Equilibrated Residual Error Estimates are p-Robust

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Abstract

Equilibrated residual error estimators applied to high order finite elements are analyzed. The estimators provide always a true upper bound for the energy error. We prove that also the efficiency estimate is robust with respect to the polynomial degrees. The result is complete for tensor product elements. In the case of simplicial elements, the theorem is based on a conjecture, for which numerical evidence is provided.

Keywords: a posteriori error estimates, hypercircle method, *hp* finite elements, equilibration

1 Introduction

Various a posteriori error estimates for numerical solutions with finite elements of low order are known to be *reliable and efficient* [13]. They are used to create locally refined meshes in an adaptive way that fit to the elliptic problem under consideration. Many theoretical and practical aspects have been considered in the literature.

The situation is less satisfactory for the p-version and the hp-version of the finite element method. Recently, Melenk and Wohlmuth [9] have shown for triangular and quadrilateral meshes that the efficiency of residual error estimators decrease only as p^{-1} if the polynomial degree p increases. They reported also, however, that numerical results do not admit to expect better theoretical results.

In this paper we will turn to a posteriori error estimates for the p-method and the hp-method by the hypercircle method that may be traced back to Prager and Synge [10]. There is actual interest in this technique, since one obtains reliable bounds without generic constants in the dominant term and cheap implementations have been described recently; see, e.g., [4, 8]. Merely local problems have to be solved; they are organized on local patches around nodes of the mesh, while local problems for other classical error estimators are oriented to elements of the triangulation.

Numerical examples show that the new estimators do not suffer from a loss of efficiency for large p. There is a complete proof for rectangular and quadrilateral meshes while a conjecture for an ingredient has to be built in in the case of simplicial meshes. A crucial tool is a result on the right inverse of the divergence operator. It can be proven for rectangles and will be left as a conjecture for triangular partitions of the domain. The advantage of the estimator by the hypercircle method is certainly that it reflects the H^{-1}-norm of the residues while the well-known residual estimators refer to weighted L_2 norms of them. Of course, we have to pay for it in the analysis, and we have to deal with distributional forms of the differential operators.

The outline of the paper follows. Section 2 describes the postprocessing which yields the error estimate. Its reliability is an immediate consequence of Prager and Synge's theorem. Section 3 is concerned with the proof of the efficiency, which emphasizes the difference to residual error estimators. It contains a result on the left inverse of the divergence operator when distributional terms are included, and it may be of independent interest. A projector on univariate polynomials that is uniformlybounden in p with respect to two norms is the topic of Section 4. The paper concludes with numerical examples.

2 Formulation of the method

Let Ω be a polygonal or polyhedral domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. The variational formulation of an elliptic differential equation of second order in divergence form is written as

$$a(u, v) = f(v) \qquad \forall\, v \in V$$

with

$$a(u, v) = \int_\Omega a\nabla u \cdot \nabla v\, dx \qquad f(v) = \int_\Omega fv\, dx.$$

We consider the finite element approximation with polynomials of higher order on a quasi-uniform triangulation of Ω with simplicial, rectangular or hexahedral elements T. Since the notation for the notation of the finite element spaces differ for the triangulations above, we will focus on triangular meshes throughout the paper unless otherwise stated. The finite element solution is given by

$$a(u_N, v) = f(v) \qquad \forall\, v \in V_N\,,$$

and in this case we have

$$V_N = \{v \in V : v_{|T} \in P^{p+1}(T)\}.$$

We assume that f is a piece-wise polynomial of degree p. Otherwise, f is assumed to be the piecewise approximation of the exact f_{ex}. As usual, this additional approximation is left to the user, and the effect of the data oscillation $f - f_{ex}$ is assumed to produce an extra error of higher order, in particular, if $\int_T (f - f_{ex})dx = 0$ for each element T.

We assume that a is piece-wise constant and for simplicity $a \approx 1$. The analysis could be extended to a highly varying coefficient a satisfying the quasi-monotonicity condition.

We will establish a postprocessing algorithm which provides an equilibrated flux $\sigma \in \Sigma_N \subset H(\mathrm{div})$, i.e., the flux satisfies pointwise

$$\operatorname{div} \sigma = f. \tag{1}$$

The difference between the *discrete flux* $a\nabla u_N$ and the postprocessed one provides a true upper bound without generic constant to the error measured in energy error. Specifically, by the theorem of Prager and Synge,

$$\|u - u_N\|_a \leq \|a\nabla u - \sigma\|_{a^{-1},L_2}; \tag{2}$$

cf. [10] or [3, Theorem III.5.1]. In the literature, also the names *hypercircle method* or *two-energies principle* are found.

Computations have shown that the overestimation of the error is bounded uniformly in the mesh-size as well as the polynomial degree. This means that the resulting estimator is *efficient*, and we will prove it in this paper.

The hypercircle method is the general framework. We propose the following specific construction for the flux σ: Following [4] the residual $r \in V^*$ defined as

$$\langle r, v\rangle := f(v) - a(u_N, v) \tag{3}$$

is written as

$$\langle r, v\rangle = \sum_T (r^T, v)_{0,T} + \sum_E (r^E, v)_{0,E}. \tag{4}$$

The well known element and edge residuals are

$$\begin{aligned} r^T &= f + \operatorname{div}(a\nabla u_N), \\ r^E &= [a\,\partial_n u_N]\,. \end{aligned} \tag{5}$$

Here the divergence is taken pointwise on each element. Let ϕ_i denote the hat basis function associated with the vertex x_i. Its support is $\omega_i = \cup\{T : x_i \in \partial T\}$. We decompose the residual by this partition of unity, i.e.,

$$\langle r_{\omega_i}, v\rangle := \langle r, \phi_i v\rangle\,. \tag{6}$$

Recalling (4) we get the element and edge terms of the local residual:

$$\begin{aligned} r^T_{\omega_i} &= \phi_i\{f + \operatorname{div}(a\nabla u_N)\}, \\ r^E_{\omega_i} &= \phi_i\,[a\partial_n u_N]\,. \end{aligned} \tag{7}$$

The Galerkin orthogonality of the hat basis functions yields

$$\langle r_{\omega_i}, 1\rangle = \langle r, \phi_i\rangle = 0, \tag{8}$$

i.e., the local residuals are bi-orthogonal to constant functions.

