Polynomial Approximation of Noninteger Powers of the Distance Function

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In the numerical treatment of elliptic equations in d-space there arises the polynomial approximation of

$$f(x) := \|x - x_0\|^{-2s} = (\sum_{i=1}^{d}(x_i - x_{0,i})^2)^{-s}$$

when the fundamental solution is to be approximated. The degree of approximation was determined for $d = 2$ in [1] by identifyeng $\mathbb{R}^2$ with $\mathbb{C}$. Therefore there arose the impression that the result is restricted to $d = 2$. A simple argument admits an entension to $d \geq 3$.

We recall that we are interested in the approximation of f on a ball in Euclidean d-space that does not contain the point x_0. After a translation and scaling, the ball is assumed to be the unit ball. Moreover, by a rotation x_0 is mapped to the point $(r, 0, \ldots, 0)$ with $r > 1$.

From (2.3) in [1] we know that there is a polynomial Q_n such that

$$|[(x_1 - r)^2 + y^2]^{-s} - Q_n(x_1, y)| \leq cn^{2s+1}r^{-n}$$

if $x_1^2 + y^2 \leq 1$. Since the polynomial $Q_n(x_1, -y)$ is an equally good approximation, the convex combination

$$Q_{n,n/2}(x_1, y^2) = \frac{1}{2}[Q_n(x_1, y) + Q_n(x_1, -y)]$$

is an equally good approximation. Now we set $y^2 = \sum_{i=2}^{d} x_i^2$ and obtain

$$|[(r - x_1)^2 + x_2^2 + \ldots + x_d^2]^{-s} - Q_{n,n/2}(x_1, x_2^2 + \ldots + x_d^2)| \leq cn^{2s+1}r^{-n}$$

if $\|x\| \leq 1$. This proves the generalization to the d-dimensional approximation problem.

References

[1] D. Braess, Note on the Approximation of Powers of the Distance in 2-Dimensional Domains, Constr. Approx. 17 (2001), 147–151.

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