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Approximation on Simplices and Orthogonal Polynomials

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Abstract

Inequalities of Jackson and Bernstein type are derived for polynomial approximation on simplices with respect to Sobolev norms. Although we do not find simple bases when looking at 120 years of research of orthogonal polynomials on triangles, sharp estimates are obtained from a decomposition into orthogonal subspaces. The formulas reflect the symmetries of simplices, but analogous estimates on rectangles show that we cannot expect rotational invariance of the terms with derivatives. An essential tool are selfadjoint differential operators that have already been used by other authors for the study of various approximation properties.

1 Introduction

The approximation of functions by polynomials with respect to a weighted L_2-norm is strongly related to orthogonal polynomials. This is well known for functions on the real interval $[-1,+1]$. The orthogonal polynomials for constant weights are the Legendre polynomials P_n which satisfy

$$\int_{-1}^{+1} P_n P_m dx = \frac{2}{2n+1}\delta_{nm}.$$

The Legendre polynomials are eigenfunctions of the singular Legendre differential operator,

$$\mathcal{L}P_n = \mu_n P_n, \quad \mu_n = n(n+1)$$

where $\mathcal{L}$ is given by $(\mathcal{L}v)(x) := -((1-x^2)v')'$. We therefore have also orthogonality of the derivatives with respect to a weight function which vanishes at the boundaries

$$\int_{-1}^{+1}(1-x^2)P_n'P_m'dx = \mu_n \frac{2}{2n+1}\delta_{nm}.$$

If we expand an L_2-function with respect to the Legendre polynomials for the natural normalization $v = \sum_{k=0}^{\infty} b_k \left(k+\frac{1}{2}\right)^{1/2} P_k$, then we have obviously,

$$\begin{aligned} \|v\|_0^2 &:= \int_{-1}^{+1} v^2 dx &&= \sum_{k=0}^{\infty} |b_k|^2, \\ |v|_{1,w}^2 &:= \int_{-1}^{+1}(1-x^2)(v')^2 dx &&= \sum_{k=1}^{\infty} \mu_k |b_k|^2 \end{aligned} \tag{1}$$

and more generally, for any $\ell \in \mathbb{N}_0$,

$$|v|_{\ell,w}^2 := (-1)^\ell \int_{-1}^{+1} v\mathcal{L}^\ell v dx = \sum_{k=1}^{\infty} (\mu_k)^\ell |b_k|^2$$

which is to be understood in the sense that the series converge if and only if $|v|_{\ell,w}$ is finite. We obtain from the definitions for v, with $|v|_{m,w} < \infty$, $\ell, m \in \mathbb{N}_0$, $m \geq \ell$, the *approximation property* (direct estimate)

$$\inf_{p\in\mathcal{P}_n} |v-p|_{\ell,w} \leq (\mu_{n+1})^{-(m-\ell)/2} |v|_{m,w} \tag{2}$$

and the *inverse estimate*

$$|p|_{m,w} \leq (\mu_n)^{(m-\ell)/2} |p|_{\ell,w} \quad \text{ for } p \in \mathcal{P}_n. \tag{3}$$

Remark 1.1 *This fits into the following general framework. Let X be a Banach space which is compactly imbedded into Y. Therefore $\|\cdot\|_X$ is a finer norm than $\|\cdot\|_Y$. Moreover, let V_m, $m \in \mathbb{N}$ be a family of finite dimensional subspaces of X. The pair X, Y is appropriate for the family (V_m) if there are parameters c_m and a constant C such that the direct approximation property*

$$\inf_{p\in V_m} \|v-p\|_Y \leq c_m \|v\|_X \quad \forall v \in X \tag{4}$$

and the inverse estimate

$$\|p\|_X \leq C\, c_m^{-1} \|p\|_Y \quad \forall p \in V_m$$

*hold. [We note that we cannot have $\|p\|_X \leq o(c_m^{-1})\|p\|_Y$ $\forall p \in V_m$ together with (**??**).] – Classical pairs of spaces that fit in this sense are given by C^0 and C^m due to Jackson's and Bernstein's theorems. Finite element spaces are another example; see* *e.g. [4, p. 85] for h-FEM, i. e. when convergence is achieved by refinements of the meshes. Recently the p-FEM has attracted much interest, i. e. the approximation is improved by increasing the degree of the polynomials [21]. Here the theory is less complete.*

Direct and inverse estimates for the rectangle are easily obtained from these results by tensor product arguments [5]. Those results show already that we cannot expect rotational invariance of the inequalities.