The element terms as well as edge terms in (7) are polynomials of degree at most $p+1$. We construct a vector function σ_{ω_i} in the *broken Raviart–Thomas space* $RT^{p+1}(\omega_i)$ [1],

$$\begin{aligned} RT^p_{-1}(\omega_i) &:= \{\tau \in L_2(\omega_i) : \tau|_T \in RT^p(T),\ T \subset \omega_i\} \\ \text{with}\quad RT^p(T) &:= \{\tau : \tau(x) = q_T + s_T x,\ q_T \in (P^p)^2,\ s_T \in P^p\} \end{aligned}$$

such that

$$\operatorname{div}\sigma_{\omega_i} = r_{\omega_i}.$$

The divergence is understood in distributional sense and is consistent with (4). Combining this with the bounding condition, it translates into

$$\begin{aligned} \operatorname{div}_T \sigma_{\omega_i} &= r^T_{\omega_i} && \text{in } T \subset \omega_i\,, \\ [\sigma_{\omega_i}\cdot n] &= r^E_{\omega_i} && \text{at } E \subset \omega_i\,, \\ \sigma_{\omega_i}\cdot n &= 0 && \text{on } \partial\omega_i\,. \end{aligned} \tag{9}$$

The vector function σ_{ω_i} with minimal L_2-norm corresponds to the gradient that minimizes the *complementary energy* and is given by the solution of a mixed method. Specifically, we can use the implementation of the Raviart–Thomas element by Arnold and Brezzi [1]. The implementation refers to the broken Raviart–Thomas space; we have only to insert the inhomogeneous condition $(9)_2$ instead of the homogeneous one: $[\sigma_{\omega_i} \cdot n] = 0$. The stability is independent of the right hand side of the equations and has been proven in [1]. We refer to [5] for elements of Raviart–Thomas type or BDM elements on quadrilateral grids or for three-dimensional domains.

The computation of these local fluxes is the crucial step of the equilibration. By adding up all the fluxes we obtain the global correction

$$\sigma^\Delta = \sum \sigma_{\omega_i}$$

as a solution of

$$\operatorname{div} \sigma^\Delta = r.$$

The postprocessed flux is $\sigma := a\nabla u_N + \sigma^\Delta$, and from (2) it follows that

$$\|u - u_N\|_a \leq \|\sigma^\Delta\|^2_{a^{-1}}$$

is a reliable error estimate without a generic constant. The main result of the present paper is to prove the p-robust efficiency of the error estimate.

Theorem 1 (p-robust efficiency). *If the mesh consists of*

- *affine quadrilateral or hexahedral elements*
- *or triangular or tetrahedral elements and Conjecture 6 is valid*

then the error estimator is locally efficient, i.e.,

$$\|\sigma_{\omega_i}\| \leq c\, \|\nabla u - \nabla u_N\|_{a,\omega_i} \tag{10}$$

holds with a constant c that is independent of h and p.

Remark 2. *A global correction with the same divergence as $\operatorname{div} \sigma^\Delta$ can be constructed by fluxes in RT^p instead of RT^{p+1}. Since $\sum_i \phi_i = 1$, the sums $\sum_i r^T_{\omega_i}$ and $\sum_i r^E_{\omega_i}$ are polynomials of maximal degree p. Therefore we can replace the local residuals by the orthogonal projections onto P^p and setting*

$$\begin{aligned} r^T_{\omega_i} &= \Pi^p(\phi_i\{f + \operatorname{div}(a\nabla u_N)\}), \\ r^E_{\omega_i} &= \Pi^p(\phi_i\, [a\partial_n u_N]). \end{aligned}$$

Numerical experiments show that the efficiency is reduced only slightly in this way. Similarly, the use of the BDM elements slightly improves the efficiency due to the larger number of local degrees of freedom.

3 Proof of the efficiency

The proof of Theorem 1 consists of several steps.

First we want to show that the dual norm of the local residual r_{ω_i} is bounded by the local error of the finite element solution. By (8) r_{ω_i} is a functional on $H^1(\omega_i)/\mathbb{R}$. In order to obtain h independent results, we set

$$\|r\|_{[H^1(\omega)/\mathbb{R}]^*} := \sup_{v\in H^1(\omega)/\mathbb{R}} \frac{\langle r, v\rangle}{\|\nabla v\|_{0,\omega}}$$

and similarly

$$\|r\|_{[H^1_0(\omega)]^*} := \sup_{v\in H^1_0(\omega)} \frac{\langle r, v\rangle}{\|\nabla v\|_{0,\omega}}\,.$$

Recalling (6) we obtain

$$\langle r_{\omega_i}, v\rangle = \langle r, \phi_i v\rangle = a(u-u_N, \phi_i v) = a(u-u_N, \phi_i(v-\overline{v}^{\omega_i}))$$

and

$$\begin{aligned}
\|r_{\omega_i}\|_{[H^1(\omega_i)/\mathbb{R}]^*} &= \sup_{v\in H^1(\omega_i),\, \|\nabla v\|_0\le 1} a(u-u_N, \phi_i(v-\overline{v}^{\omega_i})) \\
&\le \|u-u_N\|_{a,\omega_i} \sup_{v\in H^1} \frac{\|\phi_i(v-\overline{v}^{\omega_i})\|_{a,\omega_i}}{\|\nabla v\|_{0,\omega_i}} \\
&\le c\,\|u-u_N\|_{a,\omega_i}. \qquad (11)
\end{aligned}$$

The last inequality follows with standard scaling arguments from

$$\begin{aligned}
\|\phi_i(v-\bar{v})\|_a &\le c\,\|\nabla(\phi_i(v-\bar{v}))\|_0 \\
&\le c[\|\phi_i\|_{L_\infty}\|\nabla v\|_0 + \|\nabla\phi_i\|_{L_\infty}\|v-\bar{v}\|_0] \\
&\le c[\|\nabla v\|_0 + h^{-1}(h\|\nabla v\|_0)] \;\le\; c\,\|\nabla v\|_0
\end{aligned}$$

for $v\in H_1(\omega_i)$.

