

A posteriori error estimates for the EAS method

or

"Why are checkerboard modes good for avoiding locking"

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1. Introduction

There are many cases in which the EAS elements of Simo and Repai helped to avoid locking. It is easy to implement them, so they became popular.

There were however drawbacks since some people reported on checkerboard modes while other people did not. In particular when nonlinear problems were attacked, there is no uniform meaning whether the method is results are positive or negative.

D.Braess, C.Carlstensen, and B.D.Reddy

Uniform convergence and a posteriori error estimators
for the enhanced strain finite element method
Numer. Math. (2004)

I think that we are now in a position to understand the discrepancy/inconsistency. We can now provide answers to questions that we could not answer in Krakow 3 years ago. We have got positive results and results of negative character. Fortunately, the theory is so complete now that we say how to overcome/avoid the traps. You see my provoking title. It is given in this way to emphasize that there are traps and that the remedy is not trivial.

1. For which cases is volume locking eliminated
 - and for which quantities is it not?
2. In the misbehaviour of
~~Are the problems with nonlinear equations~~ problems associated to nonlinearities
 - or have we inherited a defect of the linear theory?
3. Does the EAS method provide an automatic filter for checker board modes
 - and have we to pay for it?
4. How is the relation to ~~the~~ related methods as the B-bar method or the u-p-method?

You can check the error by the a posteriori
error estimator given at the end of my talk.

2. The EAS method and variants

We consider the linear elasticity problem, and focus on nearly incompressible material

$$\Pi(u) := \int_{\Omega} \varepsilon(u) : \varepsilon(u) dx + \frac{\lambda}{2} \int_{\Omega} (\operatorname{div} u)^2 dx - \int_{\Omega} f \cdot u dx$$

$$\lambda \gg \mu, \quad \varepsilon(u) = \text{strain}$$

We start with bilinear elements - and get locking as it is well known. The following ~~several~~ variants "remedies" are equivalent or almost equivalent.

A. The EAS-method

* Extra degrees of the strain ~~soften~~ make the volume term softer.

B. In each element $\operatorname{div} u$ is replaced by a constant \rightarrow the mean value $\overline{\operatorname{div} u}$ \rightarrow it is softer

C. Explicit one-point integration

(coincides with B for rectangular elements)

D. The $n-p$ -method with piecewise constant p

Results in the $Q_1 - P_0$ element.

E. The B-bar method

F. The nonconforming element by Taylor, Wilson et al.

G. The Hellinger-Reissner approach with piecewise constant diagonal terms of stresses

For completeness we list methods for nearly incompressible materials that beta with a different behaviour.

- a. The PEERS element
- b. u-p-elements which are good for flow problems
- c. EAS element with enhanced strains on macro elements
- d. hp elements with adaptivity

*

$$\Pi_{EAS}(u_h, \tilde{\epsilon}_h) = \mu \int_{\Omega} (\epsilon(u_h) + \tilde{\epsilon}_h) : (\epsilon(u_h) + \tilde{\epsilon}_h) dx + \frac{\lambda}{2} \int_{\Omega} (\text{div } u_h + \text{tr } \tilde{\epsilon}_h)^2 dx - \int_{\Omega} f u_h dx$$

A strengthened Cauchy inequality between $\epsilon(v_h)$ and $\tilde{\epsilon}_h$ holds

3. A positive result

We start with a positive result for the algorithms A-G

Theorem (B. Carstensen, Reddy). The displacement u and the strains $\varepsilon(u)$ converge in

$$\|u - u_h\|_1 + \|\varepsilon(u) - \varepsilon(u_h)\|_0 + \|\text{enhanced strain}\|_0 \leq c h \|f\|_0$$

with a constant that is independent of the Lamé constant λ .

I will provide an argument for this result since it allows us to understand the limitations.

The ~~coercive~~ ellipticity constant of the quadratic form

$$a(v, v) := \mu \int_{\Omega} \varepsilon(v) : \varepsilon(v) dx + \frac{\lambda}{2} \int_{\Omega} (\operatorname{div} v)^2 dx$$

is μ as long as we are in the kernel of the divergence operator.

From fluid flow we know that the kernel is big.

Unfortunate, the kernel is only a five dimensional space if we are turning to Q_1 elements and

$$\|\operatorname{div} v_h\|_0 \geq \frac{h}{12 \cdot \operatorname{diam}(\Omega)} \|v_h - v_h^{\text{special}}\|_1$$

So our problem becomes too stiff,

if our solution sits outside the special low-dimensional space.

If we

The divergence is a linear function on each element.

