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An a posteriori error estimate and a Comparison Theorem for the nonconforming P_1 element

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Abstract. A posteriori error estimates for the nonconforming P_1 element are easily determined by the hypercircle method via Marini's observation on the relation to the mixed method of Raviart–Thomas. Another tool is Ainsworth's application of the hypercircle method to mixed methods. The relation on the finite element solutions is also extended to an a priori relation of the errors, and the errors of four different finite element methods can be compared.

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Thomas element

1. Introduction

Classical a posteriori error estimators for the nonconforming P_1 element are more involved than the analogous ones for the conforming element [5]. The situation is quite different when the hypercircle method is applied.

The hypercircle method [7] requires the knowledge of an equilibrated flux. Marini [6] established a connection between the nonconforming P_1 element and the mixed method of Raviart–Thomas. This relation provides a direct construction of a flux as wanted. The construction of such a flux is the most costly part when the hypercircle

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method is applied to conforming elements. Here, we only need an H^1 function that is sufficiently close to the given nonconforming P_1 element. Such a function, in turn, is given by Ainsworth's construction [1] for the Raviart–Thomas element behind which the hypercircle method is concealed.

The hypercircle method, also called *the two energies principle*, is based on the theorem of Prager and Synge; see [7] or e.g., [3, p. 181]. Here and throughout the paper we restrict ourselves to the Poisson equation on a polygonal domain $\Omega \subset \mathbb{R}^2$ with homogeneous Dirichlet boundary conditions:

$$-\Delta u = f \quad \text{in } \Omega. \tag{1}$$

Theorem 1. *(Theorem of Prager and Synge) Let $\sigma \in H(\mathrm{div})$, $v \in H_0^1(\Omega)$, and assume that*

$$\operatorname{div} \sigma + f = 0. \tag{2}$$

If u denotes the solution of the Poisson equation (1), then,

$$\|\nabla u - \nabla v\|^2 + \|\nabla u - \sigma\|^2 = \|\nabla v - \sigma\|^2. \tag{3}$$

Proof. We provide the short proof for the reader's convenience, and it also provides a hint to the name of the method. By applying Green's formula and noting that $\Delta u = \operatorname{div} \sigma = -f$ we obtain

$$\begin{aligned}
&\int_\Omega \nabla(u - v)(\nabla u - \sigma) dx \\
&= -\int_\Omega (u - v)(\Delta u - \operatorname{div} \sigma) dx + \int_{\partial\Omega} (u - v)(\frac{\partial u}{\partial n} - \sigma \cdot n) ds = 0.
\end{aligned}$$

The boundary term above vanishes, since $u - v = 0$ on $\partial\Omega$. This orthogonality relation and the Pythagorean rule yield (3).

Obviously, the estimates can be used for finite element solution $v = u_h$ in $H^1(\Omega)$ as well as for a solution $\sigma_h \in H(\mathrm{div})$ of a mixed method. The symmetry of the formula (3) is violated by the equilibration condition (2). We emphasize this asymmetry for the background of the investigation in this paper.

The hypercircle method has become popular recently, since it yields upper bounds without generic constants. There is also a case where it yields a better asymptotic than the standard residual estimator [4]. In this paper we will also obtain more results than in the usual applications. The a posteriori error estimates lead to a comparison of the nonconforming finite element solution with other finite element methods.

More information and more literature can be found in all the cited papers. We add only [2] from the early papers, since it is seldom cited.

For convenience, we restrict ourselves to the case that f is piecewise constant on the actual triangulation, and we ignore the correction which results from the data oscillation. The correction is a term of higher order, cf. Remark 1. Moreover, it can even be added to the main term in the manner of Pythagoras and not by the triangle inequality; cf. [1, Theorem 1].

2. The connection with the Raviart–Thomas element and the error estimator

Let $\mathcal{T}_h$ be a partition of Ω into triangles and denote the space of finite elements due to Crouzeix–Raviart by

$$\begin{aligned}\mathcal{M}^1_* := \{v \in L_2(\Omega);\ & v|_T \text{ is linear for every } T \in \mathcal{T}_h,\\ & v \text{ is continuous at the midpoints of the triangle edges}\}.\end{aligned}$$

The finite element solution for $\mathcal{M}^1_*$ that vanishes at the midpoints of the edges on $\partial\Omega$, is denoted as u^{CR},

$$\sum_T \int_T \nabla u^{\mathrm{CR}} \nabla v dx = \int_\Omega f v dx \qquad \forall v \in \mathcal{M}^1_*.$$