We will get at our aim, i.e., the efficiency as stated in inequality (10) showing the existence of a σ_{ω_i} on the patch such that

$$\operatorname{div} \sigma_{\omega_i} = r_{\omega_i} \qquad \text{and} \qquad \|\sigma_{\omega_i}\|_{0,\omega_i} \leq C \|r_{\omega_i}\|_{H^{-1}}.$$

This means that we have to find a continuous right inverse of the divergence that applies to distributions of the form (4) on the patch.

The constructive proof will differ from the construction given in the previous section for the use in actual computations. The estimates are based on two ingredients: One is the right-inverse of the divergence on one element, the other one is the extension of normal-traces from edges to elements that has been treated in [7] and is given in the following lemma.

Lemma 3 (Polynomial extension operators).
Let $\gamma \subset \partial T$ be the union of one, two, or three edges of T. Let $g_n \in L_2(\gamma)$ be given such that $g_n|_E \in P^p(E)$. If $\gamma = \partial T$ we additionally assume $\int_\gamma g_n = 0$. Then there exists an extension $\sigma_p \in RT^p(T)$ such that

$$\operatorname{div} \sigma_p = 0 \qquad \textit{and} \qquad \operatorname{tr}_{n,\gamma} \sigma_p = g_n,$$

and

$$\|\sigma_p\|_0 \preceq \inf_{\substack{\sigma \in L_2(T) \\ \operatorname{div} \sigma = 0,\, \operatorname{tr}_{n,\gamma} = g_n}} \|\sigma\|_0 \,.$$

A major contribution of this paper is to construct a right inverse of the divergence on quadrilateral and hexahedral elements. An essential tool is a projection operator onto univariate polynomials that is uniformly bounded in p for two norms.

Lemma 4. *Let $I = (-1, 1)$. There exist projection operators $Q^p : L_2(I) \to P^p$ which are uniformly bounded in p with respect to L_2 and simultaneously the H^1-norm.*

The proof of the lemma is postponed to the next section.

We turn to rectangular grids. Here $P^{k,\ell}$ contains the polynomials of degree k in the first variable and degree ℓ in the second one. The Raviart–Thomas elements on rectangular grids build the spaces $RT^{[k]} := P^{k+1,k} \times P^{k,k+1}$.

Theorem 5 (right inverse on tensor product elements).
Let T be a square or a cube. Let $r_T \in P^{p,p}(T)$. Then there exists a $\sigma_T \in RT^{[p]}(T)$ such that

$$\operatorname{div} \sigma_T = r_T \qquad \textit{and} \qquad \|\sigma_T\|_{0,T} \leq c \, \|r_T\|_{H_0^1(T)^*} \,.$$

$$
\begin{array}{ccc}
H^1 & \xrightarrow{d/dx} & L_2 \\
Q \big\downarrow & & \big\downarrow \tilde{Q} \\
P^{p+1} & \xrightarrow{d/dx} & P^p
\end{array}
\qquad\qquad
\begin{array}{ccc}
H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L_2 \\
Q^\Sigma \big\downarrow & & \big\downarrow \widetilde{Q}_x \otimes \widetilde{Q}_y \\
RT^{[p]} & \xrightarrow{\mathrm{div}} & P^{p,p}
\end{array}
$$

Commuting diagram properties of projectors

Proof. We restrict ourselves to the 2D-case and consider the Dirichlet problem

$$
\begin{aligned}
\Delta w &= r_T \quad \text{in } T, \\
w &= 0 \quad \text{on } \partial T.
\end{aligned}
$$

The flux $\sigma := \nabla w$ satisfies $\operatorname{div} \sigma = r_T$ and $\|\sigma\|_0 = \|r\|_{H_0^1(T)^*}$. We have to project σ into the polynomials. Take the projector $Q = Q^{p+1}$ from Lemma 4 and define another projector onto P^p by

$$
\widetilde{Q}v := (QIv)'
$$

with $Iv(x) := \int_{-1}^x v(s)\, ds$. The relation

$$
\|\widetilde{Q}v\|_0 = \|(QIv)'\|_0 \le \|(Iv)'\|_0 = \|v\|_0
$$

shows that $\widetilde{Q}$ is bounded in the L_2-norm. The two operators have the commuting diagram property

$$
\widetilde{Q}u' = (Qu)'.
$$

The tensor product projector

$$
Q^\Sigma = (Q_x \otimes \widetilde{Q}_y) \times (\widetilde{Q}_x \otimes Q_y) \;:\; L_2(T) \to RT^{[p]}(T)
$$

is bounded in $L_2(Q)$, and it has the commuting diagram property with the divergence, i.e.,

$$
\operatorname{div} Q^\Sigma = (\widetilde{Q}_x \otimes \widetilde{Q}_y)\ \operatorname{div}.
$$

We set $\sigma_N = Q^\Sigma \sigma$ to complete the proof of the lemma. $\square$

The corresponding result for BDM elements is obvious.

At the moment, an analogous result for the right inverse on simplicial elements can be posed only as a conjecture. Numerical computations with finite elements of high order and, of course, the result for rectangles support this conjecture.

Conjecture 6 (right inverse on simplicial elements).
Let T be a triangular or tetrahedral element. Let $r_T \in P^p(T)$. Then there exists a $\sigma_T \in RT^p(T)$ such that

$$\operatorname{div} \sigma_T = r_T \quad \textit{and} \quad \|\sigma_T\|_{0,T} \preceq \|r_T\|_{[H_0^1(T)]^*} .$$

For a further support of the conjecture we have computed the constants $C_{p,q}$ in Table 1 such that the inequalities

$$\min_{\substack{\sigma_T \in BDM^{p+1} \\ \operatorname{div} \sigma_T = r_T}} \|\sigma\|_0^2 \;\le\; C_{p,q} \sup_{v \in P^{p+q} \cap H_0^1(T)} \frac{(v, r_T)^2}{\|v\|_{H^1}^2} \tag{12}$$

hold for all $r_T \in P^p$. The constants can be computed by finding the largest eigenvalue of generalized eigenvalue problems. The discrete H^{-1}-norms in (12) approaches the H^{-1}-norm from below. Hence,

$$\min_{\substack{\sigma_T \in BDM^{p+1} \\ \operatorname{div} \sigma_T = r_T}} \|\sigma\|_0^2 \le C_p \|r_T\|_{[H_0^1(T)]^*}^2$$

with $C_p \le C_{p,q}$. The results indicate that C_p is bounded in p.