If we decompose it

$$\operatorname{div} v = \overline{\operatorname{div} v} + \operatorname{osc}(\operatorname{div} v)$$

then we have an orthogonality relation and

$$\int_{\Omega} (\operatorname{div} v)^2 dx = \int_{\Omega} (\overline{\operatorname{div} v})^2 dx + \int_{\Omega} \operatorname{osc}^2(\operatorname{div} v) dx$$

The softened quadratic form

$$a_h(v, v) := \mu \int_{\Omega} \varepsilon(v) : \varepsilon(v) dx + \frac{\lambda}{2} \int_{\Omega} (\overline{\operatorname{div} v})^2 dx$$

has the correct stiffness which leads to the positive result.

Consequence When we take the soft FE solution u_h ,

$a(u_h, u_h)$ is large

$a_h(u_h, u_h)$ is reasonable.

Therefore, there are large oscillations of $\lambda^{1/2} \operatorname{div} u_h$
⇒ Stress recovery:

$$\tilde{\sigma}_h := 2\mu \varepsilon(u_h) + \lambda \overline{\operatorname{div} u_h}$$

4. Limitation of the EAS method

We saw that the EAS method is favourable for the computation of displacements and strains since the locking of the original Q_1 -elements is not present. Unfortunately this was observed in actual computations. Unfortunately the positive observations conceal the limitations.

For nearly incompressible material the EAS method is equivalent with the mixed $u-p$ -method. Specifically, the bilinear elements for the displacement lead to the Q_1-P_0 element that is known the unstable. Checkerboard modes are observed in fluid mechanics. If linear the equations are linear, the checkerboard modes can be extracted by a filter.

First impression: The EAS ^{element} behaves like the Q_1-P_0 element with an automatic implementation of the filter.

Correct answer. If the inf-sup condition is not satisfied, then the Fortin interpolation operator is not bounded — the functions in the kernel are not well approximated — there is locking

In the case of "nearly incompressible material" we have a saddle point problem with penalty. The checkerboard modes have a high energy. They are not observed — which makes that the user feels safe, but the result is not correct for the volumetric part of the stress

$$\|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\mathbf{u}_h)\| \leq c h \quad \text{with } c \text{ independent of } \lambda$$

$$\boldsymbol{\sigma}_h = 2\mu \boldsymbol{\varepsilon}(\mathbf{u}_h) + \lambda \operatorname{div} \mathbf{u}_h$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \leq c h \quad \text{and } c \text{ explodes if } \lambda \rightarrow \infty !$$

Remedies (A) Add piecewise Let the given grid be a refinement of a $2h$ -grid.

Add piecewise constant strains (traces of strains) that are checkerboard modes on the $2h$ -grid

+	-
-	+

(B) Take the mean value of the divergence on ~~the $2h$ -~~ macroelements.

$$a_h(v, v) = \mu \int_{\Omega} \varepsilon(v) : \varepsilon(v) dx + \frac{\lambda}{2} \int_{\Omega} (\overline{\operatorname{div} v}^{2h})^2 dx$$

Then you will observe checkerboard modes, but they can be observed like the oscillations that we have mentioned above.

Of course there are also ~~that~~ remedies from different frameworks.

Remark If we have no macroelements, distortion locking will be present.

Rule of thumb. When you start with bilinear elements, you can model the volumetric part of the stress only on

Remark A similar analysis can be performed for shear locking. There is another feature that you find in the treatment of the Mindlin-Reissner plate

$$\bar{H} = \int_{\Omega} \epsilon(\Theta) : C \epsilon(\Theta)$$

$$T = a(\Theta_h, \Theta_h) + \frac{1}{h^2} \int_{\Omega} (\nabla \Theta_h - \Theta_h)^2 dx + \left(\frac{1}{t^2} \cdot \frac{1}{h^2} \right) \underbrace{\int_{\Omega} (\nabla \Theta_h - \Theta_h)^2 dx}_{f_h}$$

Only the second portion is softened here.

If you look at the FE-spaces for Θ_h, w_h, f_h , you see that the shear terms must live in small spaces.

In this spirit, the arguments that have been elaborated for volume locking are transported to shear locking.

5. A posteriori error estimators.

We can use the residues of the saddle-point formulation for the u-p method.

For a posteriori estimators it is sufficient that the continuous saddle point problem is stable.

Good behaviour of the FE spaces is not required

(If the spaces are not good, the estimators will be large).

Theorem (B, Causser, and Reddy)

There is a constant c that does not depend on λ, h
such that

$$c \|\sigma_h - \tilde{\sigma}_h\|_0 \leq \mu \|\tilde{\sigma}_h\|_0 + \|h(f + \operatorname{div}_h \tilde{\sigma}_h)\|_0 + \|h^{1/2} [\tilde{\sigma}_h^T \tilde{u}_e]\|_{L^2(U_e)}$$