The mixed method of Raviart-Thomas is described by the pair of spaces

$$\begin{aligned}RT_h &:= \{\tau \in H(\mathrm{div});\ \tau|_T = a_T + b_T\, x,\ a_T \in \mathbb{R}^2,\ b_T \in \mathbb{R} \text{ for } T \in \mathcal{T}_h\},\\ \mathcal{M}^0 &:= \{v \in L_2(\Omega);\ v_T = a_T,\ a_T \in \mathbb{R} \text{ for } T \in \mathcal{T}_h\},\end{aligned}$$

and the equations

$$\begin{aligned}(\sigma^{\mathrm{RT}}, \tau)_0 + (\mathrm{div}\,\tau, u^{\mathrm{RT}})_0 &= 0 & \forall \tau \in RT_h,\\ (\mathrm{div}\,\sigma^{\mathrm{RT}}, v) \qquad\qquad &= -(f, v)_0 & \forall v \in \mathcal{M}^0.\end{aligned} \tag{4}$$

Since $\mathrm{div}\,\sigma^{\mathrm{RT}} \in \mathcal{M}^0$, the relation $\mathrm{div}\,\sigma^{\mathrm{RT}} + f = 0$ holds pointwise. Here and in the following $(\cdot,\cdot)_0$ denotes the L_2 inner product and $\|\cdot\|_0$ the L_2 norm. The analogous expressions for subsets of Ω are given with the subset added to the index, and

$$\|v\|_{0,h} := \Big\{\sum_{T\in\mathcal{T}_h} \|v\|^2_{0,T}\Big\}^{1/2}, \qquad |v|_{1,h} := \Big\{\sum_{T\in\mathcal{T}_h} \|\nabla v\|^2_{0,T}\Big\}^{1/2}$$

refer to the broken Sobolev norms.

Marini [6] observed that the Raviart–Thomas finite element solution $(\sigma^{\mathrm{RT}}, u^{\mathrm{RT}})$ can be obtained from the Crouzeix–Raviart FE solution on the same triangulation:

$$\left.\begin{aligned} \sigma^{\mathrm{RT}} &= \nabla u^{\mathrm{CR}} - \tfrac{1}{2} f_T\,(x - x_T), \\ u^{\mathrm{RT}} &= \tfrac{1}{|T|}\int_T \left[u^{\mathrm{CR}} + \tfrac{1}{4} f_T\,(z - x_T)^T (z - x_T)\right] dz, \end{aligned}\right\} x \in T. \tag{5}$$

Here, $f_T := \frac{1}{|T|}\int_T f\,dx$, and x_T denotes the centroid of the triangle T. The gradient is understood as a pointwise derivative.

Following [1] we introduce an auxiliary function that consists of piecewise quadratic polynomials:

$$\left.\begin{aligned} u^0(x) &:= u^{\mathrm{CR}}(x) - \tfrac{1}{2} f_T \psi(x), \\ \psi(x) &:= \tfrac{1}{2}\,(x - x_T)^T (x - x_T) - \tfrac{1}{|T|}\int_T (z - x_T)^T (z - x_T) dz. \end{aligned}\right\} x \in T.$$

Likewise, (5) may be replaced by

$$\left.\begin{aligned} \sigma^{\mathrm{RT}}(x) &= \nabla u^0(x), \\ u^{\mathrm{RT}}(x) &= \tfrac{1}{|T|}\int_T u^0\,dz, \end{aligned}\right\} x \in T \tag{6}$$

for the definition of σ^{RT} and u^{RT}. Finally, a continuous, piecewise quadratic function u^1 is constructed by an averaging procedure. Specifically, u^1 is given by its values at the nodes that are vertices of the triangles or midpoints of their edges: Given a node ξ, let $K(\xi) := \cup_{\xi \in T} T$ and denote the number of triangles in $K(\xi)$ by N_ξ. Now set

$$u^1(\xi) = \begin{cases} 0, & \text{if } \xi \in \partial\Omega, \\ \frac{1}{N_\xi} \displaystyle\sum_{T \in K(\xi)} u^0_{|T}(\xi), & \text{otherwise.} \end{cases} \tag{7}$$

We emphasize that the finite element functions σ^{RT}, u^{RT}, u^0, and u^1 are determined from the given solution u^{CR} by local postprocessing procedures. We obtain from them a *reliable error estimate* without generic constants.