Table 1: Coefficients C_{pq} in (12).

p	$q=3$	$q=5$	$q=8$
1	1.81	1.76	1.76
2	2.05	1.92	1.92
4	2.43	1.99	1.99
8	3.23	2.00	2.00
16	4.92	2.38	2.00

Now we turn to the main theorem of the paper that guarantees the efficiency of the a posteriori error estimate for large polynomial degrees and may be also of independent interest. Here we will focus on triangular grids although we have to base the analysis now on the conjecture above.

We will refer to the subspace $RT^p_{-1,0}(\omega) := \{\tau \in RT^p_{-1}(\omega) : \tau \cdot n = 0 \text{ on } \partial\omega\}$ of the broken Raviart–Thomas space.

Theorem 7. *Let ω be the patch of elements around a vertex V. Let r be the residual*

$$\langle r, v\rangle = \sum_{T\subset\omega} \int_T r_T v + \sum_{E\subset\omega} \int_E r_E v \tag{13}$$

with $r_T \in P^p(T)$ and $r_E \in P^p(E)$. If $\langle r, 1\rangle = 0$, then

$$\inf_{\substack{\sigma\in RT^p_{-1,0}\\ \operatorname{div}\sigma = r}} \|\sigma\|_0 \leq C\,\|r\|_{[H^1(\omega)/\mathbb{R}]^*},$$

and the constant C is independent of p and h.

Proof. Step 1. Elimination of element residuals.
For each element $T\subset\omega$ we construct $\sigma_T \in RT^p(T)$ such that $\operatorname{div}_T \sigma_T = -r_T$. By Theorem 5 this is possible with

$$\|\sigma_T\|_{0,T} \preceq \|r_T\|_{H^1_0(T)^*}\,. \tag{14}$$

Now we estimate $\|r_T\|_{H^1_0(T)^*}$, and the non-locality of negative norms will be no problem. Given $v \in H^1_0(T)$ with $\|v\|_{1,T} \leq 1$, there is an extension by zero to $\tilde v \in H^1(\omega)$ and $(r_T, v)_{0,T} = \langle r, \tilde v\rangle \leq \|r\|_{H^1(\omega)^*}\|\tilde v\|_{1,\omega}$. Hence, $\|r_T\|_{H^1_0(T)^*} \leq \|r\|_{H^1(\omega)^*}$. From the assumption $\langle r, 1\rangle = 0$ it follows that $\|r\|_{H^1(\omega)^*} = \|r\|_{[H^1(\omega)/\mathbb{R}]^*}$ and

$$\|\sigma_T\|_{0,T} \preceq \|r\|_{[H^1(\omega)/\mathbb{R}]^*}\,. \tag{15}$$

Let $\sigma^{(1)} := \sum \sigma_T$ and

$$r^{(1)} := r - \operatorname{div} \sum_T \sigma_T\,, \tag{16}$$

with the divergence operator understood in the distributional sense. The new residual $r^{(1)}$ contains only edge terms including the edges on $\partial\omega$. Since $\|\operatorname{div} s\|_{H^1(\omega)^*} \leq \|s\|_{0,\omega}$ holds for all $s \in L_2(\omega)^2$, it follows from (15) that the modified functional is bounded

$$\|r^{(1)}\|_{[H^1(\omega)/\mathbb{R}]^*} \preceq \|r\|_{[H^1(\omega)/\mathbb{R}]^*}\,.$$

Recalling that (16) is understood in the distributional sense, we have

$$\langle \operatorname{div}\sigma_T, 1\rangle = -\int_{\partial T} \sigma_T\cdot n\, 1 + \int_T \operatorname{div}_T \sigma_T\, 1 = -\int_T \sigma\nabla 1 = 0$$

and again $\left\langle r^{(1)}, 1\right\rangle = 0$.

Step 2. Elimination of boundary edge residuals.
For each triangle $T \subset \omega$ consider the boundary-value problem

$$\begin{aligned} -\Delta w &= 0, \\ w &= 0 \qquad \text{on internal edges}, \\ \partial_n w &= r_E^{(1)} \qquad \text{for edges on } \partial T \cap \partial\omega, \end{aligned}$$

or in weak form with $w \in V_T := \{v \in H^1(T) : v = 0 \text{ on internal edges}\}$,

$$(\nabla w, \nabla v) = \int_{E \subset \partial\omega} r_E^{(1)} v. \tag{17}$$

Set $\sigma_T = \nabla w$ in this step (without changing the symbol). From (17) it follows that

$$\begin{aligned} \|\sigma_T\|_{0,T} &= \|\nabla w\|_{0,T} = \sup_{v \in V_T} \frac{\int_{E \subset \partial\omega} r_E^{(1)} v}{\|\nabla v\|} = \sup_{v \in \tilde{V}_T} \frac{\langle r^{(1)}, v \rangle}{\|\nabla v\|} \\ &\le \|r^{(1)}\|_{[H^1(\omega)/\mathbb{R}]^*}\ , \end{aligned}$$

where $\tilde{V}_T$ contains the extension of functions in V_T onto ω by zero.