The situation on triangles and more generally on simplices in $\mathbb{R}^d$ is more involved. There are two approaches in the literature for orthogonal polynomials on triangles/simplices, but none of them can be used directly for our purpose. We will demonstrate that by an algebraic counterpart. The remedy is that we are content with a decomposition into *orthogonal subspaces*. In particular, we will use some selfadjoint differential operators that have been discovered independently by several authors for different purposes.

2 Orthogonal Polynomials on Triangles

The one-dimensional example in the introduction showed already the relation between orthogonal polynomials and the approximation problem under consideration. There are two different approaches to orthogonal polynomials on triangles.

In 1881 Appell [1] introduced polynomials F_{mn} which give rise to a biorthogonal system F_{mn} and E_{mn} on triangles. The polynomials (and some generalizations)

$$F_{mn}(x,y) := \frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[x^m y^n (1-x-y)^{m+n} \right]$$

are now called Appell's polynomials. Obviously F_{mn} is a polynomial of degree $m+n$.

Let

$$\mathcal{P}_N := span\{x^m y^n;\ m+n \le N\} \quad and \quad \mathcal{Q}_N := \mathcal{P}_N \cap \mathcal{P}_{N-1}^{\perp}.$$

Then F_{mn} *is orthogonal to* $\mathcal{P}_{m+n-1}$.

We provide the (simple) proof since the technique (from 1881) is typical also for recent constructions. It is sufficient to verify the orthogonality for monomials $x^k y^l$ with $k+l \le m+n$. Without loss of generality we assume that $k < m$. By partial integration we obtain

$$\begin{aligned} & \int_0^{1-y} x^k y^l \frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[x^{m+\alpha} y^{n+\beta} (1-x-y)^{m+n+\gamma} \right] dx \\ = \quad & (-1)^m \int_0^{1-y} \left(\frac{\partial^m}{\partial x^m} x^k y^l \right) \frac{\partial^n}{\partial y^n} \left[x^{m+\alpha} y^{n+\beta} (1-x-y)^{m+n+\gamma} \right] dx \\ = \quad & 0 \end{aligned}$$

for $0 < y < 1$. This is a standard argument with Rodriguez' formula. After integrating over y we have the orthogonality. ■

Although Appell's polynomials $F_{mn},\ m+n \le N$, span $\mathcal{Q}_N$, the polynomials F_{mn} and F_{kl} with $k+l = m+n$ and $(k,l) \ne (m,n)$ are unfortunately not orthogonal. It is difficult to provide an orthogonal basis without destroying the symmetry of the triangle.

[The situation is comparable to that of the eigenvalue problem with the matrix

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

The matrix is invariant under permutations of the coordinates. There is an eigenvector $(1,1,1)$ with the eigenvalue 0. The orthogonal subspace consists of eigenvectors, but we cannot provide a basis without destroying the symmetry.] – We will refer to invariant subspaces due to this feature.

Investigations of orthogonal polynomials based on Appell's polynomials were done e. g. by Appell and Kampé de Fériet (1926), Gröbner (1948), Erdélyi, Magnus, Oberhettinger, and Tricomi (1953), Fackerell and Littler (1974), Derriennic (1985).

Another approach to orthogonal polynomials is obtained from a transformation of the triangle to the square; see Proriol (1957), Karlin and McGregor (1964), Szegő (1974), Koornwinder (1975), Mysovski (1981), Dunkl (1984), Suetin (1988), Dubiner (1991), Xu (1998). Consider the product

$$p_m(\frac{x}{1-y})(1-y)^m q_{n,m}(y) \tag{5}$$

where p_m is the m-th orthogonal polynomial for the weight 1 and $q_{n,m}$ is the n-th orthogonal polynomial for the weight $(1-y)^m$. Obviously the products provide orthogonal polynomial for the triangle and can be expressed in terms of Jacobi polynomials. Unfortunately these polynomials are less suited for our intention since the transformation makes that the derivatives of the fractions in (**??**) give rise to expressions that are more involved.