Theorem 2. *Let u^{CR} be the finite element solution with the nonconforming P_1 element. Then*

$$|u - u^{CR}|_{1,h} \le \|\nabla u^{CR} - \sigma^{RT}\|_{0,h} + \|\sigma^{RT} - \nabla u^1\|_0\,. \tag{8}$$

Proof. Since $u^1 \in H^1_0(\Omega)$ and σ^{RT} is equilibrated, the theorem of Prager and Synge yields $\|\sigma^{\mathrm{RT}} - \nabla u\|_0 \le \|\sigma^{\mathrm{RT}} - \nabla u^1\|_0$. The triangle inequality, applied to

$$\nabla(u - u^{\mathrm{CR}}) = [\nabla u - \sigma^{\mathrm{RT}}] + [\sigma^{\mathrm{RT}} - \nabla u^{\mathrm{CR}}],$$

completes the proof of (8).

Remark 1. We recall that f is assumed to be piecewise constant. Otherwise a higher order term $ch\|f_{orig} - f\|_0$ resulting from the data oscillation has to be added in (8); cf. [3, p. 174].

Remark 2. Due to (6) the first term of the estimator can be determined before u^{CR} and σ^{RT} are computed:

$$\|\nabla u^{\mathrm{CR}} - \sigma^{\mathrm{RT}}\|_{0,h} = \frac{1}{2}\|f_T(x - x_T)\|_{0,h} \,. \tag{9}$$

The second term of the estimator is the reliable and efficient error estimate for the Raviart–Thomas element derived by Ainsworth using (implicitly) the hypercircle method [1, Theorem 3]

$$c^{-1}\,\|\sigma^{\mathrm{RT}} - \nabla u^1\|_0 \le \|\sigma^{\mathrm{RT}} - \nabla u\|_0 \le \|\sigma^{\mathrm{RT}} - \nabla u^1\|_0 \,. \tag{10}$$

The result (8) remains true if we determine u^1 by averaging u^{CR} instead of u^0. (For this reason, we did not adopt all of the notation of [1].) The present choice, however, avoids that we have to adjust results from [1] when proving efficiency of the error estimate defined in Theorem 2.

3. Efficiency

The a posteriori estimate in Theorem 2 is considered as *efficient* if a multiple of the right-hand side of (8) is a lower bound of the error. Here, the factor in the inequality may depend only on the shape parameter of the triangulation $\mathcal{T}_h$ and possibly on the domain Ω (as generic constants usually do).

We recall that residual a posteriori error estimates for the Crouzeix–Raviart element contain area-based terms as $h_T\|f\|_{0,T}$ and appropriately scaled jumps of u^{CR} on the edges. Obviously, the contributions in (8) can be bounded by those residuals. Therefore, the efficiency of the new estimators will be no surprise.

Lemma 1. *There is a constant c that depends only on the shape parameter of $\mathcal{T}_h$ such that*

$$\|\sigma^{RT} - \nabla u^{CR}\|_{0,T} \le c\,\|\nabla(u - u^{CR})\|_{0,T} \tag{11}$$

holds for each $T \in \mathcal{T}_h$.

Proof. The proof repeats some standard arguments from the theory of a posteriori error estimates and is given only for completeness.

Let Φ_T be the cubic bubble function with the properties

$$0 \le \Phi_T \le 1 = \max_{z\in T} \Phi_T(z), \qquad \operatorname{supp} \Phi_T = T.$$

Since $\Omega \subset \mathbb{R}^2$, the normalization implies $\|\nabla\Phi_T\|_0 \le c$. It follows from the equivalence of norms on the 3-dimensional space of linear polynomials that

$$\|\Phi_T^{1/2} p\|_{0,T} \ge c\|p\|_{0,T} \quad \forall p \in P_1 .$$

Set $w := f_T\Phi_T$. Since $w \in H_0^1(T)$, partial integration yields $\int_T \nabla w dx = 0$ and

$$\begin{aligned}
c^{-1}\|f_T\|_{0,T}^2 &\le \|\Phi_T^{1/2} f_T\|_{0,T}^2 \\
&= (f_T, w)_{0,T} = (\nabla(u - u^{\mathrm{CR}}), \nabla w)_{0,T} + (\nabla u^{\mathrm{CR}}, \nabla w)_{0,T} \\
&\le \|\nabla(u - u^{\mathrm{CR}})\|_{0,T} f_T \|\nabla\Phi_T\|_{0,T} + \nabla u^{\mathrm{CR}}_{|T} \int_T \nabla w \, dx \\
&\le c\|\nabla(u - u^{\mathrm{CR}})\|_{0,T} h_T^{-1} \|f_T\|_{0,T} .
\end{aligned}$$

After dividing by $\|f_T\|_{0,T}$ we have

$$h_T \, \|f_T\|_{0,T} \le c \, |u - u^{\mathrm{CR}}|_{1,T} .$$

Combining the observation $\frac{1}{2}\|f_T(x - x_T)\|_{0,T} \le h_T\|f_T\|_{0,T}$ with (9) and summing over all triangles we complete the proof.