Since $r_E^{(1)}$ is a polynomial, by Lemma 3 there exists a polynomial $\sigma_T^{(2)}$ such that

$$\mathrm{tr}_{n,E}\, \sigma_T^{(2)} = r_E^{(1)}$$

on $E \subset \partial\omega$, and $\|\sigma_T^{(2)}\|_{0,T} \preceq \|\sigma_T\|_{0,T} \preceq \|r\|_{[H^1(\omega)/\mathbb{R}]^*}$. This construction can be done independently triangle by triangle. Next we subtract the divergences of $\sigma^{(2)} := \sum_T \sigma_T^{(2)}$ to obtain

$$r^{(2)} = r^{(1)} - \operatorname{div} \sum_T \sigma_T^{(2)}\ .$$

By construction, $r^{(2)}$ contains only edge residuals on internal edges. Moreover, $\langle r^{(2)}, 1 \rangle = 0$, and

$$\|r^{(2)}\|_{[H^1(\omega)/\mathbb{R}]^*} \preceq \|r\|_{[H^1(\omega)/\mathbb{R}]^*}.$$

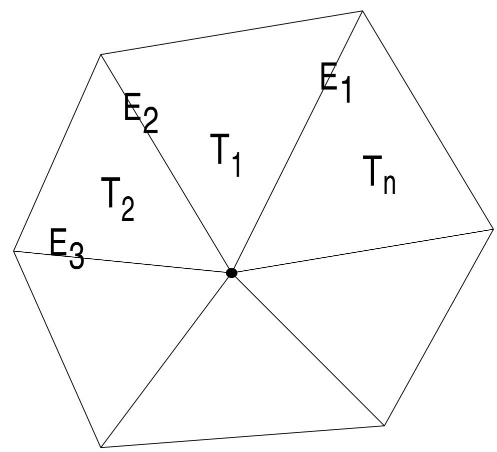

Figure 1. Enumeration of triangles and edges in the patch ω with n triangles.

Step 3. Elimination of internal edge residuals.
We will circle around the patch: $E_1, T_1, E_2, T_2, \dots, E_n, T_n, E_1$; cf. Figure 1. We start with triangle T_1. Choose

$$V_T = \{v \in H^1(T) : v = 0 \text{ on } E_2\}.$$

We pose the problem for $w \in V_T$

$$(\nabla w, \nabla v) = \int_{E_1} r^{(2)}_{E_1} v\,.$$

A function $v \in V_T$ can be extended to $\tilde{v} \in H^1(\omega)$ with $\operatorname{supp} \tilde{v} \subset T_1 \cup T_n$ and $\|\tilde{v}\|_{1,\omega} \le c\|v\|_{1,T}$, where c depends only on the shape parameter. By construction,

$$\int_{E_1} r_{E_1} v = \left\langle r^{(2)}, \tilde{v} \right\rangle.$$

We set $\sigma_T = \nabla w$, and conclude as above

$$\begin{aligned} \|\sigma_T\|_0 &\preceq \|r^{(2)}\|_{[H^1(\omega)/\mathbb{R}]^*}\,, \\ \operatorname{div} \sigma_T &= 0, \\ \operatorname{tr}_{n,\partial\omega} \sigma_T &= 0 \qquad \text{on } \partial T_1 \cap \partial\omega, \\ \operatorname{tr}_{n,E_1} \sigma_T &= r_{E,1}\,. \end{aligned}$$

By Lemma 3 there exists a polynomial $\sigma_{T_1}^3$ with the same properties, and bounded as σ_T. By subtracting, we obtain the new residual

$$r^{(3,1)} = r^{(2)} - \operatorname{div} \sigma_{T_1}^3.$$

It vanishes also on edge E_1.

We can now proceed with this procedure for all triangles in the patch except the last one. On the last triangle the residual is reduced to the edge terms on the two adjacent edges in ω. Here, we pose a pure Neumann problem. It is solvable, since $\left\langle r^{(3,n-1)}, 1\right\rangle = 0$.

We have decomposed the residual as a sum of divergences of piecewise polynomials that are bounded as stated in the theorem. □

4 A one dimensional projection operator

In order to prove Lemma 4 we start with Legendre polynomials P_i which are L_2-orthogonal on $(-1, +1)$:

$$(P_i, P_j) := (P_i, P_j)_{L2(-1,1)} = \delta_{i,j} \frac{2}{2i+1}.$$

We define the projection operator

$$(T_i u)(x) = \frac{2i+1}{2} \int_{-1}^{1} P_i(y) u(y)\, dy\, P_i(x),$$

which sends a function $u \in L^2(-1, 1)$ to the ith term of its Legendre expansion. Hence,

$$\|T_i u\|_0^2 = \left(\frac{2i+1}{2}\right)^2 (P_i, u)^2 \int_{-1}^{1} P_i(x)^2\, dx = \frac{2i+1}{2} (P_i, u)^2 \tag{18}$$

and, for $u \in L_2(-1, 1)$

$$\sum_{i=0}^{\infty} \|T_i u\|_0^2 = \|u\|_0^2.$$

The following quasi-projection operator into the space of polynomials of degree $2p - 1$ was introduced by de la Vallée-Poussin [6]:

$$S_p = \sum_{i=0}^{2p-1} c_i\, T_i \quad \text{with} \quad c_i = \begin{cases} 1, & i \le p, \\ (2p - i)/p, & p < i \le 2p - 1. \end{cases}$$

Lemma 8. *The smoothing operators S_p satisfy*
(i) S_p reproduce polynomials up to the order p.
(ii) The operators S_p are uniformly bounded in p, i.e.,

$$\|S_p\|_{L_2} = 1 \quad \textit{and} \quad \|S_p\|_{H^1} \leq 3.$$

Proof. The first assertion follows from the fact that $(T_i u)(x)$ gives the ith term in the Legendre expansion of $u(x)$. So, if u is a polynomial of degree less or equal p, then $(T_i u)(x) = 0$ for all $i > p$ and we have $(S_p u)(x) = u(x)$.

Next we estimate the L^2-norm of S_p. Note that $c_i \leq 1$. Exploiting the L^2-orthogonality of Legendre polynomials and identity (18) we calculate

$$\begin{aligned}
\|S_p u\|_0^2 &= \sum_{i,j=0}^{2p-1} c_i c_j \, \frac{2i+1}{2} \frac{2j+1}{2} \, |(P_i, u)||(P_j, u)| \, (P_i, P_j) \\
&= \sum_{i=0}^{2p-1} c_i^2 \|T_i u\|_0^2 \leq \sum_{i=0}^{2p-1} \|T_i u\|_0^2 = \|u\|_0^2 \, .
\end{aligned}$$

In the course of estimating the H^1 seminorm of $S_p u$ we will use a well-known relation between Legendre polynomials and their derivatives [12]

$$P_n(x) = \frac{1}{2n+1} (P'_{n+1}(x) - P'_{n-1}(x)), \tag{19}$$

and perform partial integration

$$\int_{-1}^{1} P_i(y) u(y) \, dy = -\frac{1}{2i+1} \int_{-1}^{1} (P_{i+1}(y) - P_{i-1}(y)) u'(y) \, dy. \tag{20}$$