For completeness we also refer to [17].

3 Estimates on the Simplex in $\mathbb{R}^d$

Now we are prepared to consider the original approximation problem on a d-simplex S^d. The simplex is the convex hull of its $d+1$ vertices $a_0, a_1, \ldots, a_d \in \mathbb{R}^d$ which do not lie on a $(d-1)$-dimensional hyperplane. In order to keep the symmetry we refer to the barycentric coordinates $\lambda_0, \lambda_1, \ldots, \lambda_d$ of the points $x = \sum_j \lambda_j a_j \in S^d$. Specifically we have

$$\lambda_j \ge 0,\ j = 0, 1, \ldots, d, \quad \sum_j \lambda_j = 1,$$

We will make use of multiindex notation, in particular

$$\lambda^m := \lambda_0^{m_0}\lambda_1^{m_1}\dots\lambda_d^{m_d}, \qquad \lambda^\alpha = \lambda_0^{\alpha_0}\lambda_1^{\alpha_1}\dots\lambda_d^{\alpha_d},$$

and $|m| = \sum_j m_j$, $|\alpha| = \sum_j \alpha_j$. We assume that $\alpha_j > -1$ for all j. Hence, $w_\alpha := \lambda^\alpha$ is a weight function for which the inner product

$$(f, g) = \int_{S^d} f g w_\alpha \tag{6}$$

and the weighted L_2-norm $\|f\|_{0,w}^2 := (f, f)$ is well defined. As before, we set

$$\mathcal{P}_n := \operatorname{span}\{\lambda^m;\ |m| \le n\} \quad \text{and} \quad \mathcal{Q}_n := \mathcal{P}_n \cap \mathcal{P}_{n-1}^\perp.$$

Due to the condition $\sum \lambda_j = 1$, the representation of a function given in terms of barycentric coordinates is not unique. Nevertheless we can write the directional derivative for the direction from a_k to a_j in the form

$$\frac{\partial}{\partial\lambda_j} - \frac{\partial}{\partial\lambda_k} \quad \text{or for short} \quad \partial_j - \partial_k.$$

Lemma 3.1 *Let $j \ne k$. Then the differential operator of second order*

$$\mathcal{L}_0 := -\lambda^{-\alpha}(\partial_j - \partial_k)\,\lambda_j\lambda_k\lambda^\alpha\,(\partial_j - \partial_k) \tag{7}$$

is selfadjoint with respect to the inner product $(\cdot,\cdot)$. It maps $\mathcal{P}_n$ into $\mathcal{P}_n$ and $\mathcal{Q}_n$ into $\mathcal{Q}_n$.

Sketch of proof. Consider a segment on a line parallel to the direction from a_k to a_j. The product $\lambda_j\lambda_k$ vanishes at the two points at which the line intersects the boundary of S^d. No boundary terms occur when performing partial integration. Therefore $\mathcal{L}_0$ is selfadjoint.

The degree of a polynomial is not augmented by the application of $\mathcal{L}_0$ since the multiplication by the quadratic polynomials is compensated by two differentiations. The arguments of Appell show that also the orthogonal complement $\mathcal{Q}_n$ is mapped into itself. ■

Now combinations of the differential operators of the form (5) have been used for several purposes [3, 5, 6, 8, 7, 14, 20, 24]. In particular,

$$\mathcal{L}_w := -\lambda^{-\alpha}\sum_{j<k}(\partial_j - \partial_k)\lambda_j\lambda_k\lambda^\alpha(\partial_j - \partial_k) \tag{8}$$

can be regarded as a Laplacian for the simplex due to its symmetry. Special cases of the eigenvalue problem (7) have already been stated by Appell and Kampé de Fériet [2] in terms of Appell's polynomials. Proofs can be found in the literature cited above. A simple proof in [5] makes use of the fact that $\mathcal{L}_w$ maps $\mathcal{Q}_n$ into itself and that it is sufficient to determine the image $\mathcal{L}_w p$ merely modulo $\mathcal{P}_{n-1}$.