Now we are in a position to establish the lower bound in order to show the efficiency of the error estimate.

Theorem 3. *Let u^{CR} be the finite element solution with the nonconforming P_1 element. Then*

$$c|u - u^{CR}|_{1,h} \ge \|\nabla u^{CR} - \sigma^{RT}\|_{0,h} + \|\sigma^{RT} - \nabla u^1\|_0 . \tag{12}$$

Proof. It follows from (10) that

$$\begin{aligned}
&\|\nabla u^{\mathrm{CR}} - \sigma^{\mathrm{RT}}\|_{0,h} + \|\sigma^{\mathrm{RT}} - \nabla u^1\|_0 \\
&\quad \le \|\nabla u^{\mathrm{CR}} - \sigma^{\mathrm{RT}}\|_{0,h} + c\|\sigma^{\mathrm{RT}} - \nabla u\|_0 \\
&\quad \le \|\nabla u^{\mathrm{CR}} - \sigma^{\mathrm{RT}}\|_{0,h} + c(\|\sigma^{\mathrm{RT}} - \nabla u^{\mathrm{CR}}\|_{0,h} + c\|\nabla(u^{\mathrm{CR}} - u)\|_{0,h}) .
\end{aligned}$$

The preceding lemma guarantees that all terms can be bounded by multiples of $\|\nabla(u - u^{\mathrm{CR}})|_{0,h}$, and the proof is complete.

There is not only the *algebraic* relation (5) between the Crouzeix–Raviart element and the Raviart–Thomas element. The errors of the solutions are also related. Eventually, we compare the errors of four finite elements. The relations in (13) are understood as inequalities modulo generic constants.

Theorem 4. *Let u^{conf1} and u^{conf2} be the solutions with linear and quadratic finite elements, respectively. If f is piecewise constant, then*

$$\|\nabla(u - u^{conf2})\|_0 \preceq \|\nabla u - \sigma^{RT}\|_0 \preceq |u - u^{CR}|_{1,h} \preceq \|\nabla(u - u^{conf1}\|_0 \,. \tag{13}$$

Proof. Since u^1 is a P_2 element, it follows from Remark 1 that

$$\begin{aligned}\|\nabla(u - u^{\mathrm{conf2}})\|_0 &\leq \|\nabla(u - u^1)\|_0 \\ &\leq \|\nabla u - \sigma^{\mathrm{RT}}\|_0 + \|\sigma^{\mathrm{RT}} - \nabla u^1\|_0 \\ &\leq \|\nabla u - \sigma^{\mathrm{RT}}\|_0 + c\|\sigma^{\mathrm{RT}} - \nabla u\|_0 \,.\end{aligned}$$

This proves the first inequality.

By Lemma 1

$$\begin{aligned}\|\nabla u - \sigma^{\mathrm{RT}}\|_0 &\leq \|\nabla u - \nabla u^{\mathrm{CR}}\|_{0,h} + \|\nabla u^{\mathrm{CR}} - \sigma^{\mathrm{RT}}\|_{0,h} \\ &\leq \|\nabla u - \nabla u^{\mathrm{CR}}\|_{0,h} + c\|\nabla u - \nabla u^{\mathrm{CR}}\|_{0,h} \,.\end{aligned}$$

This proves the second inequality.

Recalling $\nabla u^{\mathrm{CR}} = \sigma^{\mathrm{RT}} + \frac{1}{2} f_T(x - x_T)$ we have

$$\|\nabla u - \nabla u^{\mathrm{CR}}\|_{0,h} \leq \|\nabla u - \sigma^{\mathrm{RT}}\|_0 + h\|f\|_0 \,.$$

By theorem III.5.6 in [3] we know that $\|\nabla u - \sigma^{\mathrm{RT}}\|_{0,h} \leq \|\nabla u - \nabla u^{\mathrm{conf1}}\|_0$. The term $h\|f\|_0$ is a typical term in the residual estimator for the conforming P_1 element. Its efficiency implies $h\|f\|_0 \leq c\|\nabla u - \nabla u^{\mathrm{conf1}}\|_0$, and also the proof of the last inequality is complete.

Marini [6] noted that the construction of the finite element solution for the Raviart–Thomas element from the Crouzeix–Raviart element is cheaper than the direct implementation of (4). A preference for the result of the Raviart–Thomas element is consistent with the comparison in Theorem 13.

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