The boundary terms above vanish since $P_n(1) = 1$ and $P_n(-1) = (-1)^n$. Another identity we will need is

$$(P'_i, P'_j) = \begin{cases} 0, & i - j \equiv_2 1, \\ l(l+1), & i - j \equiv_2 0 \ \text{ where } l = \min\{i, j\}. \end{cases} \tag{21}$$

Now we are in the position to start calculating

$$\begin{aligned}
\|S_p u\|_1^2 &= \sum_{i,j=1}^{2p-1} c_i c_j \, \frac{(2i+1)(2j+1)}{4}(P_i', P_j')(P_i, u)(P_j, u) \\
&= \frac{1}{4}\sum_{i,j=1}^{2p-1} c_i c_j \, (P_i', P_j')(P_{i+1} - P_{i-1}, u')(P_{j+1} - P_{j-1}, u') \\
&=: \frac{1}{4}\sum_{i,j=0}^{2p} M_{i,j}(P_i, u')(P_j, u').
\end{aligned}$$

With this reordering performed in the last step we defined a symmetric matrix M and we continue estimating,

$$\begin{aligned}
4\|S_p u\|_1^2 &\leq \sum_{i,j=0}^{2p} |M_{i,j}||(P_i, u')| \, |(P_j, u')| \\
&= \sum_{i,j=0}^{2p} \frac{2|M_{i,j}|}{\sqrt{2i+1}\sqrt{2j+1}}\|T_i u'\|_0 \|T_j u'\|_0.
\end{aligned}$$

Let $M^{(0)}$ be the normalized matrix with entries $M_{i,j}^{(0)} = \frac{1}{\sqrt{2i+1}\sqrt{2j+1}}|M_{i,j}|$ and let $t = (\|T_i u'\|_0)_{i=0}^{2p}$. Then we have

$$2\|S_p u\|_1^2 \leq t^T M^{(0)} t \leq \rho(M^{(0)})^2 \|t\|_2^2 = \rho(M^{(0)})^2 \sum_{i=0}^{2p} \|T_i u'\|_0^2 \leq \rho(M^{(0)})^2 \|u'\|_0^2.$$

Thus it remains to show that the spectral radius of $M^{(0)}$ is bounded by a constant. Therefore we will estimate the row-sum norm of $M^{(0)}$ which is an upper bound for $\rho(M^{(0)})$.

We omit the lengthy computations of the coefficients of M and will merely state the resulting expressions. Furthermore, because of identity (21) only the coefficients $M_{i,j}$ with $i \equiv_2 j$ are nonzero.

We list the results for M starting with the diagonal entries,

$$\begin{aligned}
M_{i,i} &= 2(2i+1), \qquad 0 \le i \le p-1,\\
M_{p,p} &= \frac{1}{p^2}(p-1)(4p^2-p-2),\\
M_{i,i} &= \frac{1}{p^2}[(2p-i-1)^2(4i+2)+4i(i-1)],\\
&\qquad p+1 \le i \le 2p-1,\\
M_{2p,2p} &= \frac{2}{p}(2p-1).
\end{aligned}$$

Since M is symmetric, we will only consider the upper right triangular matrix. So from now on we assume $i \le j-2$. First, we have in the upper left block

$$M_{i,j} = 0, \qquad \text{for} \quad i < j \le p-1.$$

Since also $M_{i,j} = 0$ if $i-j$ is odd, we silently assume in the following formulas that $i-j$ is even. In order to deal with a special factor in the last row, we introduce the factor

$$\theta_j = \begin{cases} 1, & j \le 2p-1,\\ \frac{1}{2}, & j = 2p. \end{cases}$$

The nonzero off-diagonal entries are now given by

$$\begin{aligned}
M_{i,p} &= -\frac{2}{p}(2i+1), \qquad i < p,\\
M_{i,j} &= -\frac{4}{p}(2i+1)\,\theta_j, \qquad i < p < j,\\
M_{p,j} &= -\frac{2}{p^2}(p-1)(3p+2)\,\theta_j, \qquad p < j,\\
M_{i,j} &= \frac{4}{p^2}(1+2i+3i^2-2p(2i+1))\,\theta_j, \qquad p < i < j.
\end{aligned}$$

Now we can verify by inspection that

$$M_{i,i} \le 2(2i+1) \quad \text{and} \quad |M_{i,j}| \le \frac{4}{p}\sqrt{2i+1}\,\sqrt{2j+1},\ i \ne j.$$

Since there are at most p nonzero off-diagonal entries in each row, it follows that the rowsum of $M^{(0)}$ does not exceed 6, and the proof is complete. $\square$

It remains to find a projector from P^{2p-1} to P^p. This will be done by separating polynomials with zero boundary values. To this end we define extension operators that provide polynomials of low H^s norm, $s = 0, 1$, to given boundary data:

$$E_p^{(s)}u(x) = \operatorname*{argmin}_{\substack{v\in P^p(-1,1),\\ v(-1)=u(-1),\, v(1)=u(1)}} \|v\|_s, \qquad s = 0, 1.$$

The operators have the following properties:

Lemma 9. *There is a constant C independent of p such that*

$$\|E_p^{(s)}u\|_s \le C\,\|u\|_s\,, \qquad \forall\, u \in P^{2p-1},\ s = 0, 1.$$

Proof. We restrict ourselves to $p \ge 2$, since we may choose $E_p^{(s)}u = u$ if $p = 0$ or $p = 1$.

