Research Report on the Hypercircle Method

by Dietrich Braess and Joachim Schöberl

We got interested in the hypercircle method since it enables the derivation of reliable a posteriori error estimates without generic constants. It turned out, however, that it avoids also a loss of efficiency that we encounter when residual error estimators are used with the *hp* finite element method. Moreover, we obtain a priori error estimates for the comparison of different finite element families that are not achieved with the classical tools.

A Reliable A Posteriori Error Estimate

For simplicity, we present the theorem of Prager and Synge for the Poisson equation $-\Delta u = f$ and homogeneous boundary conditions (although there are applications to the curl-curl equation, to obstacle problems, and those discussed below). Γ_D and Γ_N are the parts of the boundary with Dirichlet and Neumann conditions, respectively. A flux σ which satisfies (1) is called *equilibrated*. All norms are L_2 norms.

Theorem of Prager and Synge [1947] (Two-Energies-Principle).
Let $\sigma \in H(\mathrm{div})$, $\sigma \cdot n = 0$ on Γ_N while $v \in H^1(\Omega)$, $v = 0$ on Γ_D and assume that

$$\mathrm{div}\,\sigma + f = 0. \tag{1}$$

Furthermore, let u be the solution of the Poisson equation. Then,

$$\|\nabla u - \nabla v\|^2 + \|\nabla u - \sigma\|^2 = \|\nabla v - \sigma\|^2. \tag{2}$$

The computation of an equilibrated flux σ_h for treating P_1 elements is performed in the broken Raviart–Thomas space of lowest order. The triangulation is the same as that one, on which the finite element solution u_h lives (in contrast to the construction of Luce and Wohlmuth). The flux σ_h is determined by the equilibration on small patches, i.e., by a cheap postprocessing.

When the well-known a posteriori estimators are analyzed, the data oscillation is separated (or a saturation assumption is incorporated). Let f denote the given right-hand side of the differential equation and denote its L_2 projection onto piecewise constant functions as $\bar{f}$. The *data oscillation* $ch\|f - \bar{f}\|$ produces an extra term of higher order that can be determined a priori. We obtain an error estimate without a generic constant in the main term:

$$\|\nabla u - \nabla u_h\| \le \|\sigma_h - \nabla u_h\| + ch\|f - \bar{f}\|\,. \tag{3}$$

Limit of the Hypercircle Method

A comparison with residual estimators shows that (3) is also efficient. Here we rather study the efficiency by returning to the theorem of Prager and Synge and take a second look at (2).

Obviously, the efficiency of the error estimate is $\|\nabla u_h - \sigma_h\| / \|\nabla u_h - \nabla u\|$, and it is the smaller the better the solution σ_h of the mixed method approximates ∇u. Now we can see a limit of the hypercircle method. If the mixed method (or another equilibration) produces an approximation that is worse than the original finite element solution, then the efficiency is not optimal. If, on the other hand, the mixed method produces a better result σ_h, then we can use σ_h as the better solution of the original problem instead of taking it merely for an error estimate.

Obviously, a value of $\sqrt{2}$ is the optimum. Indeed, various numerical experiments show that the efficiency is often close to this number. Thus, there is no *curse of the hypercircle method*.

In this context, we note that Repin's students obtained efficiencies close to 1.01 by computing equilibrated fluxes by higher order methods.

A Comparison of Finite Element Families

Classical results say that the error of the conforming P_1 element $u_h^{(1)}$, of the nonconforming P_1 element u_h^{CR}, and of the Raviart–Thomas element σ_h^{RT}, respectively, is $O(h)$. It is not excluded that one method is substantially better than the other ones for a special right-hand side f_1, while there is a different preference for some f_2. Now we get a more positive information by recalling that we have already used complementary spaces in (3). We also incorporate Ainsworth' application of the hypercircle method. As usual, $A \preceq B$ means $A \leq cB$ and $A \approx B$ that $A \preceq B$ and $B \preceq A$ holds.

Theorem. Assume that f is piecewise constant on the FE-mesh. Then

$$\|\nabla u_h^{(2)} - \nabla u\| \preceq |\nabla u_h^{CR} - \nabla u\|_{0,h} \approx \|\sigma_h^{RT} - \nabla u\| \preceq \|\nabla u_h^{(1)} - \nabla u\| \, .$$

A Posteriori Estimates for the *hp* Method

Melenk and Wohlmuth showed by theoretical and numerical investigations that the efficiency of residual estimators deteriorates as $O(p)$ when applied to the *hp* method. We observed, however, efficiency factors not far from $\sqrt{2}$ for the hypercircle method. Full efficiency could be proven for rectangular grids by the construction of uniformly bounded right inverses of the divergence operator in polynomial spaces. However, we did not succeed in treating triangular meshes.

We expect similar problems for a posteriori estimates by local Neumann problems. Uniform bounds for the discretization of the local problems are not trivial.

Validation of Plate Models

The hypercircle method was also used for the justification of plate models in order to get a priori information on the effects of dimension reduction. The solutions of the differential equations in 2-space are not only extended to 3-dimensional displacement fields, but also equilibrated stresses are constructed. In 1959 Morgenstern performed this concept for the Kirchhoff plate, while Allessandrini, Arnold, Falk, and Madureira studied the classical Mindlin plate model and generalizations in 2004.

References

[1] M. Ainsworth, *A posteriori error estimation for lowest order Raviart Thomas mixed finite elements.* Preprint 2007

[2] A.L. Alessandrini, D.N. Arnold, R.S. Falk, and A.L. Madureira, *Derivation and justification of plate models by variational methods.* In "Plates and Shells, Quebec 1996", (M. Fortin, ed.), pp. 1-20. CRM Proceeding and Lecture Notes, vol. 21, American Mathematical Society, Providence, RI 1999.

[3] D. Braess, Finite Elements: Theory, Fast Solvers and Applications in Solid Mechanics. 3rd edition. Cambridge University Press 2007, pp. 181–185

[4] D. Braess, An a posteriori error estimate for the nonconforming P1 element. (submitted)

[5] D. Braess and J. Schöberl, *Equilibrated residual error estimator for edge elements.* Math. Comp. 77 (2008), 652–677

[6] R. Luce and B. Wohlmuth, *A local a posteriori error estimator based on equilibrated fluxes.* SIAM J. Numer. Anal. **42** (2004), 1394–1414.

[7] P. Neittaanmäki and S. Repin, Reliable Methods for Computer Simulation. Error control and a posteriori estimates. Elsevier. Amsterdam 2004.