Theorem 3.2 *The operator $\mathcal{L}_w$ is selfadjoint and*

$$\mathcal{L}_w p = \mu_n p \quad \text{for all } p \in \mathcal{Q}_n. \tag{9}$$

with the eigenvalues μ_n explicitly given by

$$\mu_n = \mu_n(d,\alpha) := n(n+d+|\alpha|), \quad n = 1, 2, \dots \tag{10}$$

In accordance with (1) we now define a weighted H^1-seminorm which will form an appropriate pair together with $\|\cdot\|_{0,w}$

$$|f|_{1,w}^2 := \sum_{j<k} \int_{S^d} |(\partial_j - \partial_k) f|^2 \lambda_j \lambda_k w_\alpha.$$

We obtain our essential tool from the fact that $\mathcal{L}_w$ is selfadjoint

$$|f|_{1,w}^2 = \int_{S^d} f(\mathcal{L}_w f) w_\alpha. \tag{11}$$

In particular assume that f is expanded into polynomials from the orthogonal subspaces

$$f = \sum_{k=0}^{\infty} p_k \quad \text{with } p_k \in \mathcal{Q}_k.$$

From the orthogonality of $\mathcal{Q}_k$ and $\mathcal{Q}_l$, $k \neq l$, and Theorem 3.2 we conclude that

$$\begin{aligned} \|f\|_{0,w}^2 &= \sum_{k=0}^{\infty} \|p_k\|_{0,w}^2 \,, \\ |f|_{1,w}^2 &= \sum_{k=0}^{\infty} \int_{S^d} p_k(\mathcal{L}_w p_k) w_\alpha = \sum_{k=0}^{\infty} \mu_k \|p_k\|_{0,w}^2 \,, \end{aligned}$$

and, more generally, for any $\ell \in \mathbb{N}_0$,

$$|f|_{\ell,w}^2 := \sum_{k=0}^{\infty} \int_{S^d} p_k(\mathcal{L}_w^\ell p_k) w_\alpha = \sum_{k=0}^{\infty} (\mu_n)^\ell \|p_k\|_{0,w}^2.$$

The last equality is understood in the sense that the infinite series converges if and only if $|f|_{\ell,w}$ is finite. Similar to $|f|_{1,w}$, the seminorm $|f|_{\ell,w}$ admits the following representation in terms of f and its derivatives:

$$|f|_{\ell,w}^2 = \begin{cases} \displaystyle\int_{S^d} (\mathcal{L}_w^m f)^2 w_\alpha & \text{if } \ell = 2m, \\ \displaystyle\int_{S^d} (\mathcal{L}_w^m f) \mathcal{L}_w (\mathcal{L}_w^m f) w_\alpha & \text{if } \ell = 2m+1. \end{cases} \tag{12}$$

Accordingly, for $m \in \mathbb{N}_0$, we define the weighted spaces

$$V_w^m(S^d) := \left\{ v \in L^2(S^d);\ |f|_{\ell,w} < \infty \text{ for } \ell = 0, 1, \ldots, m \right\}.$$

In the literature cited above there are several results on the approximation by polynomials on simplices. The following theorem from [5] fits into the framework of Remark 1.1 and admits a formulation such that there is no gap between the direct and the inverse estimate.

Theorem 3.3 *Let ℓ, m be nonnegative integers and $m \geq \ell$ and denote by $\mu_n = n(n+d+|\alpha|)$ the eigenvalues of $\mathcal{L}_w$. Then, for any $v \in V_w^m(S^d)$, the approximation property*

$$\inf_{p \in \mathcal{P}_n} |v - p|_{\ell,w} \leq (\mu_{n+1})^{-(m-\ell)/2}\, |v|_{m,w} \quad n = 0, 1, 2, \ldots$$

holds, and for any $p \in \mathcal{P}_n$ we have the inverse estimate

$$|p|_{m,w} \leq (\mu_n)^{(m-\ell)/2}\, |p|_{\ell,w}.$$

Both inequalities are sharp.

The operator $\mathcal{L}_w$ annihilates constants, but its k-th power does not annihilate $\mathcal{P}_{k-1}$. Recently, Jetter and Stöckler [14] have constructed symmetric differential operators of higher order which do not have this defect. Their operators can be used for improving the results of Theorem 3.3.

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