For $s = 1$ the minimal energy extension is given by

$$E_p^{(1)}u(x) = u(-1)\frac{1-x}{2} + u(1)\frac{1+x}{2}.$$

The norm estimate follows by the trace theorem, so we have

$$\|E_p^{(1)}u\|_1 \le |u(-1)| + |u(1)| \le C\|u\|_1.$$

In order to determine the L_2-extension we consider the minimization for left and right endpoint separately,

$$w^{\pm} = \operatorname*{argmin}_{v\in P^p(-1,1), v(\pm1)=1, v(\mp1)=0} \|v\|_0^2.$$

It will be crucial that the estimate in [11, Lemma 9.1] is sharp. The ansatz $v(x) = \sum_{i=0}^p v_{\pm,i}P_i(x)$ transforms the constraint minimization problems into algebraic ones,

$$\min v^t A v,$$

with the diagonal matrix $A = \operatorname{diag}\left(\frac{2}{2i+1}\right)_{i=0}^p$. The constraints are now $\sum_{i=0}^p(-1)^i v_{-,i} = 1$ and $\sum_{i=0}^p v_{-,i} = 0$ for the left endpoint, respectively $\sum_{i=0}^p v_{+,i} = 1$ and $\sum_{i=0}^p(-1)^i v_{+,i} = 0$ for the right endpoint. Obviously the solutions are

$$v_{+,i} = \frac{2i+1}{p(p+2)}\left[1 + \frac{(-1)^{i+p+1}}{p+1}\right] \qquad \text{and} \qquad v_{-,i} = (-1)^i v_{+,i}.$$

The total extension operator is then given by

$$E_p^{(0)}u(x) = w^-(x) + w^+(x) = \sum_{i=0}^{p} [(-1)^i u(-1) + u(1)] v_{+,i} P_i(x).$$

Exploiting the L_2-orthogonality of Legendre polynomials a simple summation shows that

$$\begin{aligned} \|E_p^{(0)}u\|_0^2 =& \frac{2}{(p+1)(p+2)}[u(-1)^2 + u(1)^2] \\ &+ \frac{2}{p(p+1)(p+2)}[u(1) - (-1)^p u(-1)]^2. \end{aligned} \tag{22}$$

Examining (22) we find that $\|E_p^{(0)}u\|_0 \leq 3\|E_{2p-1}^{(0)}u\|_0$. The L_2-norm of the given function is certainly not smaller than the minimal one in P^{2p-1} and thus we obtain

$$\|E_p^{(0)}u\|_0 \leq 3\|E_{2p-1}^{(0)}u\|_0 \leq 3\|u\|_0. \tag{23}$$

□

We note that it is necessary to restrict the domain of the extension operators in Lemma 8, e.g., to P^{2p-1}. Otherwise we have no bound like (23). For this reason the map S_p will enter into the analysis.

Now we define projection operators for functions with zero boundary values. Let

$$L_i(x) = c_i \int_{-1}^{x} P_{i-1}(s)\, ds, \qquad i \geq 2,$$

where $c_i = \frac{1}{2}\sqrt{(2i-3)(2i-1)(2i+1)}$, be the ith integrated Legendre polynomial. These polynomials are orthogonal with respect to the H^1 norm and have zeros at $x = \pm 1$. The normalization has been chosen according to [2], where the the following norm estimates for $u = \sum_{i=2}^{M} u_i L_i$ have been shown:

$$\|u'\|_0^2 \approx \sum_{i=2}^{M} i^2 u_i^2, \qquad \|u\|_0^2 \approx \sum_{i=2}^{M} \left[\frac{1}{i^2}u_i^2 + (u_i - u_{i+2})^2\right]. \tag{24}$$

We set $P_0^k = P^k \cap H_0^1$ and define the projection operators

$$\begin{aligned} R_p \quad &: \quad P_0^{2p-1} \to P_0^p \\ u = \sum_{i=2}^{2p-1} u_i L_i \quad &\mapsto \quad R_p u = \sum_{i=2}^{p} (u_i - \frac{i}{p} u_{2p-i+1}) L_i\,. \end{aligned}$$

Lemma 10. *The norms $\|R_p\|_0$ and $\|R_p\|_1$ of the projection operators are uniformely bounded in p.*

Proof. Recalling (24) we obtain by a straight forward calculation

$$\begin{aligned}
\|R_p u\|_1^2 &\approx \sum_{i=2}^{p} i^2 (u_i - \frac{i}{p} u_{2p-i+1})^2 \\
&\leq 2 \left(\sum_{i=2}^{p} i^2 u_i^2 + \sum_{i=2}^{p} \frac{i^4}{p^2} u_{2p-i+1}^2 \right) \\
&\leq 2 \left(\sum_{i=2}^{p} i^2 u_i^2 + \sum_{i=2}^{p} (2p-i+1)^2 u_{2p-i+1}^2 \right) \\
&= 2 \sum_{i=2}^{2p-1} i^2 u_i^2 \leq C \|u\|_1^2.
\end{aligned}$$

The boundedness in the L^2-norm follows the same lines. We first use (24) to express $\|R_p u\|_0$, and then basic estimates yield

$$\begin{aligned}
\|R_p u\|_0^2 &\preceq \sum_{i=0}^{p} \frac{1}{i^2} u_i^2 + \sum_{i=0}^{p} \frac{1}{p^2} u_{2p-i+1}^2 \\
&\quad + \sum_{i=2}^{p} (u_i - u_{i+2})^2 + \sum_{i=2}^{p} \frac{i^2}{p^2} (u_{2p-i+1} - u_{2p-i+3})^2 \\
&\preceq \sum_{i=2}^{2p-1} \frac{1}{i^2} u_i^2 + (u_i + u_{i+2})^2 \approx \|u\|_0^2.
\end{aligned}$$

□

Now we are in the position to define the composite projection operators

$$\tilde{R}_p^{(s)} = R_p(I - E_p^{(s)}) + E_p^{(s)}, \quad s = 0, 1.$$

Since $(E_p^{(1)} - E_p^{(0)})v$ is a polynomial of degree less or equal p for $v \in P^{2p-1}$, the operators $\tilde{R}_p^{(0)}$ and $\tilde{R}_p^{(1)}$ coincide. Indeed,

$$\begin{aligned}
\tilde{R}_p^{(1)} v &= [R_p(I - E_p^{(1)}) + E_p^{(1)}]v \\
&= [R_p(I - E_p^{(0)}) + E_p^{(0)}]v + (I - R_p)(E_p^{(1)} - E_p^{(0)})v \\
&= \tilde{R}_p^{(0)} v.
\end{aligned}$$

Hence, the norm estimates of the individual operators prove the estimate for $\tilde{R}_p^{(0)} = \tilde{R}_p^{(1)}$.

