|\operatorname{div} u - \operatorname{div} u_h - \operatorname{trace} \tilde{\varepsilon}_h\|_0 + \|u - u_h\|_1 \leq c\, h \|f\|_0.$$

Finally, we emphasize that the pairs of spaces must be well balanced. *If the space $\tilde{E}_h$ is too small, then we do not achieve property (4). If the space $\tilde{E}_h$ is too large, then stability is lost.*

References

[1] Douglas N. Arnold, and L. Ridgway Scott and Michael Vogelius. Regular inversion of the divergence operator with Dirichlet boundary conditions on a polygon. Ann. Scuola Norm. Sup. Pisa Cl. Sci. Seris IV, 15, 2, 169–192 (1988).

[2] Dietrich Braess. *Finite Elements.* Cambridge University Press, Cambridge, (2001).

[3] D. Braess, C. Carstensen and B.D. Reddy, Uniform convergence and a posteriori error estimators for the enhanced strain finite element method. Numer. Math. 96, 461–479 (2004)

[4] J.C. Simo and Rifai, M. S. (1990). A class of mixed assumed strain methods and the method of incompatible modes. Internat. J. Numer. Methods Engrg., **29**, 8, 1595–1638.

[5] Seung T. Yeo and Byung C. Lee. Equivalence between enhanced assumed strain method and assumed stress hybrid method based on the Hellinger-Reissner principle. Internat. J. Numer. Methods Engrg., 39, 18, 3083–3099 (1996).