Finally we set $Q^p := \tilde{R}_p^{(0)} S_p = \tilde{R}_p^{(1)} S_p$ to complete the proof of Lemma 4.

5 Numerical example

The Poisson equation $-\Delta u = 1$ is considered on the L-shaped domain $\Omega := (-1,+1)^2 \backslash [0,1) \times (-1,0]$. We assume homogeneous Neumann boundary conditions on $\Gamma_N := (0,1) \times \{0\}$ and homogeneous Dirichlet boundary conditions on $\Gamma_D := \partial\Omega \backslash \Gamma_N$.

We have computed finite element solutions for $p = 1, 2, 4$, and 8 with h-adaptive codes. Moreover, a reference solution was obtained by computations with even higher order polynomials. The equilibrated fluxes for the a posteriori error estimates have been determined with BDM^p and BDM^{p+1} elements. There are only small differences between the two variants. The efficiency was measured by the quotients

$$\frac{\text{eror estimator}}{\text{real eror}}.$$

They were smaller than 1.6 in all cases depicted in Figures 2 and 3. A selection is listed in Tables 2 and 3.

Table 2: Error of finite elements of degree 4 via BDM^5.

unknowns	estimator	real error	quotient
45	1.165	0.8502	1.371
97	1.011	0.7576	1.334
191	0.9728	0.7491	1.299
779	0.5313	0.4056	1.310
1563	0.2139	0.1621	1.329
3131	0.03393	0.02576	1.317
4307	0.009050	0.007139	1.268

Table 3: Error of finite elements of degree 8 via BDM^9.

unknowns	estimator	real error	quotient
153	0.7873	0.5684	1.385
353	0.6779	0.5047	1.343
1009	0.5618	0.4253	1.321
2561	0.3579	0.2695	1.328
4113	0.2265	0.1701	1,331
6441	0.1135	0.08516	1.332
8769	0.05677	0.04259	1.333
10321	0.03577	0.02683	1.333
12649	0.01789	0.01342	1.333

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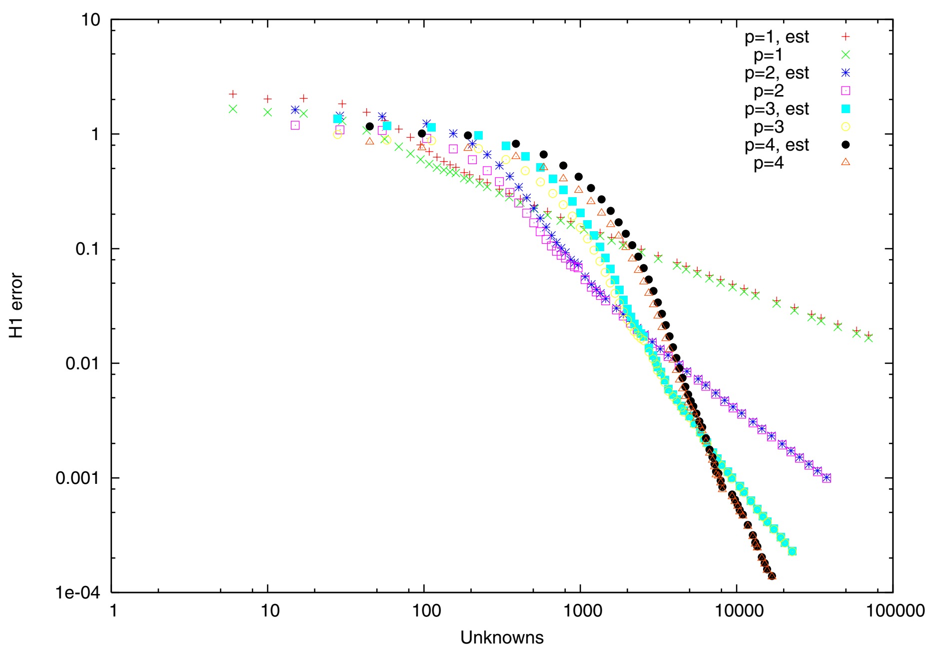

Figure 2. Results for L-shaped domain. The equilibration is performed with BDM^{p+1} elements. are

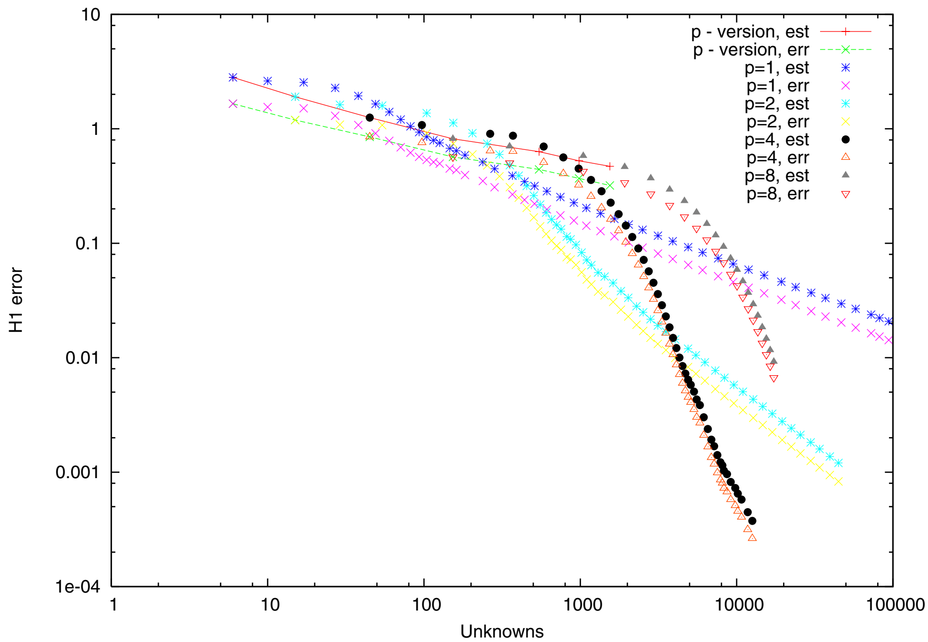

Figure 3. Results for L-shaped domain. The equilibration is performed with BDM^p elements.